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Intermediate Spaces and the Complex Method of Interpolation for Families of Banach Spaces.

EUGENIO HERNÁNDEZ (*)

1. – Introduction.

Recently, R. Coifman, M. Cwikel, R. Rochberg, Y. Sagher and G. Weiss have developed a theory of complex interpolation for families of Banach spaces ([3], [4]). They start with a family of Banach spaces associated with the boundary of the unit disk Δ in $\mathbb{C}$ (the set of complex numbers) and, for each complex number in the interior of Δ , they are able to define an intermediate space with properties that are appropriate for interpolation. (For a summary of this construction and its properties see section 2 below). This method generalizes that of Calderón for pairs of Banach spaces ([2]).

In the same papers they proved that the intermediate spaces of L^p spaces are also L^p spaces. Specifically, if p is a measurable function defined on T , the boundary of Δ , whose range is contained in $[1, \infty]$, then the intermediate space at the point z , interior to Δ , of the family of Banach spaces $\{L^{p(\xi)}\}$, $\xi \in T$, is $L^{p(z)}$, where $1/p(z)$ is the harmonic function on Δ whose boundary values are $1/p(\xi)$.

In this paper we continue the identification of other spaces of measurable functions as well as spaces of vector valued sequences (this work was suggested in [4]). More precisely, we identify the intermediate spaces of weighted L^p spaces, L^p spaces of Banach space valued functions, Lorentz spaces, l_p^s spaces of vector valued sequences, Sobolev and Besov-Lipschitz spaces. This is accomplished by developing a theory of interpolation of Banach

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lattices that generalizes that of A. P. Calderón ([2]). Since only the general theory is developed in [3], this work is a natural complement to that paper.

As for notation we systematically use the letter θ instead of $e^{i\theta}$ to denote an element of $T = \{z \in \mathbb{C}: |z| = 1\}$. Also, $P_z(\theta)$ will denote the Poisson kernel of Δ for evaluation at $z \in \Delta$, $Q_z(\theta)$ will denote the conjugate Poisson kernel and $H_z(\theta) = P_z(\theta) + iQ_z(\theta)$ will denote the Herglotz kernel.

We assume that the reader is familiar with the basic facts of the real and complex interpolation methods. Unless otherwise stated the norm on a Banach space B will be denoted by $\|\cdot\|_B$.

2. – The complex interpolation method.

We now describe the complex interpolation method for families of Banach spaces and summarize some of its properties. Let $\{B(\theta)\}$, $\theta \in T$, be a family of Banach spaces associated with the boundary of the unit disk in $\mathbb{C}$. We say that this family is an *interpolation family of Banach spaces* (or *interpolation family*, for short) if each $B(\theta)$ is continuously embedded in a Banach space $(U, \|\cdot\|_U)$, the function $\theta \rightarrow \|b\|_{B(\theta)}$ is measurable for each $b \in \bigcap_{\theta \in T} B(\theta)$, and if

$$\beta = \left\{ b \in \bigcap_{\theta \in T} B(\theta) : \int_T \log^+ \|b\|_{B(\theta)} d\theta < \infty \right\}$$

we have $\|b\|_U \leq k(\theta) \|b\|_{B(\theta)}$, for all $b \in \beta$, where $\log^+ k(\theta) \in L^1$ (the space β is called the *log-intersection space* of the given family and U is called a containing space).

We let $N^+(B(\cdot))$ be the space of all β -valued analytic functions of the form

$$g(z) = \sum_{j=1}^{\infty} \psi_j(z) b_j$$

for which $\|g\|_{\infty} = \sup \|g(\theta)\|_{B(\theta)} < \infty$, where $\psi_j \in N^+$ and $b_j \in \beta$, $j = 1, 2, \dots, m$. (N^+ denotes the positive Nevalinna class for Δ (see [5], Chapter 2)). The completion of the space $N^+(B(\cdot))$ with respect to $\|\cdot\|_{\infty}$ is denoted by $\mathcal{F}(B(\cdot))$. (It is not difficult to show that $\mathcal{F}(B(\cdot))$ is a closed subspace of a Banach space of analytic functions). The space $[B(\theta)]_z$, which will also be denoted by $B(z)$, consists of all elements of the form $f(z)$ for $f \in \mathcal{F}(B(\cdot))$. A Banach space norm is defined on $B(z)$ by

$$\|v\|_z \equiv \|v\|_{B(z)} = \{\|f\|_{\infty} : f \in \mathcal{F}(B(\cdot)), f(z) = v\}$$

$v \in B(z)$. It can be proved that $(B(z), \| \cdot \|_z)$ is a Banach space and β is dense in each $B(z)$. The space $B(z)$ is called an intermediate space of the family $\{B(\theta)\}$, $\theta \in T$.

This construction has the following two fundamental properties:

THEOREM (2.1). (*Subharmonicity*). For each $g \in \mathcal{F}(B(\cdot))$ and each $z \in \Delta$ we have

$$\|g(z)\|_{B(z)} \leq \exp \int_T \log \|g(\theta)\|_{B(\theta)} P_z(\theta) d\theta.$$

THEOREM (2.2). (*Interpolation theorem*). Let T be a linear operator which maps U continuously into V , where U and V are containing spaces for the families $\{A(\theta)\}$ and $\{B(\theta)\}$, respectively. Suppose further that T maps $\mathcal{A}$ into $\bigcap_{\theta \in T} B(\theta)$ with $\|Ta\|_{B(\theta)} \leq M(\theta) \|a\|_{A(\theta)}$ for all $a \in \mathcal{A}$, $\theta \in T$, where $\log M(\theta)$ is absolutely integrable on T and $\mathcal{A}$ is the log-intersection space of the family $\{A(\theta)\}$. Then, T maps $A(z)$ into $B(z)$ with norm not exceeding

$$M(z) = \exp \int_T (\log M(\theta)) P_z(\theta) d\theta, \quad z \in \Delta.$$

The duality and reiteration theorems hold as well as an interpolation theorem for « analytic » families of linear operators.

See [3] for details and proofs. In [3] a relation between this interpolation construction and the complex interpolation method of Calderón is also given.

3. – The fundamental inequality.

Let (M, dx) be a fixed measure space. Suppose that the function $p: \bar{\Delta} \rightarrow [1, \infty]$ is such that $1/p(z)$ is harmonic on Δ . A measurable function $F: T \times M \rightarrow \mathbb{R}$ is called p -admissible if $\int_T d\theta P_z(\theta) \|F(\theta, \cdot)\|_{L^p(\theta)} < \infty$ for some $z \in \Delta$ (and hence for all z).

PROPOSITION (3.1). For a p -admissible function F we have

$$\log \|u_F(z, \cdot)\|_{L^p(z)} \leq \int_T d\theta P_z(\theta) \log \|F(\theta, \cdot)\|_{L^p(\theta)}$$

where $u_F(z, x) = \exp \left\{ \int_T d\theta H_z(\theta) \log |F(\theta, x)| \right\}$, $z \in \Delta$.

PROOF. We start by proving the inequality for $p \equiv 1$. In other words, we assume that F is 1-admissible and we want to show

$$(3.1) \quad \log \int_M |u_F(z, x)| dx \leq \int_T d\theta P_z(\theta) \log \|F(\theta, \cdot)\|_{L^1}.$$

The right hand side of the above inequality is a harmonic function on Δ . Since F is 1-admissible, an application of Jensen's inequality and Fubini's theorem imply that $u_F(z, \cdot) \in L^1$ for all $z \in \Delta$. Since u_F is analytic, a theorem of E.M. Stein and G. Weiss (see [10]) implies that the function $\log \int_M |u_F(z, x)| dx$ is subharmonic. Since both functions have the same boundary values, namely $\log \|F(\theta, \cdot)\|_{L^1}$, the values of the subharmonic function must be smaller than the values of the harmonic one. This proves inequality (3.1).

To prove proposition (3.1), fix $z_0 \in \Delta$ and let $g \geq 0$ be a simple function on M satisfying $\|g\|_{L^{q(z_0)}} \leq 1$, where $(1/p(z)) + (1/q(z)) = 1$.

Denote by $a(z)$ the unique analytic function in Δ whose real part has boundary values $1/q(\theta)$ and $a(z_0) = 1/q(z_0)$. Consider $g(z, x) = [g(x)]^{a(z)q(z_0)}$, $z \in \bar{\Delta}$.

Simple calculations show $\int_T d\theta P_{z_0}(\theta) \log |g(\theta, x)| = \log g(x)$. From here and (3.1) we deduce

$$\int_M g(x) |u_F(z_0, x)| dx \leq \exp \left\{ \int_T d\theta P_{z_0}(\theta) \log \left(\int_M |F(\theta, x) g(\theta, x)| dx \right) \right\}.$$

Since $\|g(\theta, \cdot)\|_{L^{q(z)}} \leq 1$ for all $\theta \in T$, the above inequality together with Hölder's inequality implies

$$\int_M g(x) |u_F(z_0, x)| dx \leq \exp \left\{ \int_T d\theta P_{z_0}(\theta) \log \|F(\theta, \cdot)\|_{L^{p(\theta)}} \right\}.$$

From here, the fundamental inequality follows by observing that $\|u_F(z_0, x)\|_{L^{p(z_0)}} = \sup \left\{ \int_M g(x) |u_F(z_0, x)| dx / g \geq 0 \right\}$ simple and $\|g\|_{L^{q(z_0)}} \leq 1$.

We remark that a particular case of inequality (3.1) is Hölder's inequality. To see this take $f, g \in L^1$ and $0 \leq s \leq 1$, and apply (3.1) to $F(\theta, x) = f(x)\chi_{[0, 2\pi s)}(\theta) + g(x)\chi_{[2\pi s, 2\pi)}(\theta)$ at $z = 0$, where χ_E denotes the characteristic function of the set E . The result is Hölder's inequality with $p = 1/s$ and $q = 1/1 - s$.

4. – Banach lattices and examples.

A subclass X of the class of measurable functions on a measure space (M, dx) is called a *Banach lattice* if there exists a norm $\|\cdot\|_X$ on X such that

$(X, \|\cdot\|_X)$ is a Banach space and if $f \in X$ and g is a measurable function such that $|g(x)| \leq |f(x)|$ a.e. on M , then $g \in X$ and $\|g\|_X \leq \|f\|_X$.

Given a Banach lattice X on (M, dx) we present below a way to construct others. Let $\varphi(x, t)$ be a real valued function defined on $M \times [0, \infty]$ such that $\varphi(\cdot, 0) \equiv 0$ on M and for each $x \in M$, $\varphi(x, t)$ is a concave increasing function on t . Denote by $\varphi(X)$ the class of measurable functions g on M for which there exist $\lambda > 0$ and $f \in X$ with $\|f\|_X \leq 1$ such that

$$|g(x)| \leq \lambda \varphi(x, |f(x)|).$$

The norm of an element $g \in \varphi(X)$, denoted by $\|g\|_{\varphi(X)}$, is defined as the infimum of the values of λ for which the above inequality holds. It is well known ([2], § 13.3 and 33.3) that $(\varphi(X), \|\cdot\|_{\varphi(X)})$ is a Banach lattice.

We now give some examples of Banach lattices, which will be needed in the sequel.

EXAMPLE 1. If $X = L^1 \equiv L^1(M)$, w is a positive measurable function on M and $\varphi_{p,w}(x, t) = [w(x)]^{-1/p} t^{1/p}$, $1 \leq p \leq \infty$, then $\varphi_{p,w}(L^1)$ coincides with L^p_w , the L^p space with respect to the weight w .

EXAMPLE 2. Let $s \in \mathbb{R}$, $1 \leq p \leq \infty$ and $\varphi_{p,s}(n, t) = 2^{-sn} t^{1/p}$, $n \in \mathbb{N}$. Then, $\varphi_{p,s}(l_1)$ is the space l^s_p of all real valued sequences $a = (a_n)_{n=1}^\infty$ such that $\|a\|_{l^s_p} = \left\{ \sum_{n=1}^w [2^{sn} |a_n|]^p \right\}^{1/p} < \infty$. (When $p = \infty$ we write $\|a\|_{l^s_p} = \sup_n 2^{sn} |a_n|$). When $s = 0$ we shall write l_p instead of l^0_p for obvious reasons.

EXAMPLE 3. For $x \in (0, \infty)$, $p \in \mathbb{R} (p \neq 0)$ and $1 \leq q \leq \infty$ we define $\varphi_{p,q}(x, t) = x^{1/q-1/p} t^{1/q}$. If we consider the Lebesgue measure dx on the set $(0, \infty)$, the Banach lattice $\varphi_{p,q}(L^1) \equiv \varphi_{p,q}(L^1(0, \infty))$ is the space $X_{p,q}$ of all measurable functions g on $(0, \infty)$ such that

$$\|g\|_{X_{p,q}} = \left\{ \int_0^\infty [x^{1/p} |g(x)|]^q \frac{dx}{x} \right\}^{1/q} < \infty.$$

What needs to be proved in examples 1 and 3 is straightforward; example 2 is contained in example 1 by taking $M = \mathbb{N}$ with the discrete measure and $w(n) = 2^{sn}$, $n \in \mathbb{N}$.

5. – Interpolation of Banach lattices.

Let $\{X(\theta)\}$, with $\theta \in T$, be a family of Banach lattices on a fixed measure space (M, dx) . For $z \in \mathcal{A}$ we denote by $[X(\theta)]^z$ the class of measurable func-

tions f on M for which there exist $\lambda > 0$ and a measurable function $F: T \times M \rightarrow \mathbb{R}$ with $\|F(\theta, \cdot)\|_{X(\theta)} \leq 1$ a.e. such that

$$|f(x)| \leq \lambda \exp \left\{ \int_T d\theta P_z(\theta) \log |F(\theta, x)| \right\}.$$

We let $\|f\|^z \equiv \|f\|_{[X(\theta)]^z}$ be the infimum of the values of λ for which such an inequality holds.

LEMMA (5.1). $([X(\theta)]^z, \|\cdot\|^z)$ is a Banach lattice on (M, dx) .

PROOF. The homogeneity of the norm is clear. The subadditivity is not so clear. To prove it we proceed as follows. Let f_n be a sequence of functions in $[X(\theta)]^z$ such that $\sum_{n=1}^{\infty} \|f_n\|^z < \infty$. Then, given $\varepsilon > 0$, there exist λ_n and measurable functions $F_n: T \times M \rightarrow \mathbb{R}$ satisfying $\|F_n(\theta, \cdot)\|_{\varphi(X)} \leq 1$, $\lambda_n \leq \|f_n\|^z + \varepsilon/2^n$ and

$$|f_n(x)| \leq \lambda_n \exp \left\{ \int_T d\theta P_z(\theta) \log |F_n(\theta, x)| \right\},$$

$n = 1, 2, \dots$. Use proposition (3.1) with $M = \mathbb{N}$, the discrete measure on $\mathbb{N}$ and $p = 1$ to obtain

$$(5.1) \quad \sum_{n=1}^{\infty} |f_n(x)| \leq \lambda \exp \left\{ \int_T d\theta P_z(\theta) \log \left(\sum_{n=1}^{\infty} \frac{\lambda_n}{\lambda} F_n(\theta, x) \right) \right\},$$

where $\lambda = \sum_{n=1}^{\infty} \lambda_n$. Since $\left\| \sum_{n=1}^{\infty} (\lambda_n/\lambda) |F_n(\theta, x)| \right\|_{X(\theta)} \leq 1$, a convergence in measure argument (see [2], § 13.2 and 33.2) shows that the above series converges to an element $g(\theta, x) \in X(\theta)$ such that $\|g(\theta, \cdot)\|_{X(\theta)} \leq 1$. The inequality (5.1) then implies that $\sum_{n=1}^{\infty} |f_n(x)| \in [X(\theta)]^z$ and $\left\| \sum_{n=1}^{\infty} |f_n(x)| \right\|^z \leq \lambda \leq \sum_{n=1}^{\infty} \|f_n\|^z + \varepsilon$.

Since ε was arbitrary we deduce $\left\| \sum_{n=1}^{\infty} |f_n(x)| \right\|^z \leq \sum_{n=1}^{\infty} \|f_n\|^z$. This proves the subadditivity of the norm as a particular case. The only remaining property of the norm that is not clear is that $\|f\|_z = 0 \Rightarrow f = 0$ a.e. Assume $\|f\|_z = 0$. For each integer n , there exist functions $F_n: T \times M \rightarrow \mathbb{R}$ with $\|F_n(\theta, \cdot)\|_{X(\theta)} \leq 1$ a.e. such that

$$(5.2) \quad |f(x)| \leq \exp \left\{ \int_T d\theta P_z(\theta) \log \frac{1}{n^2} |F_n(\theta, x)| \right\}.$$

Then $\sum_{n=1}^{\infty} \|(1/2^n) F_n(\theta, x)\|_{X(\theta)} \leq \sum_{n=1}^{\infty} 1/n^2 < \infty$. As above, a convergence in measure argument shows that $(1/n^2) F_n(\theta, \cdot)$ tends to zero a.e. as $n \rightarrow \infty$. Inequality (5.2) now implies $f = 0$ a.e.

It remains to be proved that $([X(\theta)]^z, \|\cdot\|^z)$ is complete. Let f_n be a sequence of functions in $[X(\theta)]^z$ such that $\sum_{n=1}^{\infty} \|f_n\|^z < \infty$. We have proved that $\sum_{n=1}^{\infty} |f_n(x)| \in [X(\theta)]^z$ and $\left\| \sum_{n=1}^{\infty} |f_n(x)| \right\|^z \leq \sum_{n=1}^{\infty} \|f_n\|^z$. Thus, $\sum_{n=1}^{\infty} |f_n| < \infty$ a.e. and we can consider f as the pointwise sum of the series $\sum_{n=1}^{\infty} f_n$. Since $|f(x)| \leq \sum_{n=1}^{\infty} |f_n(x)|$ we see that $f \in [X(\theta)]^z$. Finally, it is easy to see that $\sum_{n=1}^{\infty} f_n$ converges to f in the space $([X(\theta)]^z, \|\cdot\|^z)$, which proves the completeness of this space. ■

We now apply this interpolation construction to particular Banach lattices. Let X be a Banach lattice on a measure space (M, dx) and let $\{\varphi_\theta\}$, $\theta \in T$, be a family of real valued functions defined on $M \times [0, \infty)$, measurable on θ , such that $\varphi_\theta(\cdot, 0) \equiv 0$ on M , a.e. θ , and for almost every θ and for each $x \in M$, $\varphi_\theta(x, t)$ is a concave increasing function of t . Suppose further, that for some $z \in \Delta$ (and hence for all)

$$(5.3) \quad \varphi_z(x, t) = \exp \left\{ \int_T d\theta P_z(\theta) \log \varphi_\theta(x, t) \right\} < \infty$$

for all $x \in M$, $t \in [0, \infty)$.

LEMMA (5.2). $\varphi_z(x, t)$ is a concave increasing function of t for all $z \in \Delta$, $x \in M$. Moreover, $\varphi_z(X) \subset [\varphi_\theta(X)]^z$ and the inclusion is norm decreasing.

PROOF. Let $0 \leq t_1 \leq t_2$ and $0 < \lambda < 1$. Inequality (3.1) applied to a two point measure space gives us

$$(1 - \lambda) \varphi_z(x, t_1) + \lambda \varphi_z(x, t_2) \leq \exp \left\{ \int_T d\theta P_z(\theta) \log |(1 - \lambda) \varphi_\theta(x, t_1) + \lambda \varphi_\theta(x, t_2)| \right\}.$$

The concavity of φ_z now follows from the concavity of each φ_θ .

To prove the inclusion, take $g \in \varphi_z(X)$ and $\varepsilon > 0$. Then, there exists $f \in X$ with $\|f\|_X \leq 1$ such that

$$(5.4) \quad |g(x)| \leq (1 + \varepsilon) \|g\|_{\varphi_z(X)} \exp \left\{ \int_T d\theta P_z(\theta) \log \varphi_\theta(x, |f(x)|) \right\}.$$

Since, clearly, $\varphi_\theta(x, |f(\cdot)|) \in X(\theta)$ and $\|\varphi_\theta(x, |f(\cdot)|)\|_{X(\theta)} \leq 1$, the definition of $[X(\theta)]^z$ and (5.4) imply $g \in [\varphi_\theta(X)]^z$ and $\|g\|^z \leq (1 + \varepsilon)\|g\|_{\varphi_z(X)}$, which allows us to obtain the desired conclusion upon letting $\varepsilon \rightarrow 0$. ■

Let now $1 \leq p(\theta) \leq \infty$ be a measurable function on T and for each $\theta \in T$ let $w_\theta(x)$ be a measurable function on M . Assume that for some $z \in \Delta$ (and hence for all)

$$(5.5) \quad w_z(x) = \exp \left\{ p(z) \int_T d\theta P_z(\theta) (1/p(z)) \log w_\theta(x) \right\} < \infty$$

a.e. $x \in M$, where $1/p(z)$ is the harmonic function on Δ whose boundary values are $1/p(\theta)$ (i.e. $1/p(z) = \int_T d\theta (1/p(\theta)) P_z(\theta)$). Lemma (5.2) together with example 1 of section 4 implies $L_{w_z}^{p(z)} \subset [L_{w_\theta}^{p(\theta)}]^z$, and the inclusion is norm decreasing. In this case the reverse inclusion is also true and it is a consequence of proposition (3.1). To see this, take $f \in [L_{w_\theta}^{p(\theta)}]^z$ and $\varepsilon > 0$. Choose a measurable function $F: T \times M \rightarrow \mathbb{R}$ such that $F(\theta, \cdot) \in L_{w_\theta}^{p(\theta)}$ with

$$\|F(\theta, \cdot)\|_{w_\theta}^{p(\theta)} \leq 1$$

and

$$|f(x)| \leq (1 + \varepsilon) \|f\|^z \exp \left\{ \int_T d\theta P_z(\theta) \log |F(\theta, x)| \right\}.$$

Proposition (3.1) now implies

$$\|f\|_{w_z}^{p(z)} \leq (1 + \varepsilon) \|f\|^z \exp \left\{ \int_T d\theta P_z(\theta) \log \|F(\theta, \cdot)\|_{w_\theta}^{p(\theta)} \right\} \leq (1 + \varepsilon) \|f\|^z.$$

This proves the following result:

PROPOSITION (5.3). *Let $1 \leq p(\theta) \leq \infty$ be a measurable function of T and for each $\theta \in T$ let $w_\theta(x) \geq 0$ be a measurable function on M . If $w_z(x) < \infty$ a.e. x , where w_z is given in (5.5), we have $[L_{w_\theta}^{p(\theta)}]^z = L_{w_z}^{p(z)}$, $z \in \Delta$, with equality of norms, where $1/p(z)$ is the harmonic function on Δ whose boundary values are $1/p(\theta)$.*

COROLLARY (5.4). *Let $1 \leq p(\theta) \leq \infty$ and $s(\theta)$ be two real values measurable functions on T such that $-\infty < s(z) = \int_T s(\theta) P_z(\theta) d\theta < \infty$. Then, $[l_{p(\theta)}^{s(\theta)}]^z = l_{p(z)}^{s(z)}$ where $1/p(z) = \int_T (1/p(\theta)) P_z(\theta) d\theta$.*

This is an easy consequence of the above proposition and example 2 of section 4. An argument similar to that used to prove proposition (5.3)

can be applied to the Banach lattices given in example 3, section 4, to obtain the following result:

PROPOSITION (5.5). *Let p, q be two measurable functions defined on T such that $1 \leq q(\theta) \leq \infty$ and $1 \leq p(\theta) \leq \infty$, $\theta \in T$. Assume that $1/p(z) = \int_T 1/p(\theta) P_z(\theta) d\theta$ and $1/q(z) = \int_T 1/q(\theta) P_z(\theta) d\theta$. Then, $[X_{p(\theta), q(\theta)}]^z = X_{p(z), q(z)}$ with equal norms, $z \in \Delta$.*

We remark that to obtain this result we need to use proposition (3.1) for $q(\theta)$ and the measure space $([0, \infty), dx/x)$.

6. – The relation with the complex method of interpolation.

Let B be a Banach space. A function defined on a measure space (M, dx) with values in B is said to be *measurable* if it is the limit almost everywhere of « simple B -values functions ». A function with values in B is said to be *simple* if it takes finitely many values, each on a measurable subset of M . Given a Banach lattice X on M we denote by $X(B)$ the class of B -values measurable functions $f(x)$ such that $\|f(x)\|_B \in X$ and we define $\|f\|_{X(B)} = \|\|f(x)\|_B\|_X < \infty$. It is known that $(X(B), \|\cdot\|_{X(B)})$ is a Banach space (see [2], 13.6 and 33.6).

We say that a Banach lattice X has the *dominated convergence property* if, given $f \in X$ and $\{f_n\}_{n=1}^\infty$ such that $|f_n| \leq |f|$, $n = 1, 2, \dots$ and $f_n \rightarrow 0$ as $n \rightarrow \infty$, then $\|f_n\|_X \rightarrow 0$ as $n \rightarrow \infty$. Notice that all the Banach lattices given in the examples of section 4 have the dominated convergence property.

A family of Banach lattices $\{X(\theta)\}$, $\theta \in T$, is called an *interpolation family* if it is an interpolation family of Banach spaces for which the containing space is also a Banach lattice and $\|f(x, \theta)\|_{X(\theta)}$ is a measurable function of θ for all measurable $f: M \times T \rightarrow \mathbb{R}$ such that $f(\cdot, \theta) \in X(\theta)$ a.e. θ .

THEOREM (6.1). *Suppose that $\{B(\theta)\}$ and $\{X(\theta)\}$, $\theta \in T$, are interpolation families of Banach spaces and that in addition each $X(\theta)$ is a Banach lattice on M and $\beta = \bigcap_{\theta \in T} B(\theta)$, where β is the log-intersection of the family $\{B(\theta)\}$.*

Then $\{X(\theta)(B(\theta))\}$, $\theta \in T$, is an interpolation family of Banach spaces and $[X(\theta)(B(\theta))]_z \subset [X(\theta)]^z(B(z))$. If, in addition, we assume that $[X(\theta)]^z$ has the dominated convergence property, the spaces $[X(\theta)(B(\theta))]_z$ and $[X(\theta)]^z(B(z))$ coincide and their norms are equal.

PROOF. We check first that $\{X(\theta)(B(\theta))\}$, $\theta \in T$, is an interpolation family of Banach spaces. If U is a containing Banach space of the family

$\{B(\theta)\}$ and V is a containing Banach lattice of the family $\{X(\theta)\}$, the Banach space $V(U)$ is a containing space for $\{X(\theta)(B(\theta))\}$. If $f \in \bigcap_{\theta \in T} X(\theta)(B(\theta))$, the function $\|f(x)\|_{B(\theta)}$ is measurable in θ for almost every $x \in M$; by hypothesis $\|f\|_{X(\theta)(B(\theta))} = \|\|f(x)\|_{B(\theta)}\|_{X(\theta)}$ is measurable in θ . Finally, if f belongs to the log-intersection space of the family $\{X(\theta)(B(\theta))\}$, we have $f(x) \in \bigcap_{\theta \in T} B(\theta) = \beta$ for almost every $x \in M$; thus, $\|f(x)\|_V \leq k_V(\theta)\|f(x)\|_{B(\theta)}$ and consequently $\|f(\cdot)\|_V \in \bigcap_{\theta \in T} X(\theta)$. Moreover, $\int_T \log^+ \|f\|_{X(\theta)(B(\theta))} d\theta < \infty$ implies

$$\int_T \log^+ \|\|f(\cdot)\|_V\|_{X(\theta)} d\theta \leq \int_T \log^+ k_V(\theta) d\theta + \int_T \log^+ \|f\|_{X(\theta)} d\theta < \infty,$$

which shows that $\|f(\cdot)\|_V \in \chi$, where χ denotes the log-intersection space of the family $\{X(\theta)\}$. Since $\{X(\theta)\}$ is an interpolation family we have $\|f\|_{V(V)} \leq \|k_V(\theta)\|f(x)\|_V\|_{X(\theta)} \leq k_V(\theta)k_U(\theta)\|f\|_{X(\theta)(B(\theta))}$ where $\int_T \log^+ k_V(\theta)k_U(\theta) d\theta < \infty$. This proves the desired result.

By an obvious density argument, the inclusion $[X(\theta)(B(\theta))]_z \subset [X(\theta)]^z(B(z))$ will follow from the inequality

$$(6.1) \quad \|g(z, \cdot)\|_{[X(\theta)]^z(B(z))} \leq \|g\|_\infty,$$

which is true for any g of the form $g(\xi, x) = \sum_{j=1}^N \Psi_j(\xi) f_j(x)$, where f_j belongs to the log-intersection space of the family $\{X(\theta)(B(\theta))\}$ and $\Psi_j \in N^+$. To prove (6.1) we observe that for almost every $x \in M$, $f_j(x) \in \bigcap_{\theta \in T} B(\theta) =$ and consequently $g(\xi, x) \in N^+(\beta)$ for a.e. $x \in M$. By theorem (2.1) we have

$$(6.2) \quad \|g(z, x)\|_{B(z)} \leq \|g\|_\infty \exp \left\{ \int_T d\theta P_z(\theta) \log \frac{\|g(\theta, x)\|_{B(\theta)}}{\|g\|} \right\}$$

a.e. $x \in M$, where $\|g\|_\infty = \operatorname{ess\,sup}_{\theta \in T} \|g(\theta)\|_{X(\theta)(B(\theta))}$ (notice that we can always assume $\|g\|_\infty \neq 0$). Since $\|\|g(\theta, x)\|_{B(\theta)}\|_{X(\theta)}\|_{X(\theta)} \leq 1$, the definition of $[X(\theta)]^z$ and (6.2) imply (6.1).

Before proving the reverse inclusion and the corresponding norm inequality we need the following lemma. The proof of this lemma is a straightforward modification of the proof of a lemma that can be found in [2] (33.6). Details can be found in [6].

LEMMA (6.2). *Assume that $[X(\theta)]^z$ has the dominated convergence property. Given $\varepsilon > 0$, let S_ε be the class of simple $k \in [X(\theta)]^z(B(z))$ such that there exists*

$K: T \times M \rightarrow \mathbb{R}$ with $\|K(\theta, \cdot)\|_{X(\theta)} \leq 1$ for all $\theta \in T$, satisfying

$$\|k(x)\|_{B(z)} = (1 + \varepsilon) \|k\|_{[X(\theta)]^z(B(z))} \exp \left\{ \int_T d\theta P_z(\theta) \log |k(\theta, x)| \right\}$$

and such that the non-zero values of each $k(\theta, \cdot)$ have positive upper and lower bounds. Then, S_ε^1 is dense in $[X(\theta)]^z(B(z))$.

We proceed now to prove the reverse inclusion. Let $k \in S_\varepsilon$ and write $k(x) = \sum_1^N \chi_j(x) a_j$ where $a_j \in B(z)$ and the χ_j are characteristic functions of disjoint measurable sets on M . We can find $\psi_j \in \mathcal{F}(B(\cdot))$ such that $\psi_j(z) = a_j / \|a_j\|_{B(z)}$, $j = 1, \dots, N$, and $\|\psi_j\|_\infty \leq 1 + \varepsilon$. Define

$$g(\xi, x) = (1 + \varepsilon) \|k\|_{[X(\theta)]^z(B(\theta))} \exp \left\{ \int_T d\theta P_z(\theta) \log |k(\theta, x)| \right\} \sum_{j=1}^N \chi_j(x) \psi_j(\xi)$$

where $k(\theta, x)$ is the function corresponding to $k \in S_\varepsilon$. Since each ψ_j is a limit of functions in $N^+(B(\cdot))$ one can show that $g \in \mathcal{F}(X(\theta))(B(\cdot))$. An elementary computation shows that $g(z, x) = k(x)$; thus, $k \in [X(\theta)(B(\theta))]_z$. Moreover, $\|\psi_j(\theta)\|_{B(\theta)} \leq \|\psi_j\|_\infty \leq 1 + \varepsilon$ implies

$$(6.3) \quad \|k\|_{[X(\theta)(B(\theta))]_z} \leq \|g\|_\infty \leq (1 + \varepsilon)^2 \|k\|_{[X(\theta)]^z(B(z))}.$$

Let now $f \in [X(\theta)]^z(B(z))$. By lemma (6.2) we construct a sequence of functions $k_m \in S_\varepsilon$ such that

$$(6.4) \quad \left\| f - \sum_{m=1}^N k_m \right\|_{[X(\theta)]^z(B(z))} \leq \frac{1}{2^N} \|f\|_{[X(\theta)]^z(B(z))}$$

and

$$(6.5) \quad \|k_m\|_{[X(\theta)]^z(B(z))} \leq \frac{1}{2^m} (1 + \varepsilon) \|f\|_{[X(\theta)]^z(B(z))}$$

$m = 1, 2, \dots$. By (6.4) the partial sum of the series $\sum_{m=1}^\infty k_m$ converges to f in $[X(\theta)]^z(B(z))$. On the other hand, (6.5) and (6.3) imply that $\sum_{m=1}^\infty k_m$ also converges in $[X(\theta)(B(\theta))]_z$ and its norm is smaller than $(1 + \varepsilon)^3 \|f\|_{[X(\theta)]^z(B(z))}$. But the two series coincide and so we have $f \in [X(\theta)(B(\theta))]_z$ with norm not exceeding $(1 + \varepsilon)^3 \|f\|_{[X(\theta)]^z(B(z))}$. The result follows from here since ε is arbitrary. ■

REMARK. We notice that, by taking $B(\theta) = \mathbf{R}$ for all $\theta \in T$, theorem (6.1) ensures us that, for $z \in \Delta$, $[X(\theta)]^z = X(z)$, provided $[X(\theta)]^z$ has the dominated convergence property.

7. – Interpolation of $L_w^p(B)$ and l_p^s spaces.

Let w be a positive measurable function on a measure space (M, dx) and $1 \leq p \leq \infty$. We say that $f \in L_w^p$ if $\|f\|_{L_w^p} = \left\{ \int_M |f(x)|^p w(x) dx \right\}^{1/p} < \infty$. If B is a Banach space $L_w^p(B)$ is defined as in section 6.

Suppose that $p: T \rightarrow [1, \infty]$ is a measurable function and $\{w_\theta\}$, $\theta \in T$, is a family of positive measurable functions on M such that

$$(7.1) \quad \theta \rightarrow w_\theta(x) \text{ is measurable for all } x \in M$$

and

$$(7.2) \quad \text{there exist } k: T \rightarrow (0, \infty) \text{ and } w: M \rightarrow \mathbf{R}_+ (w > 0) \text{ measurable such that } w(x) \leq k(\theta)w_\theta(x) \text{ a.e. } x \in M, \theta \in T, \text{ such that } \int d\theta \log^+ k(\theta) < \infty.$$

We claim that $\{L_{w_\theta}^{p(\theta)}\}$, $\theta \in T$, is an interpolation family of Banach lattices. To see this observe that if $f \in L_{w_\theta}^{p(\theta)}$ we have $f \in L_w^{p(\theta)}$ and $\|f\|_{L_w^{p(\theta)}} \leq [k(\theta)]^{1/p(\theta)} \|f\|_{L_{w_\theta}^{p(\theta)}}$. Moreover, $L_{w_\theta}^{p(\theta)} \subset L_w^1 + L^\infty$ and $\|f\|_{L_w^1 + L^\infty} \leq \|f\|_{L_{w_\theta}^{p(\theta)}}$ for all $f \in L_{w_\theta}^{p(\theta)}$ (see [13], 1.9.3). Therefore, we can take $U = L_w^1 + L^\infty$ as a containing space. The measurability of $\theta \rightarrow \|f\|_{L_{w_\theta}^{p(\theta)}}$ follows from (7.1) and the measurability of p . Finally, $\|f\|_U \leq [k(\theta)]^{1/p(\theta)} \|f\|_{L_{w_\theta}^{p(\theta)}}$ for all $f \in \bigcap_{\theta \in T} L_{w_\theta}^{p(\theta)}$ and $\int_T \log^+ [k(\theta)]^{1/p(\theta)} d\theta < \infty$.

By applying theorem (6.1) and proposition (5.3) we have the following result

PROPOSITION (7.1). *Let $p: T \rightarrow [1, \infty]$ be a measurable function and $\{w_\theta\}$, $\theta \in T$, be a family of positive measurable functions on M satisfying (7.1) and (7.2) and such that*

$$w_z(x) = \exp \left\{ p(z) \int d\theta P_z(\theta) \frac{1}{p(z)} \log w_\theta(x) \right\} < \infty.$$

Assume also that $\{B(\theta)\}$, $\theta \in T$, is an interpolation family of Banach spaces such that $\bigcap_{\theta \in T} B(\theta) = \beta$. Then $[L_{w_\theta}^{p(\theta)}(B(\theta))]_z = L_{w_z}^{p(z)}(B(z))$ and their norms coincide, where $1/p(z)$ is the harmonic function on Δ whose boundary values are $1/p(\theta)$.

REMARKS. The proposition and the interpolation theorems of [3] generalize an interpolation theorem for operators acting on L^p spaces with change of measures, due to E.M. Stein and G. Weiss (see [11]). By taking $w_\theta \equiv 1$ and $B(\theta) \equiv \mathbb{R}$ we obtain $[L^{q(\theta)}]_z = L^{q(z)}$, which has already been obtained in [3].

COROLLARY (7.2). *Let $q: T \rightarrow [1, \infty]$ and $s: T \rightarrow \mathbb{R}$ be measurable functions on T such that s is bounded below and $s(z) = \int_T s(\theta) P_z(\theta) d\theta < \infty$. If $\{B(\theta)\}, \theta \in T$, is an interpolation family of Banach spaces such that $\bigcap_{\theta \in T} B(\theta) = \beta$ we have*

$$1) [l_{q(\theta)}(B(\theta))]_z = l_{q(z)}(B(z)) \text{ and}$$

$$2) [l_{q(\theta)}^{s(\theta)}(B(\theta))]_z = l_{q(z)}^{s(z)}(B(z))$$

with equality of norms, where $1/q(z) = \int_T (1/q(\theta)) P_z(\theta) d\theta$.

PROOF. Take $M = \mathbb{N}$ with the discrete measure and $w_\theta(n) = 2^{s(\theta)np(\theta)}$ if $q(\theta) < \infty$ and $w_\theta(n) = 2^{s(\theta)n}$ if $q(\theta) = \infty$ and apply proposition (7.1). ■

8. – Interpolation of Sobolev and Besov-Lipschitz spaces.

The definitions of Sobolev and Besov-Lipschitz spaces that we shall use are taken from [1] (chapter 6). Let $\mathcal{S}$ be the class of Schwartz functions on $\mathbb{R}^n$ and let $\mathcal{S}'$, the dual of $\mathcal{S}$, be the space of tempered distributions. For $s \in \mathbb{R}$ and $f \in \mathcal{S}'$ we define $J^s f = \mathcal{F}^{-1}\{(1 + |\cdot|^2)^{s/2} \mathcal{F}f\}$, where $\mathcal{F}$ denotes the Fourier transform of f and $\mathcal{F}^{-1}$ its inverse. For $s \in \mathbb{R}$ and $1 \leq p \leq \infty$ we define the Sobolev space, $H_p^s \equiv H_p^s(\mathbb{R}^n)$ as the space of all $f \in \mathcal{S}'$ for which $\|f\|_p^s = \|J^s f\|_{L^p} < \infty$. It is known that H_p^s is a Banach space.

PROPOSITION (8.1). *Let $p: T \rightarrow (1, \infty)$ and $s: T \rightarrow \mathbb{R}$ be measurable functions on T such that s is bounded. Then, $\{H_{p(\theta)}^{s(\theta)}\}, \theta \in T$, is an interpolation family of Banach spaces and if*

$$(A) \int_T d\theta \log p(\theta) < \infty \text{ and } (B) \int_T d\theta \log(1/p(\theta) - 1) < \infty$$

we have

$$[H_{p(\theta)}^{s(\theta)}]_z = H_{p(z)}^{s(z)}$$

with equivalent norms, where

$$1/p(z) = \int_T (1/p(\theta)) P_z(\theta) d\theta \quad \text{and} \quad s(z) = \int_T s(\theta) P_z(\theta) d\theta.$$

Before proving this proposition we state the corresponding result for Besov-Lipschitz spaces. Take a function $\varphi \in \mathcal{S}$ such that $\text{supp } \varphi = \{x \in \mathbb{R}^n: 2^{-1} \leq |x| \leq 2\}$, $\varphi(x) > 0$ for $2^{-1} < |x| < 2$ and $\sum_{k=-\infty}^{\infty} \varphi(2^{-k}x) = 1$ ($x \neq 0$) (the existence of such a function is not difficult to prove). Define φ_k , $k = 0, \pm 1, \pm 2, \dots$ and ψ by

$$\varphi_k(x) = \varphi(2^{-k}x) \quad \text{and} \quad \psi(x) = 1 - \sum_{k=1}^{\infty} \varphi(2^{-k}x).$$

Evidently, $\varphi_k \in \mathcal{S}$ and $\psi \in \mathcal{S}$. Let $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$. We define the Besov-Lipschitz space $B_{p,q}^s = B_{p,q}^s(\mathbb{R}^n)$ as the set of all $f \in \mathcal{S}$ for which

$$\|f\|_{p,q}^s = \|\psi * f\|_p + \left\{ \sum_{k=1}^{\infty} |2^{ks} \|\varphi_k * f\|_p|^q \right\}^{1/q} < \infty.$$

In [12], M. Taibleson has given equivalent definitions of these spaces for $s > 0$. In particular, he was able to prove that if $0 < s < 1$, $B_{\infty,\infty}^s = \text{lip}(s)$ and $B_{p,\infty}^s = \text{lip}(s, p)$ (see [12], theorem 4).

PROPOSITION (8.2). *Let $q: T \rightarrow [1, \infty]$ and $s: T \rightarrow \mathbb{R}$ be measurable functions on T such that s is bounded below and $s(z) = \int_T s(\theta) P_z(\theta) d\theta < \infty$. Then, if $1 \leq p \leq \infty$, $\{B_{p,q(\theta)}^{s(\theta)}\}$, $\theta \in T$, is an interpolation family of Banach spaces and*

$$[B_{p,q(\theta)}^{s(\theta)}]_z = B_{p,q(z)}^{s(z)}.$$

with equivalent norms, where $1/q(z) = \int_T (1/q(\theta)) P_z d\theta$.

Before proving these two propositions we need three lemmas; these three results are well known and can be found in interpolation monographs such as [1] and [13].

LEMMA 8.1. (1) *If $s_1 < s_2$ we have $H_p^{s_2} \subset H_p^{s_1}$ ($1 \leq p \leq \infty$) and if $f \in H_p^{s_2}$, $\|f\|_p^{s_1} \leq C[1 + (2^{s_2-s_1})^{-1}] \|f\|_p^{s_2}$ where C is independent of s_1 , s_2 and p .*

(2) *If $s_1 < s_2$ we have $B_{pq}^{s_2} \subset B_{pq}^{s_1}$ ($1 \leq p, q \leq \infty$) and if $f \in B_{pq}^{s_2}$, $\|f\|_{pq}^{s_1} \leq \|f\|_{pq}^{s_2}$.*

LEMMA (8.2). Let A_0, A_1 be an interpolation couple of Banach spaces and $\alpha: T \mapsto (0, 1)$, $q: T \mapsto [1, \infty]$ be two measurable functions. Let $A(\theta) = (A_0, A_1)_{\alpha(\theta), q(\theta)}$ be the intermediate space obtained by the K -method of interpolation. Then,

$$\|a\|_{A_0 + A_1} \leq [\alpha(\theta) q(\theta)]^{1/q(\theta)} \|a\|_{A(\theta)}$$

for all $a \in \bigcap_{\theta \in T} A(\theta)$.

LEMMA (8.3). (1) Let $1 < p < \infty$, $s \in \mathbb{R}$. Then, there exist $P: H_p^s \rightarrow L^p(l_s^2)$ and $R: L^p(l_s^2) \rightarrow H_p^s$ linear and continuous such that $R \circ P$ is the identity on H_p^s . Moreover, $\|P\|, \|R\| \sim 1/(p-1)$ as $p \rightarrow 1$ and $\|P\|, \|R\| \sim p$ as $p \rightarrow \infty$.

(2) Let $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$. Then, there exist $P: B_{p,q}^s \rightarrow l_q^s(L^p)$ and $R: l_q^s(L^p) \rightarrow B_{p,q}^s$ linear and continuous such that $R \circ P$ is the identity on $B_{p,q}^s$. Moreover $\|P\| \leq 1$ and $\|R\| \sim 2^{(q-1)/q}$.

Comments on the proof of the lemmas: Lemma (8.1) can be found in [1] (theorems 6.2.3 and 6.2.4), and lemma 8.3 is theorem 6.4.3 of [1] (We notice that P maps $\mathcal{S}'$ to the space of all sequences of tempered distributions and R maps this space to $\mathcal{S}'$). To prove lemma 8.2 we assume that the reader is familiar with the K -method of interpolation. If $a \in A(\theta)$, the fact that $K(t, a)$ is an increasing function of t ([13], p. 24), together with the trivial equality $\int_1^\infty s^{-\alpha(\theta)q(\theta)} (ds/s) = 1/\alpha(\theta)q(\theta)$ imply

$$\begin{aligned} \|a\|_{A_0 + A_1} &= K(1, a) \leq [\alpha(\theta)q(\theta)]^{1/q(\theta)} \left\{ \int_1^\infty K(t, a) s^{-\alpha(\theta)q(\theta)} \frac{ds}{s} \right\}^{1/q(\theta)} \\ &= [\alpha(\theta)q(\theta)]^{1/q(\theta)} \|a\|_{A(\theta)}, \end{aligned}$$

which is the desired result.

PROOF OF PROPOSITION (8.1). Let $s_0 < \inf_{\theta \in T} s(\theta)$; lemma 8.1 (1) shows that $H_{p(\theta)}^{s(\theta)} \subset H_{p(\theta)}^{s_0}$ and $\|f\|_{p(\theta)}^{s_0} \leq C \|f\|_{p(\theta)}^{s(\theta)}$, for all $f \in H_{p(\theta)}^{s(\theta)}$, where C is independent of $s(\theta)$ and $p(\theta)$. By lemma 8.2 and $H_{p(\theta)}^{s_0} = (H_1^{s_0}, H_\infty^{s_0})_{\alpha(\theta), p(\theta)}$, where $1/p(\theta) = 1 - \alpha(\theta)$, (see theorem 6.4.5(5) of [1]) we deduce that $\|f\|_{H_1^{s_0} + H_\infty^{s_0}} \leq [p(\theta)]^{1/p(\theta)} \|f\|_{H_{p(\theta)}^{s_0}}$ for all $f \in H_{p(\theta)}^{s_0}$. Thus, we can take $U = H_1^{s_0} + H_\infty^{s_0}$ as the containing space and we have

$$\|f\|_U \leq C[p(\theta)]^{1/p(\theta)} \|f\|_{H_{p(\theta)}^{s(\theta)}}$$

for all $f \in \bigcap_{\theta \in T} H_{p(\theta)}^{s(\theta)}$. Since the measurability of $\theta \rightarrow \|f\|_{p(\theta)}^{s(\theta)} = \|J^{s(\theta)} f\|_{p(\theta)}$ is clear, we obtain the first part of the proposition. We notice that the log-intersection space of the family $\{l_2^{s(\theta)}\}$, $\theta \in T$, coincides with $l_2^{s_+} = \bigcap_{\theta \in T} l_2^{s(\theta)}$, where $s_+ = \sup_{\theta \in T} s(\theta)$.

We now prove the equality of the spaces. Since P maps $H_{p(\theta)}^{s(\theta)}$ continuously into $L^{p(\theta)}(l_2^{s(\theta)})$ with norm bounded by $M(\theta)$, where $\log M(\theta)$ is absolutely integrable on T (this is due to lemma 8.3(1) and conditions (A) and (B)), we use theorem (2.2) to deduce that P also maps $[H_{p(\theta)}^{s(\theta)}]_z$ continuously into $[L^{p(\theta)}(l_2^{s(\theta)})]_z = L^{p(z)}(l_2^{s(z)})$ where $s(z) = \int_T s(\theta) P_z(\theta) d\theta$ and $1/p(z) = \int_T (1/p(\theta) P_z(\theta) d\theta$ (see proposition 7.1). On the other hand, R maps $L^{p(z)}(l_2^{s(z)})$ continuously onto $H_{p(z)}^{s(z)}$. Consequently, $R \circ P$, which is the identity, maps $[H_{p(\theta)}^{s(\theta)}]_z$ into $H_{p(z)}^{s(z)}$. Thus, $[H_{p(\theta)}^{s(\theta)}]_z$ is continuously embedded in $H_{p(z)}^{s(z)}$.

Now, R maps $L^{p(\theta)}(l_2^{s(\theta)})$ continuously into $H_{p(\theta)}^{s(\theta)}$ and again, by theorem (2.2), it maps $L^{p(z)}(l_2^{s(z)})$ continuously into $[H_{p(\theta)}^{s(\theta)}]_z$. But the image of $L^{p(z)}(l_2^{s(z)})$ under R is $H_{p(z)}^{s(z)}$ and so $H_{p(z)}^{s(z)} \subset [H_{p(\theta)}^{s(\theta)}]_z$. Since we have already proved the reverse inclusion and its continuity, the open mapping theorem yields the desired conclusion. ■

The proof of proposition 8.2 is very similar to the proof just given, but it is obtained by using the result $(B_{p,1}^s, B_{p,\infty}^s)_{\theta,q} = B_{p,q}^s$, $1 \leq p \leq \infty$, $s \in \mathbb{R}$ (theorem 6.4.5(2) of [1]). Details are left to the reader.

9. – Interpolation of Lorentz spaces.

Let (M, μ) be a measure space and for $f \in L_{\text{loc}}(M)$ define

$$f^{**}(t) = \frac{1}{t} \sup_E \int_E |f| d\mu, \quad 0 > t > \infty$$

where the supremum is taken over all measurable sets E in M such that $\mu(E) \leq t$. If X is a Banach lattice on the halfline $0 < t < \infty$, we denote by X^* the class of measurable functions f on M such that $f^{**} \in X$ and write $\|f\|_{X^*} = \|f^{**}\|_X$. That X^* is a Banach lattice on M is a well known fact (see [2], 13.4 and 33.4).

We shall now briefly introduce the definition of Lorentz spaces, which were first studied by G. Lorentz (see [1]). For a measurable function f on a measure space (M, μ) we introduce the distribution function of f as $m(\sigma, f) = \mu\{x/|f(x)| > \sigma\}$, $\sigma > 0$. The decreasing rearrangement of f is defined as $f^*(t) = \inf\{\sigma/m(\sigma, f) \leq t\}$, $t > 0$. If $1 \leq p < \infty$ and $1 \leq q < \infty$

we let $L_{p,q}$ be the space of all measurable functions f on (M, μ) for which

$$\|f\|_{p,q} = \left\{ \int_0^\infty t^{1/p} f^*(t)^q \frac{dt}{t} \right\}^{1/q} < \infty.$$

If $1 \leq p \leq \infty$, $q = \infty$, we let $L_{p,\infty}$ be the space of all measurable f on (M, μ) such that $\|f\|_{p,\infty} = \sup_{t>0} t^{1/p} f^*(t) < \infty$. It is well known that $L_{1,1} = L^1$ and $L_{p,q}$, $1 < p \leq \infty$, $1 \leq q \leq \infty$ are Banach spaces.

As a consequence of the equality $f^{**}(t) = 1/t \int_0^t f^*(s) ds$ and Hardy's inequality one obtains the following result (see [9]), which shows that $L_{p,q}$ is a particular case of the spaces X^* introduced above.

LEMMA (9.1). *If (M, μ) is non-atomic, $1 < p < \infty$, and $1 \leq q \leq \infty$, the spaces $L_{p,q}$ and $X_{p,q}^*$, where $X_{p,q}$ is as in example 3 of section 4, coincide and their norms are equivalent.*

What we shall do now is to obtain a general interpolation theorem for Banach lattices of the type X^* and use it, together with the above lemma, to find the intermediate spaces of Lorentz spaces. For $f \in L_{\text{loc}}(0, \infty)$ we consider the operators

$$(S_1 f)(t) = \frac{1}{t} \int_0^t f(s) ds, \quad (S_2 f)(t) = \int_t^\infty \frac{f(s)}{s} ds.$$

THEOREM (9.2). *Let $\{X(\theta)\}$, $\theta \in T$, be a family of Banach lattices on $(0, \infty)$ contained in $L_{\text{loc}}(0, \infty)$. Assumed*

$$(1) \quad \|S_j f\|_{X(\theta)} \leq c_j(\theta) \|f\|_{X(\theta)}$$

for all $f \in X(\theta)$, where $\int_T (\log c_j(\theta)) d\theta < \infty$, $j = 1, 2$. Then, the spaces $[X(\theta)^]^z$ and $([X(\theta)]^z)^*$ coincide and their norms are equivalent.*

PROOF. Before starting the proof of $[X(\theta)^*]^z \subset ([X(\theta)]^z)^*$ we need the following result:

LEMMA. *Let $F: T \times M \rightarrow \mathbb{R}_+$ be measurable and assume that*

$$\int_T d\theta P_z(\theta) \log \|F(\theta, \cdot)\|_{L^1} < \infty$$

for same $z \in \Delta$ (and hence for all z). Then,

$$\left(\exp \left\{ \int_T d\theta P_z(\theta) \log F(\theta, \cdot) \right\} \right)^{**}(t) \leq \exp \left\{ \int_T d\theta P_z(\theta) \log F^{**}(\theta, t) \right\}.$$

The proof of the lemma is an easy consequence of proposition (3.1), for it follows that the left-hand side equals

$$\frac{1}{t} \sup_{\mu(E) \leq t} \int_E d\mu(x) \left\{ \exp \int_T d\theta P_z(\theta) \log F(\theta, x) \right\}$$

which is majorated by

$$\frac{1}{t} \sup_{\mu(E) \leq t} \exp \left\{ \int_T d\theta P_z(\theta) \log \left(\int_E d\mu(x) F(\theta, x) \right) \right\} \leq \exp \left\{ \int_T d\theta P_z(\theta) \log F^{**}(\theta, t) \right\}.$$

Let now $f \in [X(\theta)^*]^z$. Given $\varepsilon > 0$ we can choose $F(\theta, x)$ with $\|F(\theta, \cdot)\|_{X(\theta)} < 1$ such that

$$|f(x)| \leq (1 + \varepsilon) \|f\|^z \exp \left\{ \int_T d\theta P_z(\theta) \log |F(\theta, x)| \right\}$$

where $\|f\|^z$ denotes the norm of f as an element of $[X(\theta)^*]^z$. By the above lemma

$$f^{**}(t) \leq (1 + \varepsilon) \|f\|^* \exp \left\{ \int_T d\theta P_z(\theta) \log |F^{**}(\theta, t)| \right\}.$$

Moreover, $\|F^{**}(\theta, \cdot)\|_{X(\theta)} = \|F(\theta, \cdot)\|_{X(\theta)^*} \leq 1$ so that the above inequality implies $f^{**} \in [X(\theta)]^z$ and $\|f^{**}\|_{[X(\theta)]^z} \leq (1 + \varepsilon) \|f\|^z$. The desired inclusion and the corresponding norm inequality follow immediately.

We now prove the reverse inclusion. Given $f \in ([X(\theta)]^z)^*$ and $\lambda > \|f\|_{([X(\theta)]^z)^*}$ we can choose $F(\theta, t)$ with $\|F(\theta, \cdot)\|_{X(\theta)} \leq 1$ such that

$$f^{**}(t) \leq \lambda \exp \left\{ \int_T d\theta P_z(\theta) \log |F(\theta, t)| \right\}.$$

Proposition (3.1) implies

$$(9.1) \quad (S_2 f^{**})(t) \leq c(z) \lambda \exp \left\{ \int_T d\theta P_z(\theta) \log \left| \frac{S_2(F(\theta, \cdot))(t)}{c_1(\theta) c_2(\theta)} \right| \right\}$$

where $c(z) = \exp \left\{ \int_T d\theta P_z(\theta) \log c_1(\theta) c_2(\theta) \right\}$. Observing that

$$(S_2(S_1 g))(t) = (S_1 g)(t) + (S_2 g)(t) \text{ and } f^* \leq f^{**} = S_1 f^*$$

we deduce $f^* \leq S_1 f^* + S_2 f^* = S_2(S_1 f^*) = S_2(f^{**})$. This inequality together with (9.1) implies

$$(9.2) \quad f^*(t) \leq c(z) \lambda \exp \left\{ \int_T d\theta P_z(\theta) \log h(\theta, t) \right\}$$

where $h(\theta, t) = S_2(F(\theta, \cdot))(t)/c_1(\theta)c_2(\theta)$. Define $G(\theta, x) = h(\theta, m(|f(x)|, f))$. Using the fact that $G^*(\theta, t) \leq h(\theta, t)$ we have $G^{**}(\theta, t) = (S_1 f^*(\theta, \cdot))(t) \leq (S_1 h(\theta, \cdot))(t) = S_1 S_2 F(\theta, \cdot)(t)/c_1(\theta)c_2(\theta)$, so that condition (1) implies $\|G^{**}(\theta, \cdot)\|_{X(\theta)} \leq \|F(\theta, \cdot)\|_{X(\theta)} \leq 1$. Hence

$$G(\theta, \cdot) \in (X(\theta))^* \quad \text{and} \quad \|G(\theta, \cdot)\|_{(X(\theta))^*} \leq 1.$$

Moreover, using the inequality $|f(x)| \leq f^*(m(|f(x)|, f))$ and (9.2) we obtain

$$\begin{aligned} |f(x)| &\leq c(z) \lambda \exp \left\{ \int_T d\theta P_z(\theta) \log h(\theta, m(|f(x)|, f)) \right\} \\ &= c(z) \lambda \exp \{ d\theta P_z(\theta) \log G(\theta, x) \} \end{aligned}$$

which proves the desired result. ■

To be able to apply the theorem to Lorentz spaces we need to find a bound for the norms of the operators $S_j, j = 1, 2$ acting on $X_{p,q}$ (see the definition of $X_{p,q}$ in example 3, section 4). This is contained in the following result:

LEMMA (9.3). *If $X_{p,q}$ $1 < p < \infty, 1 \leq q < \infty$, is the Banach lattice of all measurable functions f on $(0, \infty)$ such that*

$$\|f\|_{X_{p,q}} = \left\{ \int_0^\infty [s^{1/p} |f(s)|]^q \frac{ds}{s} \right\} < \infty,$$

we have

$$\|S_1 f\|_{X_{p,q}} \leq \frac{p}{p-1} \|f\|_{X_{p,q}} \quad \text{and} \quad \|S_2 f\|_{X_{p,q}} \leq p \|f\|_{X_{p,q}}$$

for all $f \in X_{p,q}$.

PROOF. As several of the properties of Lorentz spaces, this lemma depends essentially on Hardy's inequality: if $q \leq 1$, $r \neq 0$ and $f \geq 0$,

$$(9.3) \quad \left\{ \int_0^\infty \left(\int_t^\infty f(s) ds \right)^q t^r \frac{dt}{t} \right\}^{1/q} \leq \frac{q}{|r|} \left\{ \int_0^\infty [sf(s)]^q s^r \frac{ds}{s} \right\}^{1/q}.$$

The original proof of (9.3) can be found in [7] (Chapter IX). An easier proof can be obtained as an application of Jensen's inequality and Fubini's theorem (see [8], page 256).

To prove the estimate for S_1 we use Hardy's inequality with $r = (q/p) - q < 0$ to obtain

$$\|S_1 f\|_{X_{p,q}} \leq \frac{p}{p-1} \left\{ \int_0^\infty [sf(s)]^q s^{(q/p)-q} \frac{ds}{s} \right\}^{1/q} = \frac{p}{p-1} \|f\|_{X_{p,q}}.$$

To prove the estimate for S_2 we use Hardy's inequality for $f = q/p > 0$ and $f(s)/s$ to obtain

$$\|S_2 f\|_{X_{p,q}} \leq p \left\{ \int_0^\infty [f(s)]^q s^{q/p} \frac{ds}{s} \right\}^{1/q} = \|f\|_{X_{p,q}}. \quad \blacksquare$$

PROPOSITION (9.4). Let $p: T \rightarrow (1, \infty)$ and $q: T \rightarrow [1, \infty)$ be two measurable functions on T such that

$$(1) \quad \int_T d\theta \log p(\theta) < \infty \quad \text{and} \quad \int_T d\theta \log \frac{1}{p(\theta) - 1} < \infty.$$

Then, $\{L_{p(\theta), q(\theta)}\}$, $\theta \in T$, is an interpolation family of Banach spaces and

$$[L_{p(\theta), q(\theta)}]_z = L_{p(z), q(z)}$$

with equivalent norms, where

$$1/p(z) = \int_T (1/p(\theta)) P_z(\theta) d\theta, \quad 1/q(z) = \int_T (1/q(\theta)) P_z(\theta) d\theta.$$

PROOF. To prove that $\{L_{p(\theta), q(\theta)}\}$, $\theta \in T$, is an interpolation family we observe that $(L^1, L^\infty)_{\alpha(\theta), q(\theta)} = L_{p(\theta), q(\theta)}$, where $1/p(\theta) = 1 - \alpha(\theta)$ ([1], p. 113). Then, we can take $U = L^1 + L^\infty$ as a containing space and by lemma 8.2

we have $\|f\|_U \leq [\alpha(\theta)q(\theta)]^{1/q(\theta)} \|f\|_{p(\theta), q(\theta)}$, for all $f \in \bigcap_{\theta \in T} L_{p(\theta), q(\theta)}$, where

$$\int_T \log^+ [\alpha(\theta)q(\theta)]^{1/q(\theta)} d\theta \leq \int_T \log [q(\theta)]^{1/q(\theta)} d\theta \leq 2\pi.$$

We now prove the equality of the spaces. By lemma (9.3) and condition (9.4) (1) we can use theorem (9.2) to obtain $[X_{p(\theta), q(\theta)}^*]^z = ([X_{p(\theta), q(\theta)}]^z)^*$. By lemma (9.1) and theorem (6.1) we have

$$[X_{p(\theta), q(\theta)}^*]^z = [L_{p(\theta), q(\theta)}]^z = [L_{p(\theta), q(\theta)}]_z.$$

On the other hand proposition (5.5) and lemma (9.1) imply

$$([X_{p(\theta), q(\theta)}]^z)^* = X_{p(z), q(z)}^* = L_{p(z), q(z)}.$$

This proves the desired result. ■

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