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Linear parabolic equations in Banach spaces with variable domains but constant interpolation spaces

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Linear Parabolic Equations in Banach Spaces with Variable Domains but Constant Interpolation Spaces.

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0. – Introduction.

Let E be a Banach space. We look for C^1-solutions of the Cauchy problem

$$(0.1) \qquad \begin{cases} u'(t) - A(t)u(t) = f(t)\,, & t \in [0, T] \\ u(0) = x \end{cases}$$

where $T > 0$, $\{A(t)\}_{t\in[0,T]}$ is a family of closed linear operators in E with domains $D_{A(t)}$, x is an element of E and f: $[0, T] \to E$ is (at least) a continuous function. We suppose that for each $t \in [0, T]$ $A(t)$ generates an analytic semigroup, and that its domain $D_{A(t)}$ is possibly not dense in E, i.e. the semigroup $s \to \exp[sA(t)]$ may be not strongly continuous at $s = 0$. Although the domains may vary with t, we assume that for a fixed $\varrho \in]0, 1[$ the interpolation spaces $D_{A(t)}(\varrho, \infty)$ between $D_{A(t)}$ and E are independent of t.

Many authors have studied Problem (0.1) in the parabolic case under different assumptions. The simplest situation is that of constant domains, i.e. $D_{A(t)} = D_{A(0)}$ for each $t \in [0, T]$: then a standard hypothesis is a Hölder condition on $t \to A(t)A(0)^{-1}$ (in the uniform topology). Existence and regularity results in this case are due to Tanabe [33], Sobolevskii [29], Poulsen [23], Da Prato - Grisvard [10], Da Prato - Sinestrari [12], Acquistapace - Terreni [2], [3].

When the domains depend on t, in order to obtain existence results much more smoothness for $A(t)$ is required: a standard hypothesis is now

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the differentiability of $t \to R(\lambda, A(t))$ for each fixed λ. This however is not yet sufficient to show existence of differentiable solutions of (0.1); we quote the existence and regularity results obtained, under different additional assumptions, by Kato - Tanabe [17], Tanabe [34], Yagi [37], [38], Suryanarayana [32], again Da Prato - Grisvard [10], Acquistapace - Terreni [1].

Problem (0.1) has been studied in another important situation, namely the case in which the domains still change with t, but there exists some intermediate space Y_t between $D_{A(t)}$ and E which is independent of t; this allows a considerable weakening of the smoothness assumptions about $A(t)$. Sobolevskii [27], [28] and Kato [16] consider the case $Y_t = D_{[-A(t)]^\varrho}$, for fixed $\varrho \in]0, 1[$, with a Hölder condition on $t \to [-A(t)]^\varrho[-A(0)]^{-\varrho}$ of order $\alpha \in]1-\varrho, 1[$. In concrete cases the characterizations of the fractional powers' domains $D_{[-A(t)]^\varrho}$, as well as, possibly, their constancy with respect to t, are known essentially when E is an L^p-space (see Lions [19], [20], Seeley [24], [25]). Another possible choice for Y_t is the (real) interpolation space $D_{A(t)}(\varrho, \infty)$: these spaces have been characterized in several cases (see Grisvard [13], [14], Da Prato - Grisvard [11], Lunardi [22], Acquistapace - Terreni [3], [4]), and in a large number of variable-domain examples it turns out that $D_{A(t)}(\varrho, \infty)$ is indeed constant in t for sufficiently small ϱ.

Thus in the present paper we assume $D_{A(t)}(\varrho, \infty) = D_{A(0)}(\varrho, \infty)$ for a fixed $\varrho \in]0, 1[$; moreover we require that $t \to A(t)^{-1}$ is α-Hölder continuous from E into $D_{A(0)}(\varrho, \infty)$ with $\alpha \in]1-\varrho, 1[$, in analogy with the assumptions of [27], [28], [16]. On the other hand we do not need density of domains and do not use the fundamental solution: we use a suitable « a priori » representation formula for the solution of (0.1) (if it exists), and show that if the data x, f are smooth enough, this formula indeed yields the unique solution of (0.1). In addition we prove maximal regularity of the solution both in time and in space, provided x and f satisfy some necessary and sufficient compatibility conditions: in other words, u' and $A(\cdot)u(\cdot)$ are θ-Hölder continuous in $[0, T]$ whenever f does, and are bounded with values in $D_{A(0)}(\theta, \infty)$ whenever f does.

Let us describe now the subject of the next sections. Section 1 contains some notations, assumptions and preliminary results; in Section 2 we derive uniqueness and some simple necessary conditions for existence; in Section 3 the properties of all functions and operators appearing in our representation formula are analyzed in detail; Section 4 is devoted to the study of certain problems which approximate Problem (0.1) and are useful in the proof of existence; in Section 5 we prove our main results; finally in Section 6 we describe two examples.

1. – Notations, assumptions and preliminaries.

Let E be a Banach space and fix $T > 0$. If Y is another Banach space continuously imbedded into E, we will consider the Banach function spaces $B(Y) = \{\text{bounded functions: } [0, T] \to Y\}$ and $C(Y)$, $C^\delta(Y)$ $(\delta \in]0, 1[)$, $C^1(Y)$ with their usual norms. We will also use the function spaces $B_+(Y) = \{f\colon]0, T] \to Y\colon f|_{[\varepsilon,T]} \text{ is bounded } \forall \varepsilon > 0\}$, and $C_+(Y)$, $C_+^\delta(Y)$, $C_+^1(Y)$ which are defined similarly.

If $Y, Z \subseteq E$ are Banach spaces, $\mathcal{L}(Y, Z)$ (or simply $\mathcal{L}(Y)$ if $Y = Z$) is the Banach space of bounded linear operators $Y \to Z$, with the usual norm. Let $A\colon D_A \subseteq E \to E$ be a closed linear operator. For $\sigma \in]0, 1[$ we will use the real interpolation spaces $(D_A, E)_{\sigma,\infty}$ with their usual norm (for a definition see Lions [18], Lions - Peetre [21], Butzer - Berens [8]). Obviously if $0 < \beta < \sigma < 1$ we have the continuous inclusions

$$D_A \subseteq (D_A, E)_{\beta,\infty} \subseteq (D_A, E)_{\sigma,\infty} \subseteq \overline{D}_A . \tag{1.1}$$

DEFINITION 1.1. When A is the infinitesimal generator of a strongly continuous (except possibly at 0) semigroup $\{\exp[\xi A]\}_{\xi \geq 0}$, we will set

$$D_A(\beta, \infty) := (D_A, E)_{1-\beta,\infty} \qquad \forall \beta \in]0, 1[.$$

Now we list our assumptions.

HYPOTHESIS I. For each $t \in [0, T]$ $A(t)$ is a closed linear operator in E, with domain $D_{A(t)} \subseteq E$, which is the infinitesimal generator of an analytic semigroup $\{\exp[\xi A(t)]\}_{\xi \geq 0}$. More precisely:

(i) there exists $\theta_0 \in]\pi/2, \pi]$ such that

$$\varrho(A(t)) \supseteq S_{\theta_0} := \{z \in \mathbf{C} - \{0\}\colon |\arg z| < \theta_0\} \cup \{0\} \qquad \forall t \in [0, T] ,$$

(ii) there exists $M > 0$ such that

$$\|R(\lambda, A(t))\|_{\mathcal{L}(E)} \leq \frac{M}{1 + |\lambda|} \qquad \forall \lambda \in S_{\theta_0}, \quad \forall t \in [0, T] .$$

REMARK 1.2. The domains $D_{A(t)}$ are not supposed to be dense in E, so that the semigroups $\{\exp[\xi A(t)]\}$ are not necessarily strongly continuous at $\xi = 0$; however if Hypothesis I holds and E is reflexive (or, more generally, if E is locally sequentially weakly compact) then $\overline{D}_{A(t)} = E$ for each $t \in [0, T]$ (see Kato [15]).

HYPOTHESIS II. There exists $\varrho \in]0, 1[$ such that $D_{A(t)}(\varrho, \infty) \cong D_{A(0)}(\varrho, \infty)$, $\forall t \in [0, T]$, set-theoretically and topologically; more precisely:

(i) $D_{A(t)}(\varrho, \infty) = D_{A(0)}(\varrho, \infty)$, $\forall t \in [0, T]$;

(ii) there exists $K > 0$ such that

$$\|x\|_{D_{A(t)}(\varrho,\infty)} \leqslant K \|x\|_{D_{A(0)}(\varrho,\infty)} \qquad \forall t \in [0, T], \ \forall x \in D_{A(0)}(\varrho, \infty).$$

REMARK 1.3. By Hypothesis II and by the Reiteration Theorem (Triebel [36, Theorem 1.10.2]) it follows that for each $\beta \in]0, \varrho]$ we have

$$D_{A(t)}(\beta, \infty) \cong D_{A(0)}(\beta, \infty)$$

set-theoretically and topologically; moreover there exists $H_\beta > 0$ such that

$$(1.2) \qquad H_\beta^{-1} \|x\|_{D_{A(0)}(\beta,\infty)} \leqslant \|x\|_{D_{A(t)}(\beta,\infty)} \leqslant H_\beta \|x\|_{D_{A(0)}(\beta,\infty)}$$
$$\forall t \in [0, T], \ \forall x \in D_{A(0)}(\beta, \infty).$$

Moreover by (1.1) the closures $\overline{D}_{A(t)}$ coincide with $\overline{D}_{A(0)}$ for each $t \in [0, T]$. It is then justified the following

DEFINITION 1.4. We set

$$\overline{D}_A := \overline{D}_{A(t)}, \qquad D_A(\beta, \infty) := D_{A(t)}(\beta, \infty) \qquad \forall \beta \in]0, \varrho], \ \forall t \in [0, T];$$

formulas (1.8), (1.9) and (1.10) below define a class of norms in $D_A(\beta, \infty)$ which are all equivalent uniformly in t.

HYPOTHESIS III. There exist $\alpha \in]1 - \varrho, 1[$ and $L > 0$ such that

$$\|A(t)^{-1} - A(s)^{-1}\|_{\mathcal{L}(E, D_A(\varrho,\infty))} \leqslant L|t - s|^\alpha \qquad \forall t, s \in [0, T].$$

Let us define now the strict solutions of problem (0.1).

DEFINITION 1.5. Let $x \in D_{A(0)}$, $f \in C(E)$. We say that a function $u \in C(E)$ is a strict solution of (0.1) if $u \in C^1(E)$, $u(t) \in D_{A(t)}$, $\forall t \in [0, T]$ and

$$u'(t) - A(t)u(t) = f(t) \qquad \forall t \in [0, T], \ u(0) = x.$$

Thus we have $A(\cdot)u(\cdot) \in C(E)$ for any strict solution u of (0.1). If we set

$$(1.3) \quad C(D_{A(\cdot)}) := \{u \in C(E) \colon u(t) \in D_{A(t)}, \ \forall t \in [0, T] \text{ and } A(\cdot)u(\cdot) \in C(E)\},$$

then any strict solution u of (0.1) satisfies $u \in C^1(E) \cap C(D_{A(\cdot)})$.

Let us recall now some properties of the semigroups $\{\exp[\xi A(t)]\}_{\xi\geqslant 0}$ and of the interpolation spaces $D_{A(t)}(\beta,\infty)$.

By the well-known representation of the analytic semigroup $\exp[\xi A(t)]$

$$(1.4)\qquad A(t)^k \exp[\xi A(t)] = \\ = \frac{1}{2\pi i}\int_\gamma \exp[\xi\lambda]\lambda^k R(\lambda, A(t))\,d\lambda\,, \qquad \forall\xi>0,\ \forall t\in[0,T],\ \forall k\in\mathbf{N},$$

where

$$(1.5)\qquad \begin{cases} \gamma \ := \gamma^-\cup\gamma_0\cup\gamma^+\,, \\ \gamma_0 := \{z\in\mathbf{C}\colon |z|=1,\ |\arg z|\leqslant\theta\}\,, \\ \gamma^\pm := \{z\in\mathbf{C}\colon |z|\geqslant 1,\ \arg z = \pm\theta\}\,, \end{cases}$$

(with fixed $\theta\in\,]\pi/2,\theta_0[$), we get the usual estimate

$$(1.6)\qquad \|A(t)^k\exp[\xi A(t)]\|_{\mathcal{L}(E)}\leqslant\frac{C_k}{\xi^k}\qquad \forall\xi>0,\ \forall t\in[0,T],\ \forall k\in\mathbf{N}\,.$$

As $\{\exp[\xi A(t)]\}_{\xi\geqslant 0}$ is a bounded analytic semigroup, we can characterize the interpolation spaces $D_{A(t)}(\beta,\infty)$ in several ways (see [8] for the dense-domain case, [26] and [1] for the general case). Namely we have:

$$(1.7)\qquad \begin{aligned} D_{A(t)}(\beta,\infty) &= \Big\{x\in E\colon \sup\{\xi^{-\beta}\|(\exp[\xi A(t)]-1)x\|_E\colon \xi>0\}<\infty\Big\} \\ &= \Big\{x\in E\colon \sup\{\xi^{1-\beta}\|A(t)\exp[\xi A(t)]x\|_E\colon \xi>0\}<\infty\Big\} \\ &= \Big\{x\in E\colon \sup\{|\lambda|^\beta\|A(t)R(\lambda,A(t))x\|_E\colon \lambda\in S_{\theta_0}\}<\infty\Big\}; \end{aligned}$$

the corresponding norms

$$(1.8)\qquad \|x\|^{(1,t)} := \|x\|_E + \sup_{\xi\geqslant 0}\{\xi^{-\beta}\|(\exp[\xi A(t)]-1)x\|_E\}\,,$$

$$(1.9)\qquad \|x\|^{(2,t)} := \|x\|_E + \sup_{\xi\geqslant 0}\{\xi^{1-\beta}\|A(t)\exp[\xi A(t)]x\|_E\}\,,$$

$$(1.10)\qquad \|x\|^{(3,t)} := \|x\|_E + \sup_{\lambda\in S_{\theta_0}}\{|\lambda|^\beta\|A(t)R(\lambda,A(t))x\|_E\}$$

are all equivalent to the usual norm of $D_{A(t)}(\beta,\infty)$ as an interpolation space (see [8]). If in addition $\beta\in\,]0,\varrho]$, by (1.2) these equivalences hold uniformly in t: thus, as claimed in Definition 1.4, each one of the norms (1.8), (1.9) and (1.10) can be taken as a norm in $D_A(\beta,\infty)$, with equivalences holding uniformly in t.

The above characterizations are meaningful also in the extreme cases $\beta = 0$ and $\beta = 1$. In these cases one has $D_A(0, \infty) = E$ and $D_{A(t)}(1, \infty) \supseteq D_{A(t)}$ for each $t \in [0, T]$ (without equality in general). However we will adopt the following convention:

CONVENTION 1.6. We set $D_A(0, \infty) = E$, $D_{A(t)}(1, \infty) = D_{A(t)}$, $\forall t \in [0, T]$.

Let us recall some other useful properties.

LEMMA 1.7. *Under Hypothesis I, we have:*

(i) *if* $\delta \in [0,1]$, *then* $\|\exp[tA(0)]\|_{\mathcal{L}(E, D_{A(0)}(\delta,\infty))} \leqslant \dfrac{C}{t^\delta} \quad \forall t > 0$;

(ii) $t \to \exp[tA(0)]x \in C(E)$ *if and only if* $x \in \bar{D}_{A(0)}$;

(iii) *if* $\delta \in]0,1[$, $t \to \exp[tA(0)]x \in C^\delta(E)$ *if and only if* $x \in D_{A(0)}(\delta, \infty)$;

(iv) *if* $\delta \in [0, 1]$, $t \to \exp[tA(0)]x \in B(D_{A(0)}(\delta, \infty))$ *if and only if* $x \in D_{A(0)}(\delta, \infty)$.

PROOF. Parts (i), (iii) and (iv) are easy consequences of (1.6), (1.9) and (1.8). Part (ii) is proved in [26, Proposition 1.2(i)]. ///

LEMMA 1.8. *Under Hypotheses I, II, III let* $0 \leqslant \tau \leqslant t \leqslant T$ *and* $\lambda \in S_{\theta_0}$: *Then*:

(i) $$R(\lambda, A(t)) - R(\lambda, A(\tau)) = -A(t)R(\lambda, A(t))[A(t)^{-1} - A(\tau)^{-1}]A(\tau)R(\lambda, A(\tau));$$

(ii) *if* $\beta \in [0, \varrho]$
$$\|R(\lambda, A(t)) - R(\lambda, A(\tau))\|_{\mathcal{L}(D_A(\beta,\infty), E)} \leqslant \frac{c(\beta)(t-\tau)^\alpha}{(1+|\lambda|)^{\varrho+\beta}};$$

(iii) *if* $\beta \in]\varrho, 1]$,
$$\|R(\lambda, A(t)) - R(\lambda, A(\tau))\|_{\mathcal{L}(D_{A(0)}(\beta,\infty), E)} \leqslant \frac{c(\beta)(t-\tau)^\alpha}{(1+|\lambda|)^{\varrho+\beta}}(1 + |\lambda|^{1-\varrho}\tau^\alpha).$$

PROOF. (i) It is a straightforward computation.

(ii) It follows by (i), Hypothesis III and (1.10).

(iii) Taking into account (i) we can write

$$R(\lambda, A(t)) - R(\lambda, A(\tau)) = -A(t)R(\lambda, A(t))(A(t)^{-1} - A(\tau)^{-1}) \cdot [-\lambda A(\tau)R(\lambda, A(\tau))(A(\tau)^{-1} - A(0)^{-1}) + 1]A(0)R(\lambda, A(0));$$

the result then follows easily by Hypothesis III and (1.10). ///

LEMMA 1.9. *Under Hypotheses I, II, III let* $0 \leqslant \tau \leqslant t \leqslant T$. *Then we have:*

(i) *if* $\beta \in [0, \varrho]$,

$$\|A(t)^k \exp[\xi A(t)] - A(\tau)^k \exp[\xi A(\tau)]\|_{\mathcal{L}(D_A(\beta,\infty),E)} \leqslant \frac{c(\beta, k)(t-\tau)^\alpha}{\xi^{k+1-\varrho-\beta}} \quad \forall \xi > 0, \quad \forall k \in \mathbf{N};$$

(ii) *if* $\beta \in [\varrho, 1]$,

$$\|A(t)^k \exp[\xi A(t)] - A(\tau)^k \exp[\xi A(\tau)]\|_{\mathcal{L}(D_{A(0)}(\beta,\infty),E)} \leqslant \frac{c(\beta, k)(t-\tau)^\alpha}{\xi^{k+1-\varrho-\beta}}\left(1 + \frac{\tau^\alpha}{\xi^{1-\varrho}}\right) \quad \forall \xi > 0, \quad \forall k \in \mathbf{N}.$$

PROOF. The results follow by (1.6) and Lemma 1.8 (ii)-(iii). ///

LEMMA 1.10. *Under Hypotheses I, II, III let* $\beta \in [0, \varrho]$; *then*

$$\|A(t)^k \exp[\xi A(t)]\|_{\mathcal{L}(D_A(\beta,\infty),E)} \leqslant \frac{c(\beta, k)}{\xi^{(k-\beta)\vee 0}} \quad \forall \xi > 0, \quad \forall k \in \mathbf{N}, \quad \forall t \in [0, T].$$

PROOF. It follows by (1.6), (1.9) and the semigroup property. ///

2. – Necessary conditions.

In this section we derive some easy necessary conditions for the existence of strict solutions of (0.1), and prove uniqueness of such solutions. Hypotheses I, II, III are always assumed.

PROPOSITION 2.1. *Let* $x \in D_{A(0)}$, $f \in C(E)$ *and suppose that* $u \in C^1(E) \cap C(D_{A(\cdot)})$ *is a strict solution of* (0.1). *Then we must have* $A(0)x + f(0) \in \bar{D}_A$.

PROOF. We have as $t \to 0^+$

$$\bar{D}_A \supseteq D_A(\varrho, \infty) \ni \frac{u(t)-x}{t} \to u'(0) = A(0)x + f(0). \quad ///$$

PROPOSITION 2.2. *For each* $x \in D_{A(0)}$ *and* $f \in C(E)$ *there exists at most one strict solution of* (0.1).

PROOF. Let w be a strict solution of (0.1) with $x=0$, $f\equiv 0$. Fix $t\in\,]0,T]$ and set

$$v(s):=\exp\,[(t-s)A(t)]\,w(s)\,,\qquad s\in[0,t]\,.$$

Then

$$v'(s)=-A(t)\exp\,[(t-s)A(t)]w(s)+\exp\,[(t-s)A(t)]A(s)w(s)\,,\qquad s\in[0,t[\,;$$

by integrating from 0 to t and operating with $A(t)$ we get

$$A(t)w(t)=\int_0^t A(t)^2\exp\,[(t-s)A(t)]\big(A(t)^{-1}-A(s)^{-1}\big)A(s)w(s)\,ds\,,\qquad \forall t\in\,]0,T]. \tag{2.1}$$

By Hypothesis III

$$\|A(t)w(t)\|_E\leqslant C\int_0^t(t-s)^{\alpha+\varrho-2}\,\|A(s)w(s)\|_E\,ds\qquad \forall t\in\,]0,T]\,,$$

so that by Gronwall's Lemma (see e.g. Amann [7]) we deduce $w\equiv 0$. ///

REMARK 2.3. For each $\varphi\in C(E)$ set

$$(Q\varphi)(t):=\int_0^t A(t)^2\exp\,[(t-s)A(t)]\big(A(t)^{-1}-A(s)^{-1}\big)\varphi(s)\,ds\,,\quad t\in[0,T]; \tag{2.2}$$

then equation (2.1) becomes

$$A(\cdot)w(\cdot)=Q\big(A(\cdot)w(\cdot)\big)\,.$$

Repeat (just formally) the above argument for the solution u of (0.1) with non-zero data x, f (if it exists): it follows that

$$A(\cdot)u(\cdot)=Q\big(A(\cdot)u(\cdot)\big)+L(f,x)\,,$$

where

$$L(f,x)(t):=A(t)\int_0^t\exp\,[(t-s)A(t)]f(s)\,ds+A(t)\exp\,[tA(t)]x\,. \tag{2.3}$$

If $f\in C(E)$ and $x\in D_{A(0)}$ the function (2.3) is not meaningful in general: but we will see in Section 3 that for slightly more regular data $L(f,x)$

makes sense, and it will be possible to get for any strict solution u of (0.1) the following representation formula:

$$u = A(\cdot)^{-1}(1-Q)^{-1}L(f, x)\,. \tag{2.4}$$

3. – The representation formula.

In this section we analyze the properties of the operator Q and of the function $L(f, x)$, respectively defined by (2.2) and (2.3); as a consequence we will prove the representation formula (2.4) for any strict solution of (0.1) with sufficiently regular data. Hypotheses I, II and III are always assumed.

(*a*) *The function* $L(f, x)$.

PROPOSITION 3.1. *Fix* $\delta \in\,]0, \alpha + \varrho - 1]$, *and let* $x \in D_{A(0)}$, $f \in C^{\delta}(E)$. *Then*:

(i) $L(f, x) \in C^{\delta}_{+}(E)$ *and* $L(f, x) + f \in B_{+}(D_A(\delta, \infty))$;

(ii) $L(f, x) \in C(E)$ *if and only if* $A(0)x + f(0) \in \bar{D}_A$;

(iii) $L(f, x) \in C^{\delta}(E)$ *if and only if* $A(0)x + f(0) \in D_A(\delta, \infty)$;

(iv) $L(f, x) + f \in B(D_A(\delta, \infty))$ *if and only if* $A(0)x + f(0) \in D_A(\delta, \infty)$.

PROOF. (ii)-(iii) We can write

$$L(f, x) = \int_0^t A(t) \exp[(t-s)A(t)]\big(f(s) - f(t)\big)\, ds + \big(\exp[tA(t)] - 1\big) f(t) + A(t)\exp[tA(t)]x;$$

hence if $0 \leqslant \tau \leqslant t \leqslant T$

$$\begin{aligned} L(f, x)(t) - L(f, x)(\tau) &= \int_\tau^t A(t) \exp[(t-s)A(t)]\big(f(s) - f(t)\big)\, ds \\ &+ \int_0^\tau \big(A(t)\exp[(t-s)A(t)] - A(\tau)\exp[(t-s)A(\tau)]\big)\big(f(s) - f(t)\big)\, ds \\ &+ \big(\exp[tA(\tau)] - \exp[(t-\tau)A(\tau)]\big)\big(f(\tau) - f(t)\big) \\ &+ \int_0^\tau \int_{\tau-s}^{t-s} A(\tau)^2 \exp[\sigma A(\tau)]\big(f(s) - f(\tau)\big)\, d\sigma\, ds \end{aligned}$$

$$+ \big(\exp[tA(t)]-1\big)\big(f(t)-f(\tau)\big) + \big(\exp[tA(t)]-\exp[tA(\tau)]\big)f(\tau)$$

$$+\int_\tau^t A(\tau)\exp[\sigma A(\tau)]\big(f(\tau)-f(0)\big)\,d\sigma$$

$$+\int_\tau^t \big(A(\tau)\exp[\sigma A(\tau)]-A(0)\exp[\sigma A(0)]\big)f(0)\,d\sigma$$

$$+ \big(A(t)\exp[tA(t)]-A(\tau)\exp[tA(\tau)]\big)x$$

$$+\int_\tau^t \big(A(\tau)^2\exp[\sigma A(\tau)]-A(0)^2\exp[\sigma A(0)]\big)x\,d\sigma$$

$$+ \big(\exp[tA(0)]-\exp[\tau A(0)]\big)\big(A(0)x+f(0)\big)\,.$$

We estimate separately each term on the right-hand side by using Lemmata 1.10 and 1.9 (i)-(ii). Tedious but easy calculations yield

$$\begin{aligned}(3.1)\qquad L(f,x)(t)-L(f,x)(\tau) &= O\big((t-\tau)^\delta\big)\big(\|f\|_{C^\delta(E)}+\|A(0)x\|_E\big)\\ &\quad+\big(\exp[tA(0)-\exp[\tau A(0)]\big)\big(A(0)x+f(0)\big)\qquad \text{as } t-\tau\to 0^+,\end{aligned}$$

and (ii)-(iii) follow by Lemma 1.7 (ii)-(iii). As evidently $t\to\exp[tA(0)]$ belongs to $C^1_+(\mathcal{L}(E))$, (3.1) also implies the first part of (i).

(iv) We can write for $0\leqslant t\leqslant T$

$$\begin{aligned}L(f,x)(t)+f(t) &= \int_0^t A(t)\exp[(t-s)A(t)]\big(f(s)-f(t)\big)\,ds\\ &\quad+\exp[tA(t)]\big(f(t)-f(0)\big)+\big(\exp[tA(t)]-\exp[tA(0)]\big)\big(f(0)+A(0)x\big)\\ &\quad+A(t)\exp[tA(t)]\big(A(0)^{-1}-A(t)^{-1}\big)A(0)x+\exp[tA(0)]\big(A(0)x+f(0)\big)\,.\end{aligned}$$

Again we estimate separately each term, using Lemmata 1.10, 1.9 (i)-(ii) and 1.7 (i). Therefore we obtain for each $t\in[0,T]$

$$\begin{aligned}(3.2)\qquad \big\|[L(f,x)(t)+f(t)]-\big[\exp[tA(0)]\big(A(0)x+f(0)\big)\big]\big\|_{D_A(\delta,\infty)}&\\ \leqslant C\big(\|f\|_{C^\delta(E)}+\|A(0)x\|_E\big)\,,&\end{aligned}$$

and taking into account (ii), (iv) follows by Lemma 1.7 (iv). As, clearly, $t\to\exp[tA(0)]\in B_+\big(\mathcal{L}(E,D_{A(0)})\big)$, (3.2) also implies the second part of (i). ///

PROPOSITION 3.2. *Fix* $\delta \in]0, \alpha + \varrho - 1]$, *and let* $x \in D_{A(0)}$, $f \in C(E) \cap B(D_A(\delta, \infty))$. *Then*:

(i) $L(f, x) \in C_+^\delta(E) \cap B_+(D_A(\delta, \infty))$;

(ii) $L(f, x) \in C(E)$ *if and only if* $A(0)x \in \bar{D}_A$;

(iii) $L(f, x) \in C^\delta(E)$ *if and only if* $A(0)x \in D_A(\delta, \infty)$;

(iv) $L(f, x) \in C(E) \cap B(D_A(\delta, \infty))$ *if and only if* $A(0)x \in D_A(\delta, \infty)$.

PROOF. (ii)-(iii) We can write

$$L(f, x)(t) = \int_0^t A(t) \exp[(t-s)A(t)] f(s)\, ds + A(t) \exp[tA(t)]x, \qquad t \in [0, T].$$

By splitting $L(f, x)(t) - L(f, x)(\tau)$ and using Lemmata 1.10 and 1.9 (i)-(ii) we easily check

$$\begin{aligned}(3.3)\qquad L(f, x)(t) - L(f, x)(\tau) &= O\big((t-\tau)^\delta\big)\big(\|f\|_{B(D_A(\delta,\infty))} + \|A(0)x\|_E\big)\\ &\quad + \big(\exp[tA(0)] - \exp[\tau A(0)]\big)A(0)x \qquad \text{as } t - \tau \to 0^+,\end{aligned}$$

so that (ii) and (iii) follow by Lemma 1.7 (ii)-(iii); (3.3) also implies the first part of (i).

(iv) We can write for $0 \leqslant t \leqslant T$

$$\begin{aligned}L(f, x)(t) = \int_0^t A(t) &\exp[(t-s)A(t)] f(s)\, ds\\ &+ A(t) \exp[tA(t)]\big(A(0)^{-1} - A(t)^{-1}\big)A(0)x\\ &+ \big(\exp[tA(t)] - \exp[tA(0)]\big)A(0)x + \exp[tA(0)]A(0)x.\end{aligned}$$

Estimate once more each term separately: by Lemma 1.10 and Hypothesis III we deduce for each $t \in [0, T]$

$$(3.4)\qquad \|L(f, x)(t) - \exp[tA(0)]A(0)x\|_{D_A(\delta,\infty)} \leqslant C\big(\|f\|_{B(D_A(\delta,\infty))} + \|A(0)x\|_E\big),$$

and taking into account (ii), (iv) follows by Lemma 1.7 (iv); (3.4) also implies the second part of (i). ///

(*b*) *The operator* Q.

PROPOSITION 3.3. *We have*:

(i) $Q\in \mathfrak{L}\big(C(E),\ C^{\alpha+\varrho-1}(E)\big)$;

(ii) $Q\in \mathfrak{L}\Big(C(E),\ C(E)\cap B\big(D_A(\alpha+\varrho-1,\ \infty)\big)\Big)$.

PROOF. (i) If $0\leqslant\tau\leqslant t\leqslant T$ we have for any $\varphi\in C(E)$

$$Q\varphi(t)-Q\varphi(\tau)=\int_\tau^t A(t)^2\exp\left[(t-s)A(t)\right]\big(A(t)^{-1}-A(s)^{-1}\big)\varphi(s)\,ds$$

$$+\int_0^\tau A(t)^2\exp\left[(t-s)A(t)\right]\big(A(t)^{-1}-A(\tau)^{-1}\big)\varphi(s)\,ds$$

$$+\int_0^\tau \big(A(t)^2\exp\left[(t-s)A(t)\right]-A(\tau)^2\exp\left[(t-s)A(\tau)\right]\big)\big(A(\tau)^{-1}-A(s)^{-1}\big)\varphi(s)\,ds$$

$$+\int_0^\tau\int_{\tau-s}^{t-s} A(\tau)^3\exp\left[\sigma A(\tau)\right]\big(A(\tau)^{-1}-A(s)^{-1}\big)\varphi(s)\,d\sigma\,ds\,.$$

By Lemmata 1.10 (i) and 1.9 (i) it follows easily that

$$\|Q\varphi\|_{C^{\alpha+\varrho-1}(E)}\leqslant C\|\varphi\|_{C(E)}\qquad \forall\varphi\in C(E)\,.$$

(ii) Let $\varphi\in C(E)$; we already know that $Q\varphi\in C(E)$ and $\|Q\varphi\|_{C(E)}\leqslant C\|\varphi\|_{C(E)}$. Now if $t\in[0,T]$ Lemma 1.10 easily yields for any $\xi>0$

$$\|\xi^{2-\alpha-\varrho}A(t)\exp\left[\xi A(t)\right]Q\varphi(t)\|_E\leqslant C\xi^{2-\alpha-\varrho}\int_0^t\frac{(t-s)^\alpha}{(\xi+t-s)^{3-\varrho}}\,ds\,\|\varphi\|_{C(E)}\leqslant C\|\varphi\|_{C(E)}\,,$$

which implies the result. ///

(*c*) *The operator* $(1-Q)^{-1}$.

PROPOSITION 3.4. *We have*:

i) *the operator* $(1-Q)^{-1}$ *exists and belongs to* $\mathfrak{L}\big(C(E)\big)$;

(ii) *if* $\delta\in\,]0,\alpha+\varrho-1]$ *and* $\varphi\in C(E)\cap C^\delta_+(E)$ $\big($*resp.* $\varphi\in C(E)\cap B_+\big(D_A(\delta,\ \infty)\big)\big)$, *then the same holds for* $(1-Q)^{-1}\varphi$;

(iii) $(1-Q)^{-1}\in\mathfrak{L}\big(C^\delta(E)\big)$ *for each* $\delta\in\,]0,\alpha+\varrho-1]$;

(iv) $(1-Q)^{-1}\in\mathfrak{L}\Big(C(E)\cap B\big(D_A(\delta,\ \infty)\big)\Big)$ *for each* $\delta\in\,]0,\alpha+\varrho-1]$.

PROOF. (i) Pick $\omega > 0$ and define a new norm in $C(E)$ by

$$\|f\|_\omega := \sup_{t\in[0,T]} \|\exp[-\omega t]f(t)\|_E, \qquad f \in C(E).$$

Obviously

$$\|f\|_\omega \leqslant \|f\|_{C(E)} \leqslant \exp[\omega T]\,\|f\|_\omega \qquad \forall f \in C(E). \tag{3.5}$$

On the other hand it is easily seen that

$$\|Q\varphi\|_\omega \leqslant C\,\frac{\Gamma(\alpha+\varrho-1)}{\omega^{\alpha+\varrho-1}}\,\|\varphi\|_\omega;$$

this clearly implies that $(1-Q)$ is an isomorphism in $C(E)$ with respect to the norm $\|\cdot\|_\omega$ (for large ω), and (i) then follows by (3.5).

(iii) Let $\varphi \in C^\delta(E)$; by (i), $\psi := (1-Q)^{-1}\varphi \in C(E)$ and $\psi = Q\psi + \varphi$. By Proposition 3.3 (i), $Q\psi \in C^{\alpha+\varrho-1}(E)$ and

$$\|Q\psi\|_{C^{\alpha+\varrho-1}(E)} \leqslant C\|\psi\|_{C(E)}.$$

Hence $\psi = Q\psi + \varphi \in C^\delta(E)$ and

$$\|\psi\|_{C^\delta(E)} \leqslant \|Q\psi\|_{C^{\alpha+\varrho-1}(E)} + \|\varphi\|_{C^\delta(E)} \leqslant C\|\psi\|_{C(E)} + \|\varphi\|_{C^\delta(E)} \leqslant C\|\varphi\|_{C^\delta(E)}.$$

(iv) Let $\varphi \in C(E) \cap B(D_A(\delta, \infty))$; by (*i*), $\psi := (1-Q)^{-1}\varphi \in C(E)$ and $\psi = Q\psi + \varphi$. By Proposition 3.3 (ii), $Q\psi \in C(E) \cap B(D_A(\alpha+\varrho-1, \infty))$ and

$$\|Q\psi\|_{B(D_A(\alpha+\varrho-1,\infty))} \leqslant C\|\psi\|_{C(E)}.$$

Hence, as in the proof of (iii) we get

$$\|\psi\|_{B(D_A(\delta,\infty))} + \|\psi\|_{C(E)} \leqslant C\big(\|\varphi\|_{B(D_A(\delta,\infty))} + \|\varphi\|_{C(E)}\big).$$

(ii) Let $\varphi \in C(E) \cap C^\delta_+(E)$ $\big($resp. $\varphi \in C(E) \cap B_+(D_A(\delta, \infty))\big)$; then by (i) $\psi := (1-Q)^{-1}\varphi \in C(E)$ and $\psi = Q\psi + \varphi$. By Proposition 3.3,

$$Q\psi \in C^{\alpha+\varrho-1}(E) \cap B(D_A(\alpha+\varrho-1, \infty)),$$

so that ψ has the same regularity as φ. ///

Now we are ready to give sense to the heuristic argument used in Remark 2.3 to introduce the representation formula for the strict solution of (0.1). Indeed, we have:

PROPOSITION 3.5. *Fix* $\delta \in]0, \alpha + \varrho - 1]$; *let* $x \in D_{A(0)}$, $f \in C(E)$ *and suppose that* f, x *fulfil any of the following conditions*: (i) $f \in C^{\delta}(E)$ *and* $A(0)x + f(0) \in \bar{D}_A$, *or* (ii) $f \in C(E) \cap B(D_A(\delta, \infty))$ *and* $A(0)x \in \bar{D}_A$. *Then if* u *is a strict solution of* (0.1) *the following representation formula holds*:

$$u(t) = A(t)^{-1}\big((1-Q)^{-1}L(f, x)\big)(t)\,, \qquad t \in [0, T]\,, \tag{3.6}$$

where the operator Q *and the function* $L(f, x)$ *are defined by* (2.2) *and* (2.3).

PROOF. Proceeding as in the proof of Proposition 2.2 (see also Remark 2.3) we deduce that

$$A(t)u(t) = Q\big(A(\cdot)u(\cdot)\big)(t) + L(f, x)(t)\,, \qquad t \in]0, T]\,,$$

and (3.6) follows by Proposition 3.4 (i). ///

We have to show now that the function u given by (3.6) is in fact the strict solution of (0.1), provided the data x, f are sufficiently regular. We will obtain u as the limit in $C^1(E)$ of a suitable sequence $\{u_n\}_{n \in \mathbf{N}^+}$, where the functions u_n solve certain problems which in some sense approach problem (0.1) as $n \to \infty$. Such problems have the same form as (0.1) with $A(t)$ replaced by the bounded operator $A_n(t) := nA(t)R(n, A(t))$ (the Yosida approximation of $A(t)$). This will be done in the next sections.

4. – The approximating problems.

We analyze here the properties of the solutions u_n of the approximating problems mentioned at the end of the preceding section; we prove a representation formula for u_n which is analogous to (3.6) and study the convergence as $n \to \infty$. Hypotheses I, II, III are always assumed.

We start with a review of the main properties of the Yosida approximations

$$A_n(t) = nA(t)R\big(n, A(t)\big)\,. \tag{4.1}$$

LEMMA 4.1. *Fix* $\theta \in]\pi/2, \theta_0[$. *Then*:

(i) $$\frac{1}{|\lambda + n|} \leqslant \frac{3}{\sin\theta(1 + |\lambda|)} \qquad \forall \lambda \in S_\theta, \quad \forall n \in \mathbf{N}^+;$$

(ii) $$\frac{1}{|\lambda + n|} \leqslant \frac{(1 + \operatorname{tg}^2\theta)^{\frac{1}{2}}}{|\operatorname{tg}\theta| n} \qquad \forall \lambda \in S_\theta, \quad \forall n \in \mathbf{N}^+.$$

PROOF. Tedious but elementary. ///

LEMMA 4.2. *Let $A_n(t)$ be given by* (4.1). *If $\theta \in]\pi/2, \theta_0[$, then:*

(i) $\varrho(A_n(t)) \supseteq S_\theta$ *and*

$$R(\lambda, A_n(t)) = \frac{1}{\lambda + n}(n - A(t))\,R\left(\frac{\lambda n}{\lambda + n}, A(t)\right) \quad \forall \lambda \in S_\theta,\ \forall n \in \mathbf{N}^+,\ \forall t \in [0, T];$$

(ii) $\|R(\lambda, A_n(t))\|_{\mathcal{L}(E)} \leqslant \dfrac{C(\theta)}{1 + |\lambda|} \quad \forall \lambda \in S_\theta, \quad \forall n \in \mathbf{N}^+, \quad \forall t \in [0, T]$;

(iii) $$R(\lambda, A_n(t)) - R(\lambda, A(t)) = \frac{1}{\lambda + n} A(t) R\left(\frac{\lambda n}{\lambda + n}, A(t)\right) A(t) R(\lambda, A(t))$$
$$\forall \lambda \in S_\theta, \quad \forall n \in \mathbf{N}^+, \quad \forall t \in [0, T];$$

(iv) *if* $\beta \in [0, \varrho]$,

$$\|A_n(t)^k \exp[\xi A_n(t)] - A(t)^k \exp[\xi A(t)]\|_{\mathcal{L}(D_A(\beta,\infty),E)} \leqslant \frac{C(\theta, k)}{n^\sigma \xi^{k-\beta+\sigma}}$$
$$\forall n \in \mathbf{N}^+, \quad \forall k \in \mathbf{N}, \quad \forall t \in [0, T], \quad \forall \xi > 0, \quad \forall \sigma \in [0, 1];$$

(v) $$\lim_{n\to\infty} \sup_{t\in[0,T]} \sup_{\xi>0} \xi^k \|A_n(t)^k \exp[\xi A_n(t)]y - A(t)^k \exp[\xi A(t)]y\|_E = 0$$
$$\forall y \in \overline{D}_A, \quad \forall k \in \mathbf{N}.$$

PROOF. Part (i), (ii) and (iii) are straightforward.

(iv) By (ii) we easily find

$$\|A_n(t)^k \exp[\xi A_n(t)]y\|_E \leqslant C(\theta, k)\xi^{-k}\|y\|_E \tag{4.2}$$
$$\forall \xi > 0, \quad \forall t \in [0, T], \quad \forall n \in \mathbf{N}^+, \quad \forall k \in \mathbf{N}.$$

Now by (iii) we have

$$A_n(t)^k \exp[\xi A_n(t)]y - A(t)^k \exp[\xi A(t)]y \tag{4.3}$$
$$= \frac{1}{2\pi i}\int_\gamma \exp[\xi\lambda]\,\frac{\lambda^k}{\lambda + n} A(t) R(\lambda, A(t))\, A(t) R\left(\frac{\lambda n}{\lambda + n}, A(t)\right) y\, d\lambda \qquad \forall y \in E,$$

with γ given by (1.5); hence if $\beta \in [0, \varrho]$ and we choose $y \in D_A(\beta, \infty)$, by Lemma 4.1 it is not difficult to check for each $\sigma \in [0, 1]$

$$\|A_n(t)^k \exp[\xi A_n(t)]y - A(t)^k \exp[\xi A(t)]y\|_E \leqslant \frac{C(\theta, k)}{\xi^{k-\beta+\sigma} n^\sigma}\|y\|_{D_A(\beta,\infty)}.$$

(v) Let $y \in \overline{D}_A$ and fix any $\varepsilon > 0$; choose $z \in D_A(\varrho, \infty)$ such that $\|y - z\|_E < \varepsilon$. Then by (1.4), (4.2) and (iv)

$$\|A_n(t)^k \exp[\xi A_n(t)]y - A(t)^k \exp[\xi A(t)]y\|_E \leqslant \frac{C(\theta, k)\,\varepsilon}{\xi^k} + \frac{C(\theta, k)}{\xi^k n^\varrho}\|z\|_{D_A(\varrho,\infty)};$$

hence

$$\limsup_{n\to\infty} \sup_{t\in[0,T]} \sup_{\xi>0} \xi^k \|A_n(t)^k \exp[\xi A_n(t)]y - A(t)^k \exp[\xi A(t)]y\|_E \leqslant C(\theta, k)\varepsilon \qquad \forall \varepsilon > 0 . \qquad ///$$

LEMMA 4.3. *If $\theta \in]\pi/2, \theta_0[$ and $0 \leqslant \tau \leqslant t \leqslant T$ we have:*

(i) $R(\lambda, A_n(t)) - R(\lambda, A_n(\tau))$

$$= -\frac{n^2}{(\lambda+n)^2} A(t) R\left(\frac{\lambda n}{\lambda+n}, A(t)\right)\left(A(t)^{-1} - A(\tau)^{-1}\right) A(\tau) R\left(\frac{\lambda n}{\lambda+n}, A(\tau)\right)$$
$$\forall \lambda \in S_\theta, \quad \forall n \in \mathbf{N}^+;$$

(ii) *if* $\beta \in [0, \varrho]$,

$$\|A_n(t)^k \exp[\xi A_n(t)] - A_n(\tau)^k \exp[\xi A_n(\tau)]\|_{\mathcal{L}(D_A(\beta,\infty),E)} \leqslant C(\theta, k) \frac{(t-\tau)^\alpha}{\xi^{k+1-\varrho-\beta}}$$
$$\forall n \in \mathbf{N}_+, \quad \forall k \in \mathbf{N}, \quad \forall \xi > 0 .$$

PROOF. Part (i) is straightforward; part (ii) follows by using (i) and Lemma 4.1 (i), exactly as in the proof of Lemma 1.9 (i). ///

Fix now $x \in D_{A(0)}$, $f \in C(E)$, and consider for each $n \in \mathbf{N}^+$ the problem

$$(4.4)_n \qquad \begin{cases} u_n'(t) - A_n(t) u_n(t) = f(t), & t \in [0, T], \\ u_n(0) = x_n = x - \dfrac{1}{n} A(0)x, \end{cases}$$

where $A_n(t)$ is given by (4.1).

PROPOSITION 4.4. *Let $x \in D_{A(0)}$, $f \in C(E)$; then for each $n \in \mathbf{N}^+$ problem $(4.4)_n$ has a unique solution $u_n \in C^1(E)$, given by*

$$(4.5)_n \qquad u_n(t) = A_n(t)^{-1}\left((1-Q_n)^{-1} L_n(f, x_n)\right)(t), \qquad t \in [0, T],$$

where the operator Q_n and the function $L_n(f, x_n)$ are defined by

$$(4.6)_n \qquad (Q_n\varphi)(t) := \int_0^t A_n(t)^2 \exp[(t-s)A_n(t)]\left(A(t)^{-1} - A(s)^{-1}\right)\varphi(s)\,ds, \qquad \varphi \in C(E),$$

$$(4.7)_n \qquad L_n(f, x_n)(t) := \int_0^t A_n(t) \exp[(t-s)A_n(t)] f(s)\,ds + A_n(t) \exp[tA_n[t)]x_n .$$

PROOF. For fixed $n \in \mathbf{N}^+$, by Lemma 1.8 (ii) we get

$$\|A_n(t) - A_n(s)\|_{\mathcal{L}(E)} \leqslant Cn^{2-\varrho}|t-s|^{\alpha} \qquad \forall t, s \in [0, T].$$

Thus the method of successive approximations yields a unique solution of $(4.4)_n$. Formula $(4.5)_n$ follows as in the proof of Proposition 3.5. ///

We study now the regularity and convergence properties of the functions $L_n(f, x_n)$ and of the operators Q_n, defined respectively by $(4.7)_n$ and $(4.6)_n$.

(*a*) *The functions* $L_n(f, x_n)$.

PROPOSITION 4.5. *If* $x \in D_{A(0)}$ *and* $f \in C(E)$ *then* $L_n(f, x_n) \in C(E)$, $\forall n \in \mathbf{N}^+$.

PROOF. It is quite easy since, for fixed n, $(t, s) \to A_n(t) \exp[(t-s)A_n(t)]$ and $t \to \exp[tA_n(t)]$ are continuous functions with values in $\mathcal{L}(E)$. ///

PROPOSITION 4.6. *Fix* $\delta \in]0, \alpha + \varrho - 1]$. *If* $x \in D_{A(0)}$, $f \in C^{\delta}(E)$ *and* $A(0)x + f(0) \in \overline{D}_A$, *then*

$$L_n(f, x_n) \to L(f, x) \qquad \text{in } C(E) \text{ as } n \to \infty.$$

PROOF. We have for each $t \in [0, T]$

$$\begin{aligned} L_n(f, x_n)(t) &- L(f, x)(t) \\ &= \int_0^t \big(A_n(t) \exp[(t-s)A_n(t)] - A(t) \exp[(t-s)A(t)]\big)\big(f(s) - f(t)\big)\, ds + \\ &\qquad + \big(\exp[tA_n(t)] - \exp[tA(t)]\big) f(t) \\ &\qquad + \big(A_n(t) \exp[tA_n(t)]x_n - A(t) \exp[tA(t)]x\big) = J_1 + J_2 + J_3 . \end{aligned}$$

Now by Lemma 4.2 (iv)

$$(4.8) \qquad \|J_1\|_E \leqslant Cn^{-\delta/2} \int_0^t \frac{ds}{(t-s)^{1-\delta/2}} \|f\|_{C^{\delta}(E)} \leqslant Cn^{-\delta/2} \|f\|_{C^{\delta}(E)} \qquad \forall t \in [0, T];$$

to estimate the other two terms we fix any $\varepsilon \in [0, T]$ and distinguish two cases: (*a*) $t \in [0, \varepsilon]$, (*b*) $t \in]\varepsilon, T]$.

If $t \in]\varepsilon, T]$ we have, by Lemma 4.2 (iv) and (4.1)

$$(4.9)\quad \begin{cases} \|J_2\|_E \leqslant \dfrac{C}{n\varepsilon}\|f\|_{C(E)}, \\ \|J_3\|_E \leqslant \|\big(A_n(t)\exp[tA_n(t)] - A(t)\exp[tA(t)]\big)x\|_E \\ \qquad + \dfrac{1}{n}\|A_n(t)\exp[tA_n(t)]A(0)x\|_E \leqslant \dfrac{C}{n\varepsilon^2}\|x\|_E + \dfrac{C}{n\varepsilon}\|A(0)x\|_E ; \end{cases}$$

this, together with (4.8), implies

$$(4.10)\qquad \lim_{n\to\infty} \sup_{t\in[\varepsilon,T]} \|L_n(f, x_n)(t) - L(f, x)(t)\|_E = 0 .$$

If $t \in [0, \varepsilon]$, as $A_n(0)x_n = A(0)x$ we can write

$$\begin{aligned} J_2 + J_3 = {} & \big(\exp[tA_n(t)] - \exp[tA(t)]\big)\big(f(t) - f(0)\big) \\ & + \big(\exp[tA_n(t)] - \exp[tA_n(0)]\big)f(0) - \big(\exp[tA(t)] - \exp[tA(0)]\big)f(0) \\ & + \big(A_n(t)\exp[tA_n(t)] - A_n(0)\exp[tA_n(0)]\big)x_n \\ & - \big(A(t)\exp[tA(t)] - A(0)\exp[tA(0)]\big)x \\ & + \big(\exp[tA_n(0)] - \exp[tA(0)]\big)\big(A(0)x + f(0)\big) = \sum_{i=1}^{6} I_i . \end{aligned}$$

Now by Lemmata 4.2 (iv), 4.3 (ii) and 1.9 (i)-(ii) we have easily

$$(4.11)\quad \begin{cases} \|I_1\|_E \leqslant \dfrac{C}{n^\delta}\|f\|_{C^\delta(E)}, \\ \|I_2 + I_3\|_E \leqslant C\varepsilon^{\alpha+\varrho-1}\|f\|_{C(E)}, \\ \|I_5\|_E \leqslant C\varepsilon^{\alpha+\varrho-1}\|A(0)x\|_E ; \end{cases}$$

on the other hand, noting that $A_n(0)x_n = A(0)x$ and using Lemma 4.2 (vi) we find

$$(3.12)\qquad \|I_4\|_E \leqslant C\varepsilon^{\alpha+\varrho-1}\|A(0)x\|_E , \qquad \sup_{t\in[0,\varepsilon]} \|I_6\|_E = o(1) \qquad \text{as } n\to\infty .$$

Hence we get

$$\limsup_{n\to\infty} \sup_{t\in[0,\varepsilon]} \|J_2 + J_3\|_E \leqslant C\varepsilon^{\alpha+\varrho-1}\big(\|f\|_{C(E)} + \|A(0)x\|_E\big).$$

Summing up, recalling (4.8) we have shown that

$$(4.13)\qquad \limsup_{n\to\infty} \sup_{t\in[0,T]} \|L_n(f, x_n)(t) - L(f, x)(t)\|_E \leqslant C\varepsilon^{\alpha+\varrho-1} \qquad \forall \varepsilon > 0 ,$$

and the result follows. ///

PROPOSITION 4.7. *Fix* $\delta \in]0, \alpha + \varrho - 1]$. *If* $x \in D_{A(0)}$, $f \in C(E) \cap B(D_A(\delta, \infty))$ *and* $A(0)x \in \bar{D}_A$, *then*

$$L_n(f, x_n) \to L(f, x) \qquad \text{in } C(E) \text{ as } n \to \infty .$$

PROOF. We have for each $t \in [0, T]$

$$L_n(f, x_n)(t) - L(f, x)(t) = \int_0^t \big(A_n(t) \exp[(t-s)A_n(t)] - A(t) \exp[(t-s)A(t)]\big) f(s)\, ds + \big(A_n(t) \exp[tA_n(t)]x_n - A(t) \exp[tA(t)]x\big) = J_1 + J_2 .$$

Now by Lemma 4.2 (iv)

$$\|J_1\|_E \leqslant \frac{C}{n^{\delta/2}} \int_0^t \frac{ds}{(t-s)^{1-\delta/2}} \|f\|_{B(D_A(\delta,\infty))} \leqslant \frac{C}{n^{\delta/2}} \|f\|_{B(D_A(\delta,\infty))} . \tag{4.14}$$

Again to estimate the second term we fix any $\varepsilon \in]0, T[$ and distinguish two cases: (a) $t \in [0, \varepsilon]$, (b) $t \in]\varepsilon, T]$.

If $t \in]\varepsilon, T]$, wr still have (4.9), so that we still get (4.10). If $t \in [0, \varepsilon]$ we write (as $A_n(0)x_n = A(0)x$):

$$J_2 = \big(A_n(t) \exp[tA_n(t)] - A_n(0) \exp[tA_n(0)]\big) x_n - \big(A(t) \exp[tA(t)] - A(0) \exp[tA(0)]\big)x + \big(\exp[tA_n(0)] - \exp[tA(0)]\big)A(0)x ,$$

and as in the proof of Proposition 4.6 we obtain (4.11) and (4.12); on the other hand by Lemma 4.2 (vi)

$$\sup_{t\in[0,\varepsilon]} \big\|\big(\exp[tA_n(0)] - \exp[tA(0)]\big)A(0)x\big\|_E = o(1) \qquad \text{as } n \to \infty .$$

Hence by taking into account (4.14) we deduce (4.13). ///

(b) *The operators* Q_n.

PROPOSITION 4.8. (i) $Q_n \in \mathcal{L}(C(E))$ *and*

$$\|Q_n\|_{\mathcal{L}(C(E))} \leqslant C \qquad \forall n \in \mathbf{N}^+ ; \tag{4.15}$$

(ii) $Q_n \to Q$ *in* $\mathcal{L}(C(E))$ *as* $n \to \infty$.

PROOF. (i) It is easily seen that, for fixed n, $Q_n \in \mathcal{L}(C(E))$, since the kernel of Q_n belongs to $C(\mathcal{L}(E))$. Moreover (4.15) easily follows by (4.3) and Lemma 4.1 (ii).

(ii) If $\varphi \in C(E)$, by Lemma 4.2 (iv) we readily obtain

$$\|Q_n\varphi(t) - Q\varphi(t)\|_E \leqslant \frac{C}{n^{(\alpha+\varrho-1)/2}}\|\varphi\|_{C(E)}. \qquad ///$$

(*c*) *The operators* $(1-Q_n)^{-1}$.

PROPOSITION 4.9. (i) *The operator* $(1-Q_n)^{-1}$ *exists and belongs to* $\mathcal{L}(C(E))$, *and moreover*

$$\|(1-Q_n)^{-1}\|_{\mathcal{L}(C(E))} \leqslant C \qquad \forall n \in \mathbb{N}^+ ; \tag{4.16}$$

(ii) $(1-Q_n)^{-1} \to (1-Q)^{-1}$ *in* $\mathcal{L}(C(E))$ *as* $n \to \infty$.

PROOF. (i) Exactly as in the proof of Proposition 3.4 (i); estimate (4.16) is a consequence of (4.15).

(ii) Set $\psi_n = (1-Q_n)^{-1}\varphi$, $\psi = (1-Q)^{-1}\varphi$: then it is easily seen that $\psi_n - \psi = (1-Q_n)^{-1}(Q_n - Q)\psi$, so that the result follows by (i) and Proposition 4.8 (ii). ///

5. – Strict solutions.

We are now ready to show that the function $u(t)$ defined by (3.6) is in fact the strict solution of (0.1); we will prove here also its maximal regularity properties. As usual, we always assume Hypotheses I, II and III.

THEOREM 5.1. *Fix* $\delta \in]0, \alpha + \varrho - 1]$, *let* $x \in D_{A(0)}$, $f \in C^\delta(E)$ *and suppose that* $A(0)x + f(0) \in \overline{D}_A$. *Then*:

(i) *the function* u *defined by* (3.6) *is the unique strict solution of* (0.1);

(ii) $u' \in C^\delta_+(E) \cap B_+(D_A(\delta, \infty))$ *and* $A(\cdot)u(\cdot) \in C^\delta_+(E)$;

(iii) $u' \in C^\delta(E) \cap B(D_A(\delta, \infty))$ *and* $A(\cdot)u(\cdot) \in C^\delta(E)$ *if and only if* $A(0)x + f(0) \in D_A(\delta, \infty)$.

PROOF. (i) For each $n \in \mathbb{N}^+$ let $u_n \in C^1(E)$ be the strict solution of problem $(4.4)_n$. By Proposition 4.4 u_n is given by

$$u_n(t) = A_n(t)^{-1}[(1-Q_n)^{-1}L_n(f, x_n)](t) , \qquad t \in [0, T] , \tag{5.1$_n$}$$

where $A_n(t) = nA(t)R(n, A(t))$, $x_n = x - (1/n)A(0)x$ and Q_n, $L_n(f, x_n)$ are defined by $(4.6)_n$ and $(4.7)_n$. By Propositions 4.6 and 4.9 (ii), and taking into account that $A_n(t)^{-1} = A(t)^{-1} - 1/n$, we deduce that $u_n \to u$ in $C(E)$ as $n \to \infty$, where u is the function (3.6) (which belongs to $C(D_{A(\cdot)})$ by Propositions 3.1 (ii) and 3.4 (i)). On the other hand

$$u_n'(t) = A_n(t)u_n(t) + f(t) = [(1-Q_n)^{-1}L_n(f, x_n)](t) + f(t)\,, \qquad t \in [0, T]\,,$$

and hence, by Propositions 4.6 and 4.9 (ii), $u_n' \to v$ in $C(E)$ as $n \to \infty$, where

$$v(t) := [(1-Q)^{-1}L(f, x)](t) + f(t) = A(t)u(t) + f(t)\,, \qquad t \in [0, T]\,.$$

This implies that $u \in C^1(E)$ and

$$u'(t) = v(t) = A(t)u(t) + f(t)\,, \qquad t \in [0, T]\,.$$

As $u(0) = A(0)^{-1}[(1-Q)^{-1}L(f, x)](0) = A(0)^{-1}L(f, x)(0) = x$, we have shown that u is a strict solution of (0.1).

Uniqueness follows by Proposition 2.2.

(ii) By Proposition 3.1 (i), $L(f, x) \in C_+^\delta(E)$, which by Proposition 3.4 (ii) implies $A(\cdot)u(\cdot) = (1-Q)^{-1}L(f, x) \in C_+^\delta(E)$; as $u' = A(\cdot)u(\cdot) + f$, we also get $u' \in C_+^\delta(E)$. Next, Proposition 3.1 (i) also yields $L(f, x) + f \in B_+(D_A(\delta, \infty))$, so that by Proposition 3.4 (ii)

$$(1-Q)^{-1}\big(L(f, x) + f\big) \in B_+\big(D_A(\delta, \infty)\big)\,. \tag{5.2}$$

On the other hand we can write

$$u' = (1-Q)^{-1}L(f, x) + f = (1-Q)^{-1}\big(L(f, x) + f\big) - Q(1-Q)^{-1}f\,; \tag{5.3}$$

as $(1-Q)^{-1}f \in C(E)$ (Proposition 3.4 (ii)), we have

$$Q(1-Q)^{-1}f \in B\big(D_A(\alpha + \varrho - 1, \infty)\big)$$

(Proposition 3.3. (ii)) and therefore (5.3) and (5.2) imply that

$$u' \in B_+\big(D_A(\delta, \infty)\big)\,.$$

(iii) We have $A(\cdot)u(\cdot) = (1-Q)^{-1}L(f, x) \in C^\delta(E)$ if and only if $L(f, x) \in C^\delta(E)$ (Proposition 3.4 (ii)), i.e. if and only if $A(0)x + f(0) \in D_A(\delta, \infty)$

(Proposition 3.1 (iii)); as $u' = A(\cdot)u(\cdot) + f$, this is also equivalent to $u' \in C^\delta(E)$. In addition by Proposition 3.1 (iv) $A(0)x + f(0) \in D_A(\delta, \infty)$ if and only if $L(f, x) + f \in B(D_A(\delta, \infty))$, which by Proposition 3.4 (iv) is equivalent to $(1 - Q)^{-1}(L(f, x) + f) \in B(D_A(\delta, \infty))$; by (5.3) and Proposition 3.3 (ii) it follows that this is true if and only if $u' \in B(D_A(\delta, \infty))$. ///

THEOREM 5.2. *Fix* $\delta \in]0, \alpha + \varrho - 1]$, *let* $x \in D_{A(0)}$, $f \in C(E) \cap B(D_A(\delta, \infty))$ *and suppose that* $A(0)x \in \bar{D}_A$. *Then*:

(i) *the function* u *defined by* (3.6) *is the unique strict solution of* (0.1);

(ii) $u' \in B_+(D_A(\delta, \infty))$ *and* $A(\cdot)u(\cdot) \in C_+^\delta(E) \cap B_+(D_A(\delta, \infty))$;

(iii) $u' \in B(D_A(\delta, \infty))$ *and* $A(\cdot)u(\cdot) \in C^\delta(E) \cap B(D_A(\delta, \infty))$ *if and only if* $A(0)x \in D_A(\delta, \infty)$.

PROOF. (i) As in the proof of Theorem 5.1, let u_n be the strict solution of problem $(4.4)_n$; then u_n is given by $(5.1)_n$. By Propositions 4.7 and 4.9 (ii), $u_n \to u$ in $C(E)$ as $n \to \infty$, where u is the function (3.6) (which belongs to $C(D_{A(\cdot)})$ by Propositions 3.1 (ii) and 3.4 (i)); similarly we have

$$u_n' = (1 - Q_n)^{-1}L_n(f, x_n) + f \to (1 - Q)^{-1}L(f, x) + f = A(\cdot)u(\cdot) + f \quad \text{in } C(E) \text{ as } n \to \infty .$$

This means that $u \in C^1(E)$ and $u' = A(\cdot)u(\cdot) + f$; as

$$u(0) = A(0)^{-1}[(1 - Q)^{-1}L(f, x)](0) = A(0)^{-1}L(f, x)(0) = x ,$$

we have shown that u is a strict solution of (0.1). Uniqueness follows by Proposition 2.2.

(ii) By Proposition 3.2 (i), $L(f, x) \in B_+(D_A(\delta, \infty))$, which by Proposition 3.4 (ii) implies

$$A(\cdot)u(\cdot) = (1 - Q)^{-1}L(f, x) \in B_+(D_A(\delta, \infty));$$

as $u' = A(\cdot)u(\cdot) + f$, we also get $u' \in B_+(D_A(\delta, \infty))$. Next, Proposition 3.2 (i) also yields $L(f, x) \in C_+^\delta(E)$, so that by Proposition 3.4 (ii)

$$A(\cdot)u(\cdot) = (1 - Q)^{-1}L(f, x) \in C_+^\delta(E) .$$

(iii) We have $A(\cdot)u(\cdot) = (1 - Q)^{-1}L(f, x) \in B(D_A(\delta, \infty))$ if and only if $L(f, x) \in B(D_A(\delta, \infty))$ (Proposition 3.4 (iv)), i.e. if and only if $A(0)x \in D_A(\delta, \infty)$ (Proposition 3.2 (iv)); as $u' = A(\cdot)u(\cdot) + f$, this is also equivalent to $u' \in B(D_A(\delta, \infty))$. In addition by Proposition 3.2 (iii) $A(0)x \in D_A(\delta, \infty)$ if and only if $L(f, x) \in C^\delta(E)$, which by Proposition 3.4 (iii) is equivalent to $A(\cdot)u(\cdot) \in C^\delta(E)$. ///

6. – Examples.

We apply here the results of the preceding sections to partial differential equations of parabolic type in a bounded open set $\Omega \subseteq R^n$, in the cases $E = L^p(\Omega)$, with $1 < p < \infty$, or $E = C(\bar{\Omega})$. In these cases, as remarked in the Introduction, concrete characterizations of the interpolation spaces $D_{A(t)}(\beta, \infty)$ are known (see Grisvard [13], [14], Da Prato - Grisvard [11], Lunardi [22], Acquistapace - Terreni [3], [4]).

First example.

Fix $m \in \mathbf{N}^+$, and let Ω be a bounded connected open set of R^n with boundary $\partial\Omega$ of class C^{2m}. Consider the differential operator with complex-valued coefficients

$$E(t, x, D) := \sum_{|\gamma| \leqslant 2m} a_\gamma(t, x) D^\gamma, \qquad (t, x) \in [0, T] \times \bar{\Omega}, \tag{6.1}$$

under the following assumptions:

$$a_\gamma \in C([0, T] \times \bar{\Omega}) \qquad \forall \gamma \in \mathbf{N}^n \text{ with } |\gamma| \leqslant 2m; \tag{6.2}$$

$$\text{(strong uniform ellipticity)} \sum_{|\gamma| = 2m} \operatorname{Re} a_\gamma(t, x) \xi^\gamma \geqslant N |\xi|^{2m} \quad \forall (t, x) \in [0, T] \times \bar{\Omega}, \ \forall \xi \in \mathbf{R}^n; \tag{6.3}$$

(6.4) $\left\{\begin{array}{l}\text{(root condition) there exists } \theta_0 \in]\pi/2, \pi] \text{ such that if } \lambda = \varrho \exp(i\theta) \\ \text{with } \varrho \geqslant 0,\ |\theta| < \theta_0 \text{ then for each pair } (\xi, \zeta) \text{ of linearly independent} \\ \text{vectors of } R^n, \text{ the polynomial in the complex variable } \eta \\ \qquad \eta \to \sum_{|\gamma| = 2m} a_\gamma(t, x)(\xi + \eta\zeta)^\gamma + \lambda \\ \text{has exactly } m \text{ roots } \eta_j^+(t, x, \xi, \zeta, \lambda) \text{ with positive imaginary part.}\end{array}\right.$

Consider also the boundary differential operators

$$\Gamma_j(t, x, D) := \sum_{|\beta| \leqslant m_j} b_{j\beta}(t, x) D^\beta, \quad (t, x) \in [0, T] \times \partial\Omega, \ m_j \leqslant 2m - 1, \ j = 1, \dots, m, \tag{6.5}$$

under the following assumptions:

$$\begin{cases} b_{j\beta} \in C([0,T]\times\partial\Omega) \quad \text{and} \quad b_{j\beta}(t,\cdot) \in C^{2m-m_j}(\partial\Omega) \quad \text{with} \\ \sup_{t\in[0,T]} \left(\sum_{j=1}^{m} \sum_{|\beta|\leqslant m_j} \|b_{j\beta}(t,\cdot)\|_{C^{2m-m_j}(\partial\Omega)}\right) < \infty\,; \end{cases}$$

$$(6.7)\quad \begin{cases} \text{(normality condition) } m_i < m_j \text{ if } i<j, \text{ and} \\ \sum_{|\beta|=m_j} b_{j\beta}(t,x)\nu(x)^\beta \neq 0 \qquad \forall (t,x)\in[0,T]\times\partial\Omega\,,\ \forall j=1,\dots,m\,, \end{cases}$$

where $\nu(x)$ is the unit outward normal vector at $x\in\partial\Omega$;

$$(6.8)\quad \begin{cases} \text{(complementing condition) if } \lambda=\varrho\exp(i\theta) \text{ with } \varrho\geqslant 0,\ |\theta|<\theta_0 \text{ then} \\ \text{for each } x\in\partial\Omega \text{ and for each vector } \tau(x) \text{ tangent to } \partial\Omega \text{ at } x, \text{ the} \\ \text{polynomials in the complex variable } \eta \\ \qquad \eta \to \sum_{|\beta|=m_j} b_{j\beta}(t,x)\big(\tau(x)+\eta\nu(x)\big)^\beta\,, \qquad j=1,\dots,m \\ \text{are linearly independent modulo the polynomial (compare with} \\ \text{(6.4))} \\ \qquad \eta\to\prod_{k=1}^{m}[\eta-\eta_k^+(t,x,\tau(x),\nu(x),\lambda)]\,. \end{cases}$$

We want to study the parabolic initial-boundary value problem

$$(6.9)\quad \begin{cases} u_t(t,x)-E(t,x,D)u(t,x)=f(t,x)\,, \qquad (t,x)\in[0,T]\times\Omega\,, \\ u(0,x)=\varphi(x)\,, \qquad x\in\Omega\,, \\ \Gamma_j(t,x,D)u(t,x)=0\,, \qquad (t,x)\in[0,T]\times\partial\Omega\,,\ j=1,\dots,m\,, \end{cases}$$

where $\varphi\in L^p(\Omega)$ and $f\in C([0,T],L^p(\Omega))$ are prescribed data.

Set $E=L^p(\Omega)$ and for each $t\in[0,T]$ define

$$(6.10)\quad \begin{cases} D_{A(t)}=\{u\in H^{2m,p}(\Omega)\colon\ \Gamma_j(t,\cdot,D)u=0 \text{ on } \partial\Omega,\ j=1,\dots,m\}, \\ A(t)u=E(t,\cdot,D)u, \end{cases}$$

where for $s\geqslant 0 H^{s,p}(\Omega)$ is the usual Sobolev space.

We will verify that there exists $\omega>0$ such that $\{A(t)-\omega\}_{t\in[0,T]}$ satisfies Hypotheses I, II and III of Section 1.

A basic tool is the following result (Agmon - Douglis - Nirenberg [6], Agmon [5]):

PROPOSITION 6.1. *Under assumptions* (6.1)-(6.8), *there exist* $\theta_0 \in]\pi/2, \pi]$ *and* $\omega > 0$ (*independent of* p) *such that if* $\lambda - \omega \in S_{\theta_0}$ *then the problem*

$$\begin{cases} \lambda u - E(t, \cdot, D)u = f \in L^p(\Omega)\,, \\ \Gamma_j(t, \cdot, D)u = g_j \in H^{2m-m_j-1/p,p}(\partial\Omega)\,, \qquad j = 1, \dots, m\,, \end{cases}$$

has a unique solution $u(t) \in H^{2m,p}(\Omega)$. *Moreover there exists* $C_p > 0$ (*independent of* t) *such that*

$$\begin{aligned} (6.11) \qquad & \sum_{k=0}^{2m} (1 + |\lambda - \omega|)^{1-k/2m} \|\nabla^k u(t)\|_{L^p(\Omega)} \\ & \leqslant C_p\Big(\|f\|_{L^p(\Omega)} + \sum_{j=1}^{m} \sum_{k=0}^{2m-m_j} (1 + |\lambda - \omega|)^{1-m_j/2m-k/2m} \|G_j\|_{H^{k,p}(\Omega)}\Big)\,. \end{aligned}$$

where G_j *is any function in* $H^{2m-m_j,p}(\Omega)$ *satisfying* $G_j|_{\partial\Omega} = g_j$.

PROOF. It follows easily by Tanabe [35, Lemma 3.8.1] and Triebel [36, Theorem 5.5.2(*b*)]. ///

By Proposition 6.1 it is clear that Hypothesis I is fulfilled. Concerning Hypothesis II, we need the characterization of $D_{A(t)}(\beta, \infty) = D_{A(t)-\omega}(\beta, \infty)$ proved by Grisvard [13], [14]. First of all we have:

DEFINITION 6.2. For $s > 0$ the Besov-Nikolsky space $B^{s,p}_\infty(\Omega)$ is defined as follows:

(i) if $s \in]0, 1[$,

$$B^{s,p}_\infty(\Omega) := \left\{u \in L^p(\Omega) \colon \sup_{h \in R^n} \left[\int_{\Omega_h} \frac{|u(x+h) - u(x)|^p}{|h|^{sp}} dx\right]^{1/p} =: [u]_{B^{s,p}_\infty(\Omega)} < \infty\right\},$$

where $\Omega_h := \{x \in \Omega \colon x + h \in \Omega\}$;

(ii) if $s = 1$,

$$B^{1,p}_\infty(\Omega) := \left\{u \in L^p(\Omega) \colon \sup_{h \in R^n} \left[\int_{\Omega'_h} \frac{|u(x+h) + u(x-h) - 2u(x)|^p}{|h|^p} dx\right]^{1/p} =: [u]_{B^{1,p}_\infty(\Omega)} < \infty\right\},$$

where $\Omega'_h := \{x \in \Omega \colon x \pm h \in \Omega\}$;

(iii) if $s = k + \sigma$ with $k \in \mathbf{N}$, $\sigma \in]0, 1]$,

$$B^{s,p}_\infty(\Omega) := \{u \in H^{k,p}(\Omega) \colon D^\gamma u \in B^{\sigma,p}_\infty(\Omega) \text{ for } |\gamma| \leqslant k\}\,.$$

A norm in $B^{s,p}_{\infty}(\Omega)$, where $s = k + \sigma$ with $k \in \mathbf{N}$ and $\sigma \in]0, 1]$, is given by

$$\|u\|_{B^{s,p}_{\infty}(\Omega)} := \|u\|_{H^{k,p}(\Omega)} + \sum_{|\gamma|=k} [D^{\gamma}u]_{B^{\sigma,p}_{\infty}(\Omega)}. \tag{6.12}$$

It is known that if $s > 1/p$ the functions of $B^{s,p}_{\infty}(\Omega)$ have a trace on $\partial\Omega$. The characterization of the spaces $D_{A(t)}(\beta, \infty)$ is the following:

PROPOSITION 6.3. *Under assumptions* (6.1)-(6.8) *let* $\{A(t)\}_{t\in[0,T]}$ *be defined by* (6.10). *Then for each* $t \in [0, T]$ *we have*

$$D_{A(t)}(\beta, \infty) = \{u \in B^{2m\beta,p}_{\infty}(\Omega) \colon \Gamma_j(t, \cdot, D)u = 0 \text{ for } m_j < 2m\beta - 1/p\}$$

provided $2m\beta - 1/p \neq m_j$ *for* $j = 1, \ldots, m$. *In particular if* $\beta < 1/2mp$ *(actually, if* $\beta < (m_1 + 1/p)/2m$*), we have*

$$D_{A(t)}(\beta, \infty) = B^{2m\beta,p}_{\infty}(\Omega) \qquad \forall t \in [0, T]$$

with equivalent norms; moreover such equivalence is uniform in t.

PROOF. See Grisvard [13], [14], Triebel [36, Theorem 4.3.3 (*a*)]. The equivalence of norms when $\beta < 1/2mp$ is uniform in t since all assumptions concerning $E(t, x, D)$ and $\{\Gamma_j(t, x, D)\}_{1\leq j\leq m}$ are uniform in t. ///

By Proposition 6.3 Hypothesis II follows easily with any $\varrho < 1/2mp$; if $\beta \in]0, \varrho]$ a norm in $D_A(\beta, \infty) = D_{A-\omega}(\beta, \infty)$ is given by (6.12) with $s = 2m\beta$.

To verify Hypothesis III we need a further assumption. Extend the coefficients of $\Gamma_j(t, x, D)$ to functions $\tilde{b}_{j\beta} \colon [0, T] \times \bar{\Omega} \to \mathbf{C}$ such that

$$\begin{cases} \tilde{b}_{j\beta} \in C([0, T] \times \bar{\Omega}) \quad \text{and} \quad \tilde{b}_{j\beta}(t, \cdot) \in C^{2m-m_j}(\bar{\Omega}) \quad \text{with} \\ \sup\limits_{t\in[0,T]} \Big(\sum\limits_{j=1}^{m} \sum\limits_{|\gamma|\leq 2m} \|\tilde{b}_{j\beta}(t, \cdot)\|_{C^{2m-m_j}(\bar{\Omega})}\Big) < \infty\,; \end{cases} \tag{6.13}$$

next, assume that

$$\begin{cases} \text{there exists } \alpha \in]1 - 1/(2mp), 1[\text{ such that } a_{\gamma}(\cdot, x),\ D^{\delta}b_{j\beta}(\cdot, x) \\ \in C^{\alpha}([0, T]) \text{ and} \\ \sup\limits_{x\in\bar{\Omega}} \Big(\sum\limits_{|\gamma|\leq 2m} \|a_{\gamma}(\cdot, x)\|_{C^{\alpha}([0,T])} \\ \qquad + \sum\limits_{j=1}^{m} \sum\limits_{|\beta|\leq m_j} \sum\limits_{|\delta|\leq 2m-m_j} \|D^{\delta}\tilde{b}_{j\beta}(\cdot, x)\|_{C^{\alpha}([0,T])} \Big\} < \infty\,. \end{cases} \tag{6.14}$$

Then we have:

PROPOSITION 6.4. *Under assumptions* (6.1)-(6.8), (6.13) *and* (6.14), *let* $\{A(t)\}_{t\in[0,T]}$ *be defined by* (6.10) *and let* ω *be the number introduced in Proposition* 6.1. *Then*

$$\|\big(R(\omega, A(t)) - R(\omega, A(s))\big) f\|_{H^{2m,p}(\Omega)} \leq C_p |t-s|^\alpha \|f\|_{L^p(\Omega)} \qquad \forall t, s \in [0, T],\ \forall f \in L^p(\Omega)\,.$$

PROOF. Set $u(t) = R(\omega, A(t))f$, $t \in [0, T]$. Then $u(t) - u(s)$ solves

$$\begin{cases} [\omega - E(t, \cdot, D)][u(t) - u(s)] = [E(t, \cdot, D) - E(s, \cdot, D)]u(s) & \text{in } \Omega\,, \\ \Gamma_j(t, \cdot, D)[u(t) - u(s)] = -[\Gamma_j(t, \cdot, D) - \Gamma_j(s, \cdot, D)]u(s)\,, & \\ & j = 1, \dots, m \text{ on } \partial\Omega\,; \end{cases}$$

hence by (6.11)

$$\begin{aligned} \|u(t) - u(s)\|_{H^{2m,p}(\Omega)} \leq C_p\Big(&\|[E(t, \cdot, D) - E(s, \cdot, D)]u(s)\|_{L^p(\Omega)} \\ &+ \sum_{j=1}^m \|[\Gamma_j(t, \cdot, D) - \Gamma_j(s, \cdot, D)]u(s)\|_{H^{2m-m_j-1/p,p}(\partial\Omega)}\Big)\,. \end{aligned} \tag{6.15}$$

Denoting by $\tilde{\Gamma}_j(t, x, D)$ the differential operator whose coefficients are $\tilde{b}_{j\beta}$, we have

$$\begin{aligned} &\sum_{j=1}^m \|[\Gamma_j(t, \cdot, D) - \Gamma_j(s, \cdot, D)]u(s)\|_{H^{2m-m_j-1/p,p}(\partial\Omega)} \\ &\qquad\leq C_p \sum_{j=1}^m \|[\tilde{\Gamma}(t, \cdot, D) - \tilde{\Gamma}_j(s, \cdot, D)]u(s)\|_{H^{2m-m_j,p}(\Omega)} \end{aligned}$$

hence by (6.15) we easily deduce that

$$\|u(t) - u(s)\|_{H^{2m,p}(\Omega)} \leq C_p |t-s|^\alpha \|u(s)\|_{H^{2m,p}(\Omega)}\,,$$

and the result follows since

$$\|u(s)\|_{H^{2m,p}(\Omega)} \leq C_p \|f\|_{L^p(\Omega)}\,. \qquad ///$$

Now Hypothesis III clearly follows by the continuous inclusion $H^{2m,p}(\Omega) \subseteq B^{s,p}_\infty(\Omega)$, $\forall s \in]0, 2m[$.

Thus we have shown that under assumptions (6.1)-(6.8), (6.13) and (6.14) the operators $\{A(t)-\omega\}_{t\in[0,T]}$ defined by (6.10) fulfil Hypotheses I, II and III with any $\varrho\in]0,1/2mp[$ and any $\alpha\in]1-\varrho,1[$; therefore Theorems 5.1 amd 5.2 are applicable to problem (6.9): we omit the explicit statement of the results.

Second example.

Choose $m=1$ and let Ω be a bounded connected open set of $\mathbb{R}^n$ with $\partial\Omega$ of class C^2. Consider again the differential operator $E(t,x,D)$ defined by (6.1), under assumptions (6.2), (6.3), (6.4). Concerning the boundary operator $\Gamma_1(t,x,D)=:\Gamma(t,x,D)$, we take an oblique derivative operator, i.e.

$$\Gamma(t,x,D)=\gamma(t,x)I+\big(\beta(t,x)|\nabla\big)_n\,,\qquad (t,x)\in[0,T]\times\partial\Omega\,, \tag{6.16}$$

where

$$\begin{cases}\gamma,\beta_i\in C\big([0,T]\times\partial\Omega\big)\,;\quad \gamma(t,\cdot),\beta_i(t,\cdot)\in C^2(\partial\Omega)\quad\text{with}\\ \sup\limits_{t\in[0,T]}\Big(\|\gamma(t,\cdot)\|_{C^2(\partial\Omega)}+\sum\limits_{i=1}^n\|\beta_i(t,\cdot)\|_{C^2(\partial\Omega)}\Big)<\infty\,.\end{cases} \tag{6.17}$$

and

$$\begin{cases}\gamma,\ \beta_i\ \text{are real-valued, and}\\ \gamma(t,x)\geqslant 0\,,\quad \big(\beta(t,x)|\nu(x)\big)_n>0\qquad \forall(t,x)\in[0,T]\times\partial\Omega\,,\end{cases} \tag{6.18}$$

(here $\nu(x)$ is the unit outward normal vector at $x\in\partial\Omega$).

We want to study the problem

$$\begin{cases}u_t(t,x)-E(t,x,D)u(t,x)=f(t,x)\,;\qquad (t,x)\in[0,T]\times\bar\Omega\,,\\ u(0,x)=\varphi(x)\,,\qquad x\in\bar\Omega\,,\\ \Gamma(t,x,D)u(t,x)=0\,,\qquad (t,x)\in[0,T]\times\partial\Omega\,.\end{cases} \tag{6.19}$$

Set $E=C(\bar\Omega)$ and

$$\begin{cases}D_{A(t)}=\Big\{u\in\bigcap\limits_{q\in]1,\infty[}H^{2,q}(\Omega):E(t,\cdot,D)u\in C(\bar\Omega)\\ \qquad\qquad\qquad\qquad\text{and } \Gamma(t,\cdot,D)u=0\ \text{on}\ \partial\Omega\Big\}\,,\\ A(t)u=E(t,\cdot,D)u\,.\end{cases} \tag{6.20}$$

Let us verify Hypotheses I, II, III. Hypothesis I follows by a result of Stewart [31, Theorem 1]:

PROPOSITION 6.5. *Let* $m = 1$ *and assume* (6.1)-(6.4), (6.16)-(6.18). *There exist* $\theta_0 \in]\pi/2, \pi]$ *and* $\omega > 0$ *such that if* $\lambda - \omega \in S_{\theta_0}$, *then the problem*

$$\lambda u - E(t, \cdot, D)u = f \in C(\bar{\Omega}), \qquad \Gamma(t, \cdot, D)u = 0 \qquad \text{on } \partial\Omega,$$

has a unique solution $u(t) \in \bigcap_{q \in]1,\infty[} H^{2,q}(\Omega)$; *moreover there exists* $C > 0$ (*independent of* t) *such that*

$$(1 + |\lambda - \omega|)\|u(t)\|_{C(\bar{\Omega})} + (1 + |\lambda - \omega|)^{\frac{1}{2}} \|\nabla u(t)\|_{C(\bar{\Omega})} \leq C\|f\|_{C(\bar{\Omega})}.$$

PROOF. See [31]. ///

The characterization of $D_{A(t)}(\beta, \infty)$ needed to get Hypothesis II is proved in Acquistapace - Terreni [4, Theorem 6.2]. Namely, we have:

PROPOSITION 6.6. *Let* $m = 1$, *assume* (6.1)-(6.4), (6.16)-(6.18) *and define* $\{A(t)\}_{t \in [0,T]}$ *by* (6.20). *Then for each* $t \in [0, T]$ *we have* (*with equivalent norms*):

$$D_{A(t)}(\beta, \infty) = \left| \begin{array}{ll} C^{2\beta}(\bar{\Omega}), & \text{if } \beta \in]0, \tfrac{1}{2}[\ ; \\ C^{*,1}_{\beta(t,\cdot)}(\bar{\Omega}), & \text{if } \beta = \tfrac{1}{2}\ ; \\ \{u \in C^{1,2\beta-1}(\bar{\Omega}) \colon \Gamma(t, \cdot, D)u = 0 \ \text{ on } \partial\Omega\}, & \text{if } \beta \in]\tfrac{1}{2}, 1[\ ; \end{array} \right.$$

here the Zygmund class $C^{*,1}(\bar{\Omega})$ *is defined by*

$$C^{*,1}(\bar{\Omega}) := \left\{ u \in C(\bar{\Omega}) \colon \sup \left\{ \frac{|u(x) + u(y) - 2u((x+y)/2)|}{|x - y|} \colon x, y, \frac{x+y}{2} \in \bar{\Omega},\ x \neq y \right\} =: [u]_{C^{*,1}(\bar{\Omega})} < \infty \right\}$$

and normed by

$$\|u\|_{C^{*,1}(\bar{\Omega})} := \|u\|_{C(\bar{\Omega})} + [u]_{C^{*,1}(\bar{\Omega})},$$

whereas the space $C^{*,1}_{\beta(t,\cdot)}(\bar{\Omega})$ *is defined by*

$$C^{*,1}_{\beta(t,\cdot)}(\bar{\Omega}) := \left\{ u \in C^{*,1}(\bar{\Omega}) \colon \sup \left(\frac{|u(x - \sigma\beta(t, x)) - u(x)|}{\sigma} \colon \sigma > 0,\ x \in \partial\Omega,\ x - \sigma\beta(t, x) \in \bar{\Omega} \right) =: [u]_{*,1,\beta(t,\cdot)} < \infty \right\}$$

and normed by

$$\|u\|_{C^{*,1}_{\beta(t,\cdot)}} := \|u\|_{C^{*,1}(\Omega)} + [u]_{*,1,\beta(t,\cdot)}.$$

Moreover the equivalence of norms is uniform in t. In particular, if $\beta \in]0, \frac{1}{2}[$ $D_{A(t)}(\beta, \infty)$ *does not depend on t.*

PROOF. See [4]. The uniformity in t of the equivalences follows as in Proposition 6.3. ///

Hence Hypothesis II holds with any $\varrho \in]0, \frac{1}{2}[$.

To get Hypothesis III introduce, as in (6.14), the additional requirement that

$$\text{(6.21)} \quad \begin{cases} \text{there exists } \alpha \in]\frac{1}{2}, 1[\text{ such that } a_\gamma(\cdot, x),\ \beta_i(\cdot, x) \in C^\alpha([0, T]) \text{ and} \\ \sup\limits_{x \in \bar{\Omega}} \Big(\sum\limits_{|\gamma| \leqslant 2} \|a_\gamma(\cdot, x)\|_{C^\alpha([0,T])} \Big) \\ \qquad\qquad + \sup\limits_{x \in \partial\Omega} \|\gamma(\cdot, x)\|_{C^\alpha([0,T])} + \sum\limits_{i=1}^{n} \|\beta_i(\cdot, x)\|_{C^\alpha([0,T])} < \infty . \end{cases}$$

Set $u(t) = R(\omega, A(t)) f$, $t \in [0, T]$, where $f \in C(\bar{\Omega})$. Then

$$u(t) - u(s) \in \bigcap_{q \in]1,\infty[} H^{2,q}(\Omega)$$

and solves

$$\begin{cases} [\omega - E(t, \cdot, D)][u(t) - u(s)] = [E(t, \cdot, D) - E(s, \cdot, D)] u(s) & \text{in } \Omega , \\ \Gamma(t, \cdot, D)[u(t) - u(s)] \qquad = - [\Gamma(t, \cdot, D) - \Gamma(s, \cdot, D)] u(s) & \text{on } \partial\Omega , \end{cases}$$

so that choosing $p > n$ and using (6.11) we check

$$\|u(t) - u(s)\|_{H^{1,p}(\Omega)} \leqslant C_p |t - s|^\alpha \|f\|_{C(\bar{\Omega})} ,$$

and by Sobolev's Theorem

$$\|u(t) - u(s)\|_{C^{1-n/p}(\bar{\Omega})} \leqslant C_p |t - s|^\alpha \|f\|_{C(\bar{\Omega})} .$$

Thus for each $\varrho \in]1 - \alpha, \frac{1}{2}[$ we get

$$\|[R(\omega, A(t)) - R(\omega, A(s))] f\|_{D_A(\varrho,\infty)} \leqslant C_\varrho |t - s|^\alpha \|f\|_{C(\bar{\Omega})}$$

which is Hypothesis III.

Hence if $m = 1$, under assumptions (6.1)-(6.4), (6.16)-(6.18), (6.21) the operators $\{A(t) - \omega\}_{t \in [0,T]}$ introduced in (6.20) satisfy Hypotheses I, II and III with any $\varrho \in]0, \frac{1}{2}[$ and any $\alpha \in]1 - \varrho, 1[$, so that Theorems 5.1 and 5.2 are applicable to problem (6.19). We omit the details.

REMARK 6.7. The above argument applies to Dirichlet boundary conditions (when $m = 1$ and $\partial\Omega$ is of class C^2). In this case Hypothesis I follows by Stewart [30, Theorem 1] (if $\partial\Omega \in C^{2,\mu}$) and Cannarsa - Terreni - Vespri [9, Theorem 6.1] (general case); the characterization of $D_{A(t)}(\beta, \infty)$, which is due to Lunardi [22, Theorem 2.7] (if $\partial\Omega \in C^{2,\mu}$ and $a_\gamma \in C^{\mu}(\bar{\Omega})$ for $|\gamma| = 2$) and to Acquistapace - Terreni [4, Theorem 6.3] (general case), is the following:

$$D_{A(t)}(\beta, \infty) = \begin{cases} \{u \in C^{2\beta}(\bar{\Omega}) \colon u|_{\partial\Omega} = 0\} & \text{if } \beta \in]0, \tfrac{1}{2}[\,, \\ \{u \in C^{*,1}(\bar{\Omega}) \colon u|_{\partial\Omega} = 0\} & \text{if } \beta = \tfrac{1}{2}\,, \\ \{u \in C^{1,2\beta-1}(\bar{\Omega}) \colon u|_{\partial\Omega} = 0\} & \text{if } \beta \in]\tfrac{1}{2}, 1[\,. \end{cases}$$

Thus $D_{A(t)}(\beta, \infty)$ is constant in t for all $\beta \in]0, 1[$ (whereas $D_{A(t)}$ does depend on t). Finally Hypothesis III follows exactly as before.

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