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Allard Type Regularity Results for Varifolds with Free Boundaries.

MICHAEL GRÜTER - JÜRGEN JOST

1. – Introduction.

In this paper, we investigate the behaviour of rectifiable n-varifolds near the free boundary. Assuming the varifold V intersects a hypersurface in $\mathbb{R}^{n+k}$ orthogonally (in a generalized sense) and that its mean curvature vector lies in some L^p with $p > n$, we show that V is a submanifold of $\mathbb{R}^{n+k}$ (with boundary) with tangent spaces varying Hölder continuously, locally diffeomorphic to a half ball provided it is already close to a half ball in the sense that the mass of V inside small balls centered at the free boundary is not much larger than half the volume of the n-dimensional ball with the same radius and the density of V is at least one at almost all points of its support. For a precise statement of the main result, see Theorem 4.13. Some of our lemmata, like the reflection principle 4.11 (iii), or the monotonicity formulae, 3.1, 3.4, 4.11 (ii), should also be of independent interest.

As the title already suggests, our work may be considered as an extension of earlier results of Allard [AW1], [AW2]. We also use some of the techniques presented in the lecture notes [SL] of Leon Simon.

Recently, free boundary value problems for minimal surfaces have attracted much interest, and with the help of the results of the present paper, we can actually solve a well-known geometric problem, namely to show the existence of a nontrivial minimal embedded disk inside a given convex body in $\mathbb{R}^3$ which meets the boundary of this body orthogonally. This is carried out in our companion paper [GJ].

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2. – Notation and basic definitions.

We shall deal with rectifiable n-varifolds in $\mathbb{R}^{n+k}$, which can be described as follows. For further information the reader is referred to [AW1], [AW2] and especially to [SL]; the general reference concerning geometric measure theory is of course [FH].

If $M \subset \mathbb{R}^{n+k}$ is countably n-rectifiable and $\mathcal{H}^n$-measurable, and $\theta \geq 0$ a locally $\mathcal{H}^n$-integrable function on M we define the *rectifiable n-varifold* $\boldsymbol{v}(M, \theta)$ as the equivalence class of all pairs $(\tilde{M}, \tilde{\theta})$, where $\tilde{M}$ is countably n-rectifiable with $\mathcal{H}^n((M \sim \tilde{M}) \cup (\tilde{M} \sim M)) = 0$ and where $\theta = \tilde{\theta}$, $\mathcal{H}^n$ - a.e. on $M \cap \tilde{M}$.

As usual we call θ the *multiplicity function* of $\boldsymbol{v}(M, \theta)$.

If $B \subset \mathbb{R}^m$ is a C^2-submanifold of dimension $m-1$, we use the following notation.

For $b \in B$ let

$$\tau(b) := \operatorname{Tan}_b B \quad \text{and} \quad \nu(b) := \operatorname{Nor}_b B = \tau(b)^{\perp},$$

where $A^{\perp}$ denotes the orthogonal complement of a subspace $A \subset \mathbb{R}^m$.

For convenience we denote by $\tau(b)$ and $\nu(b)$ also the orthogonal projections of $\mathbb{R}^m$ onto these spaces. We assume that $\varkappa \geq 0$ is the smallest number such that

$$|\nu(b)(b'-b)| \leq \frac{\varkappa}{2} |b'-b|^2,$$

whenever b, $b' \in B$, and if $a \in \mathbb{R}^m$ we set $\varrho(a) := \operatorname{dist}(a, B)$. $\varkappa^{-1}$ may be considered as a radius of curvature for B. If $\varrho(a) \leq \varkappa^{-1}$ and if there exists $b \in B$ with $\varrho(a) = |a - b|$, then b is unique and is denoted by $\xi(a)$, so that ξ is the projection onto B.

For the proof of the following lemma the reader is referred to [AW2], Lemma 2.2.

2.1 LEMMA:

(i) *ξ is defined and continuously differentiable on an open set.*

(ii) $Q(a) := D\xi(a) - \tau(\xi(a))$ *is symmetric,* $Q(a)\circ\nu(\xi(a)) = 0$ *and*

$$(*)\qquad \|Q(a)\| \leqslant \varkappa\varrho(a)[1 - \varkappa\varrho(a)]^{-1}.$$

(iii) $\|D\nu(b)\| = \|D\tau(b)\| \leqslant \varkappa$.

2.2 Remark. We use two equivalent norms on Hom $(\mathbf{R}^m, \mathbf{R}^m)$. If $A \in$ Hom $(\mathbf{R}^m, \mathbf{R}^m)$ we set

$$|A|^2 = \operatorname{trace}(A^*\circ A) \qquad \text{and} \qquad \|A\| = \sup_{|u|=1} |Au| ;$$

we have the inequalities

$$\|A\| \leqslant |A| \leqslant \sqrt{m}\, \|A\| .$$

We now assume $\varkappa \leqslant \frac{1}{2}$, $0 \in B$, $\bar{B} \cap B_1(0) = B$, so that $\xi : B_1(0) \to B$ is well defined.

In this case we set

$$\tilde{x} := 2\xi(x) - x ;$$

then $\tilde{x}$ is the reflection of x across B and we note that $x = \tilde{x}$ if $x \in B$, $x = x$, and $\xi(x) = \xi(\tilde{x})$.

If $w \in \mathbf{R}^m$ we set

$$i_x(w) = i_{\tilde{x}}(w) = \tau(\xi(x))\, w - \nu(\xi(x))\, w .$$

Then i_x is an isometry with $i_x^2 = \mathrm{id}_{\mathbf{R}^m}$.

If $b \in B$ we get from (*)

$$|\tilde{x} - b| \leqslant \{1 + 2\varkappa|x - b|[1 - \varkappa|x - b|]^{-1}\}|x - b| ,$$

which implies for $\sigma \leqslant \frac{1}{2}$, $l_1(\sigma) = 1 + 2\varkappa\sigma/(1 - \varkappa\sigma)$, $l_2(\sigma) = 1 - 3\varkappa\sigma$.

$$B_{l_2(\sigma)\sigma}(b) \subset \tilde{B}_\sigma(b) \subset B_{l_1(\sigma)\sigma}(b) ,$$

where

$$\tilde{B}_\sigma(b) = \{x : |\tilde{x} - b| < \sigma\} .$$

If $a \in B_1(0)$, $\sigma \geqslant \varrho(a)$ we have for $x \in B_\sigma(a)$

$$\varrho(x) \leqslant 4\sigma ,$$

because $\varkappa \leqslant \frac{1}{2}$.

This on the other hand implies $(\sigma \geqslant \varrho(a))$

$$\tilde{B}_\sigma(a) \subset B_{9\sigma}(a)$$

as well as

$$B_\sigma(a) \subset \tilde{B}_{9\sigma}(a) .$$

We denote by ω_n the volume of the n-dimensional ball of radius one.

In the course of the paper C will denote different constants. The parameters on which C depends will be obvious from the context.

We often write $x \in \mathbb{R}^{n+k}$ as

$$x = (x^1, \dots, x^n, 0, \dots, 0) + (0, \dots, 0, x^{n+1}, \dots, x^{n+k})$$

and denote this orthogonal splitting of $\mathbb{R}^{n+k}$ by

$$\mathbb{R}^{n+k} = \mathbb{R}^n \oplus \mathbb{R}^k .$$

Also in this notation

$$B^n_r(0) = B_r(0) \cap \mathbb{R}^n .$$

3. – Monotonicity results.

Let $B \subset \mathbb{R}^{n+k} (n, k \geqslant 1)$ be a C^2-submanifold of dimension $n + k - 1$ having the following properties (compare section 2 for notation):

(1) $$\varkappa \leqslant \tfrac{1}{2} ,$$

(2) $$0 \in B ,$$

(3) $$\bar{B} \cap B_1(0) = B \quad \big(\text{i.e. } B \text{ has no boundary inside } B_1(0)\big) .$$

Then for each $a \in B_1(0)$, $\xi(a) \in B$ is defined and thus $B_1(0) \sim B$ consists of two open components $B_1'(0)$ and $B_1''(0)$.

In this section we assume $V = \boldsymbol{v}(M, \theta)$ is a rectifiable n-varifold such that

$$\operatorname{spt}\mu_V = \operatorname{spt}(\mathcal{H}^n \llcorner \theta) \subset \overline{B_1'(0)}\,, \tag{4}$$

$$0 \in \operatorname{spt}\mu_V\,, \tag{5}$$

$$\mu_V(B_1(0)) < \infty\,, \tag{6}$$

$$\int \operatorname{div}_M X\, d\mu_V = -\int X \cdot \boldsymbol{H}\, d\mu_V\,, \tag{7}$$

whenever $X \in C_c^1(B_1(0), \mathbb{R}^{n+k})$ with $X(b) \in \tau(b)$ for $b \in B$ and where

$$\boldsymbol{H} \in L^1_{\mathrm{loc}}(B_1(0), \mathbb{R}^{n+k}, \mu)\,.$$

From now on we write $\mu = \mu_V$.

The aim of this section is to prove monotonicity results for V (up to error terms involving integral norms of $\boldsymbol{H}$ and curvature bounds for B). Compared to the interior regularity theory, here an additional difficulty arises from the fact that the balls we are looking at in general intersect the free boundary but are not necessarily centered on it. Since for several reasons it is not convenient at this stage to reflect the varifold across the free boundary, our idea is to reflect the balls across this boundary and add the mass of V in such a ball and its mass in the reflected ball and prove monotonicity formulae for this sum.

We now fix $a \in B_1'(0) \cup B$ and choose for (7) the vector field

$$X(x) = \gamma(r)(x - a) + \gamma(\tilde{r})\big(i_{\tilde{x}}(\tilde{x} - a)\big)\,,$$

with $r = |x - a|$, $\tilde{r} = |\tilde{x} - a| = |2\xi(x) - x - a|$ and where $\gamma \in C^1(\mathbb{R})$ is such that

$$\gamma'(t) \leqslant 0 \quad \forall t\,, \qquad \gamma(t) \equiv 1 \quad \text{for } t \leqslant \varrho/2\,,$$
$$\gamma(t) \equiv 0 \quad \text{for } t \geqslant \varrho\,,$$

and where $\varrho > 0$ is such that $\operatorname{dist}\big(\operatorname{spt}\mu \cap (B_\varrho(a) \cup \tilde{B}_\varrho(a)), \partial B_1(0)\big) > 0$. Note that because of the definition of $\tilde{x}$, $i_{\tilde{x}}$ this vector field is admissible in (7).

Using the properties of the previous section we get by a simple calculation analogous to the one in [SL, § 17] for any $x \in M$ for which $T_x M$ exists

$$\operatorname{div}_M X(x) = n[\gamma(r) + \gamma(\tilde r)] + r\gamma'(r)\,[1 - |(Dr)^\perp|^2] + \tilde r\gamma'(\tilde r)\left[1 - \left|\left(i_{\tilde x}\left(\frac{\tilde x - a}{\tilde r}\right)\right)^\perp\right|^2\right] + \gamma(\tilde r)\,\varepsilon_1(x) - \tilde r\gamma'(\tilde r)\,\varepsilon_2(x)\,,$$

where $P_x = P_{T_x M}$, $(\ )^\perp = P_x^\perp(\ )$ and where

$$\varepsilon_1(x) = 2\operatorname{tr}\left(P_x \circ \boldsymbol{Q}(x)\right) + \left\langle P_x \circ \left(\tau(\xi(x)) + \boldsymbol{Q}(x)\right), (D\tau - D\nu)(\xi(x)), \tilde x - a\right\rangle,$$

$$\varepsilon_2(x) = -2\boldsymbol{Q}(x)\left(\frac{\tilde x - a}{\tilde r}\right)\cdot P_x \circ i_{\tilde x}\left(\frac{\tilde x - a}{\tilde r}\right);$$

$\langle\ ,\ ,\ \rangle$ has to be interpreted appropriately.

Thus (7) leads to the following integral identity.

$$\begin{aligned}(8)\qquad n\int \gamma(r) + \gamma(\tilde r)\,d\mu + \int r\gamma'(r) + \tilde r\gamma'(\tilde r)\,d\mu &= -\int \boldsymbol{H}\cdot[\gamma(r)(x-a) + \gamma(\tilde r)\,i_{\tilde x}(\tilde x - a)]\,d\mu \\ &\quad + \int r\gamma'(r)\left|\left(\frac{x-a}{r}\right)^\perp\right|^2 + \tilde r\gamma'(\tilde r)\left|\left(i_{\tilde x}\left(\frac{\tilde x - a}{\tilde r}\right)\right)^\perp\right|^2 d\mu \\ &\quad - \int \gamma(\tilde r)\,\varepsilon_1(x)\,d\mu + \int \tilde r\gamma'(\tilde r)\,\varepsilon_2(x)\,d\mu\,.\end{aligned}$$

Now take $\phi \in C^1(\mathbb{R})$ such that $\phi(t) \equiv 1$ for $t \leqslant \frac{1}{2}$, $\varphi(t) = 0$ for $t \geqslant 1$ and $\phi'(t) \leqslant 0$ for all t and use (8) with $\gamma(r) = \phi(r/\varrho)$. Noting that $r\gamma'(r) = -\varrho(\partial/\partial\varrho)\phi(r/\varrho)$ we get

$$nI(\varrho) - \varrho I'(\varrho) = -\left\{L(\varrho) + \varrho J'(\varrho) + L_1(\varrho) + \varrho L_2'(\varrho)\right\},$$

where

$$I(\varrho) = \int \phi(r/\varrho) + \phi(\tilde r/\varrho)\,d\mu\,,$$

$$L(\varrho) = \int \boldsymbol{H}\cdot[\phi(r/\varrho)(x-a) + \phi(\tilde r/\varrho)\,i_{\tilde x}(\tilde x - a)]\,d\mu\,,$$

$$J(\varrho) = \int \phi(r/\varrho)\left|\left(\frac{x-a}{r}\right)^\perp\right|^2 + \phi(\tilde r/\varrho)\left|\left(i_{\tilde x}\left(\frac{\tilde x - a}{r}\right)\right)^\perp\right|^2 d\mu\,,$$

$$L_i(\varrho) = \int \phi(\tilde r/\varrho)\,\varepsilon_i(x)\,d\mu\,,\qquad i = 1, 2\,.$$

Multiplying by ϱ^{-n-1} we get

$$\text{(9)}\qquad \frac{d}{d\varrho}[\varrho^{-n}I(\varrho)] = \varrho^{-n}[J'(\varrho) + L_2'(\varrho)] + \varrho^{-n-1}[L(\varrho) + L_1(\varrho)].$$

By letting ϕ increase to $\chi_{(-\infty,1)}$ we obtain (χ always denotes characteristic functions)

$$\text{(10)}\qquad \frac{d}{d\varrho}\left[\varrho^{-n}\big(\mu(B_\varrho(a)) + \mu(\tilde{B}_\varrho(a))\big)\right]$$
$$= \varrho^{-n}\frac{d}{d\varrho}\left[\int_{B_\varrho(a)}\left|\left(\frac{x-a}{r}\right)^{\perp}\right|^2 d\mu + \int_{\tilde{B}_\varrho(a)}\left|\left(i_{\tilde{x}}\left(\frac{\tilde{x}-a}{r}\right)\right)^{\perp}\right|^2 d\mu\right]$$
$$+ \varrho^{-n-1}\left[\int_{B_\varrho(a)}(x-a)\cdot\boldsymbol{H}\,d\mu + \int_{\tilde{B}_\varrho(a)}\big(i_{\tilde{x}}(\tilde{x}-a)\cdot\boldsymbol{H} + \varepsilon_1(x)\big)\,d\mu\right]$$
$$+ \varrho^{-n}\frac{d}{d\varrho}\int_{\tilde{B}_\varrho(a)}\varepsilon_2(x)\,d\mu.$$

This is the fundamental *monotonicity identity* which reduces to the well known corresponding formula used in the interior regularity in case $\varrho < \varrho(a) = \operatorname{dist}(a, B)$, because then $\mu(\tilde{B}_\varrho(a)) = 0$.

By the inequalities of section 2 we have the following estimates

$$\text{(11)}\qquad \begin{cases} |\varepsilon_1(x)| \leqslant c\varkappa\{\varrho(x) + \tilde{r}(1 + \varkappa\varrho(x))\}, \\ |\varepsilon_2(x)| \leqslant c\varkappa\varrho(x). \end{cases}$$

As a first consequence of (9) we get

3.1 THEOREM. *If* $a \in B_1'(0) \cup B$,

$$\left[\omega_n^{-1}\int_{B_1(0)}|\boldsymbol{H}|^p\,d\mu\right]^{1/p} \leqslant \Gamma, \qquad R > 0,$$

where

$$p > n \quad \text{and} \quad \operatorname{dist}\big(\operatorname{spt}\mu \cap (B_R(a) \cup \tilde{B}_R(a)),\ \partial B_1(0)\big) > 0,$$

then

$$\text{(12)}\qquad \left[\omega_n^{-1}\sigma^{-n}\big(\mu(B_\sigma(a)) + \mu(\tilde{B}_\sigma(a))\big)\right]^{1/p} + \frac{\Gamma}{p-n}\sigma^{1-n/p}$$
$$\leqslant \left[\omega_n^{-1}\varrho^{-n}\big(\mu(B_\varrho(a)) + \mu(\tilde{B}_\varrho(a))\big)\right]^{1/p} + \frac{\Gamma}{p-n}\varrho^{1-n/p}$$

whenever $\sigma < \varrho \leqslant \min\{R, \varrho(a)\}$; *and if* $\varrho(a) \leqslant \sigma < \varrho \leqslant R$ *we have*

$$(13)\qquad \Big[\omega_n^{-1}\sigma^{-n}\Big(\mu(B_\sigma(a)) + \mu(\tilde{B}_\sigma(a))\Big)\Big]^{1/p}\left(1 + c\varkappa\sigma\left(1 + \frac{1}{p-n}\right)\right) + \frac{\Gamma}{p-n}\sigma^{1-n/p}$$
$$\leqslant \Big[\omega_n^{-1}\varrho^{-n}\Big(\mu(B_\varrho(a)) + \mu(\tilde{B}_\varrho(a))\Big)\Big]^{1/p}\left(1 + c\varkappa\varrho\left(1 + \frac{1}{p-n}\right)\right) + \frac{\Gamma}{p-n}\varrho^{1-n/p}\,.$$

PROOF. We only have to show (13), because (12) is just the statement of [SL], 17.7.

From (9) we get, using Hölder's inequality,

$$\frac{d}{d\varrho}\left[\varrho^{-n}I(\varrho)\right] \geqslant \varrho^{-n}L_2'(\varrho) + \varrho^{-n-1}L_1(\varrho) - \varrho^{-n}\omega_n^{1/p}\Gamma I(\varrho)^{1-1/p}\,.$$

By (11) we have

$$|\varrho^{-n-1}L_1(\varrho)| \leqslant \varrho^{-n-1}c\varkappa\varrho(1+\varkappa\varrho)\int\phi(\tilde{r}/\varrho)\,d\mu \leqslant c\varkappa\varrho^{-n}I(\varrho)$$

and

$$|\varrho^{-n}L_2'(\varrho)| \leqslant c\varkappa\varrho^{1-n}I'(\varrho)\,.$$

Thus we get the differential inequality

$$\frac{d}{d\varrho}\left[\varrho^{-n}I(\varrho)\right]^{1/p} \geqslant -\frac{1}{p}\left\{\varrho^{-n/p}\omega_n^{1/p}\Gamma + c\varkappa\varrho^{-n/p}I(\varrho)^{1/p} + c\varkappa\varrho^{1-n/p}I(\varrho)^{1/p-1}I'(\varrho)\right\}.$$

Integration from σ to ϱ yields

$$\left[\varrho^{-n}I(\varrho)\right]^{1/p} - \left[\sigma^{-n}I(\sigma)\right]^{1/p} \geqslant$$
$$-\omega_n^{1/p}\frac{\Gamma}{p-n}\left(\varrho^{1-n/p} - \sigma^{1-n/p}\right) - \frac{c\varkappa}{p}\int_\sigma^\varrho \tau^{-n/p}I(\tau)^{1/p}\,d\tau - c\varkappa\int_\sigma^\varrho \tau^{1-n/p}\frac{d}{d\tau}\left(I(\tau)\right)^{1/p}d\tau\,.$$

Integrating by parts we may estimate

$$\left[\varrho^{-n}I(\varrho)\right]^{1/p} - \left[\sigma^{-n}I(\sigma)\right]^{1/p} \geqslant -\omega_n^{1/p}\frac{\Gamma}{p-n}\left(\varrho^{1-n/p} - \sigma^{1-n/p}\right)$$
$$- c\varkappa\left(1 + \frac{1}{p-n}\right)\left(\varrho^{1-n/p}I(\varrho)^{1/p} - \sigma^{1-n/p}I(\sigma)^{1/p}\right).$$

Rearranging terms and letting ϕ increase to $\chi_{(-\infty,1)}$ we get (13). □

3.2 COROLLARY. *If* $H \in L^p_{\mathrm{loc}}(B_1(0), \mathbb{R}^{n+k}, \mu)$ *for some* $p > n$, *then the density*

$$\theta^n(\mu, x) := \lim_{\varrho \downarrow 0} \frac{\mu\left(\overline{B_\varrho(x)}\right)}{\omega_n \varrho^n}$$

exists at every point $x \in B_1(0)$, *and*

$$\tilde{\theta}^n(\mu, x) = \begin{cases} \theta^n(\mu, x)\,, & x \notin B\,, \\ 2\theta^n(\mu, x)\,, & x \in B\,, \end{cases}$$

is an upper-semi-continuous function in $B_1(0)$.

PROOF. For $x \in B_1''(0)$ we have $\tilde{\theta}^n(\mu, x) = 0$.

If $x \in B_1'(0)$ we get by (12) (note that $\mu(\tilde{B}_\sigma(x)) = 0$ for $\sigma < \varrho(x)$) that

$$[\omega_n^{-1} \sigma^{-n} \mu(B_\sigma(x))]^{1/p} + \frac{\Gamma}{p-n} \sigma^{1-n/p}$$

is non-decreasing for $\sigma < \varrho(x)$, and thus $\lim_{\sigma \downarrow 0} \mu(B_\sigma(x))/(\omega_n \sigma^n)$ exists and equals $\theta^n(\mu, x)$.

Now if $x \in B$ we argue as follows.

First we note that by (13)

$$\lim_{\sigma \downarrow 0} \left[\omega_n^{-1} \sigma^{-n}\left(\mu(\overline{B_\sigma(x)}) + \mu(\overline{\tilde{B}_\sigma(x)})\right)\right]^{1/p}$$

exists.

With $l_1(\sigma)$ and $l_2(\sigma)$ as in section 2 we get

$$\overline{B_{l_2(\sigma)\sigma}(x)} \subset \overline{\tilde{B}_\sigma(x)} \subset \overline{B_{l_1(\sigma)\sigma}(x)}\,.$$

Using $\lim_{\sigma \downarrow 0} l_i(\sigma) = 1$, $i = 1, 2$, we can now easily check that $\theta^n(\mu, x)$ exists and that

$$\lim_{\sigma \downarrow 0} \left[\omega_n^{-1} \sigma^{-n}\left(\mu(\overline{B_\sigma(x)}) + \mu(\overline{\tilde{B}_\sigma(x)})\right)\right]^{1/p} = [2\theta^n(\mu, x)]^{1/p}\,.$$

It remains to show the upper-semi-continuity of $\tilde{\theta}^n(\mu, x)$. This is well known if $x \notin B$.

Now for $x \in B$, $y \in B \cup B_1'(0)$ (as $\tilde{\theta} \geqslant 0$, we only have to consider such y) and for $\sigma < \varrho$, $|x - y| < \varepsilon$ and ϱ and ε small enough we get from (12) and (13)

$$\begin{aligned} &\left[\omega_n^{-1} \sigma^{-n}\left(\mu(B_\sigma(y)) + \mu(\tilde{B}_\sigma(y))\right)\right]^{1/p} \\ &\quad \leqslant \left[\omega_n^{-1} \varrho^{-n}\left(\mu(B_\varrho(y)) + \mu(\tilde{B}_\varrho(y))\right)\right]^{1/p} (1 + c\varkappa\varrho) + c\varrho^{1-n/p} \\ &\quad \leqslant \left[\omega_n^{-1} (\varrho + \varepsilon)^{-n}\left(\mu(B_{\varrho+\varepsilon}(x)) + \mu(\tilde{B}_{\varrho+\varepsilon}(x))\right)\right]^{1/p} \left(1 + \frac{\varepsilon}{\varrho}\right)^{n/p} (1 + c\varkappa\varrho) + c\varrho^{1-n/p}\,. \end{aligned}$$

Letting $\sigma\downarrow 0$ we get on the left hand side of this inequality $[\tilde{\theta}^n(\mu, y)]^{1/p}$.

Now let $\delta > 0$ be given and choose first $\varrho > 0$ and then $0 < \varepsilon < \varrho$ so that the right hand side of the inequality can be estimated by

$$[\tilde{\theta}^n(\mu, x)]^{1/p} + \delta .$$

This yields

$$[\tilde{\theta}^n(\mu, y)]^{1/p} \leqslant [\tilde{\theta}^n(\mu, x)]^{1/p} + \delta$$

provided $|y - x| < \varepsilon(\delta)$. Thus the corollary is proved. □

3.3 REMARKS:

(i) If $\theta \geqslant 1$ μ - a.e. in $B_1(0)$, then $\tilde{\theta}^n(\mu, x) \geqslant 1$ for *each* $x \in B_1(0) \cap \operatorname{spt} \mu$.

(ii) If $\left[\omega_n^{-1} \int\limits_{B_1(0)} |H|^p \, d\mu\right]^{1/p} \leqslant \Gamma$, $p > n$, $a \in B_1(0)$ with $\tilde{\theta}^n(\mu, a) \geqslant 1$ and $R > 0$ such that $\operatorname{dist}\big(\operatorname{spt} \mu \cap (B_R(a) \cup \tilde{B}_R(a)), \partial B_1(0)\big) > 0$, then we have for $\varrho \leqslant R$

$$\int\limits_{B_\varrho(a)} |H| \, d\mu + \int\limits_{\tilde{B}_\varrho(a)} |H| \, d\mu \leqslant c\Gamma\big(\mu(B_\varrho(a)) + \mu(\tilde{B}_\varrho(a))\big) \varrho^{-n/p} \tag{14}$$

provided $\big(\Gamma/(p - n)\big) \varrho^{1-n/p} \leqslant \frac{1}{2}$, where $c = c(n, k, p)$.

In fact by Hölder's inequality we have

$$\int\limits_{B_\varrho(a)} |H| \, d\mu + \int\limits_{\tilde{B}_\varrho(a)} |H| \, d\mu \leqslant 2\omega_n^{1/p} \Gamma\big(\mu(B_\varrho(a)) + \mu(\tilde{B}_\varrho(a))\big)^{1-1/p} .$$

On the other hand letting $\sigma\downarrow 0$ in (12) and (13) we get

$$\mu(B_\varrho(a)) + \mu(\tilde{B}_\varrho(a)) \geqslant \omega_n \varrho^n \cdot \left\{\left(1 - \frac{\Gamma}{p - n} \varrho^{1-n/p}\right)\left[1 + c\varkappa\varrho\left(1 + \frac{1}{p - n}\right)\right]^{-1}\right\}^p \geqslant \omega_n \varrho^n \tilde{C} .$$

Another application of the monotonicity identity (10) is

3.4 THEOREM. *If* $a \in B_1'(0) \cup B$, $0 < \alpha < 1$, $\Lambda > 0$ *and if*

$$\int\limits_{B_\varrho(a)} |H| \, d\mu + \int\limits_{\tilde{B}_\varrho(a)} |H| \, d\mu \leqslant \alpha\Lambda(\varrho/R)^{\alpha-1}\big(\mu(B_\varrho(a)) + \mu(\tilde{B}_\varrho(a))\big) \tag{15}$$

for all $0 < \varrho \leqslant R$, *where* $\operatorname{dist}\big(\operatorname{spt} \mu \cap (B_R(a) \cup \tilde{B}_R(a)), \partial B_1(0)\big) > 0$, *then*

$$\exp\left[c\varkappa\sigma + \Lambda R^{1-\alpha} \sigma^\alpha\right] \sigma^{-n}\big(\mu(B_\sigma(a)) + \mu(\tilde{B}_\sigma(a))\big)$$

is non-decreasing on $(0, R]$ *and furthermore we have*

$$\text{(16)}\quad \exp[\Lambda R^{1-\alpha}\sigma^{\alpha}]\sigma^{-n}\big(\mu(B_\sigma(a)) + \mu(\tilde{B}_\sigma(a))\big)$$
$$\leqslant \exp[\Lambda R^{1-\alpha}\varrho^{\alpha}]\varrho^{-n}\big(\mu(B_\varrho(a)) + \mu(\tilde{B}_\varrho(a))\big)$$
$$-\Bigg\{\int\limits_{B_\varrho(a)\sim B_\sigma(a)} \left|\left(\frac{x-a}{r}\right)^{\perp}\right|^2 r^{-n}\,d\mu + \int\limits_{\tilde{B}_\varrho(a)\sim \tilde{B}_\sigma(a)} \left|\left(i_{\tilde{x}}\left(\frac{\tilde{x}-a}{\tilde{r}}\right)\right)^{\perp}\right|^2 \tilde{r}^{-n}\,d\mu\Bigg\},$$

whenever $\sigma < \varrho \leqslant \min\{R, \varrho(a)\}$; *while if* $\varrho(a) \leqslant \sigma < \varrho \leqslant R$ *we get*

$$\text{(17)}\quad \exp[c\varkappa\sigma + \Lambda R^{1-\alpha}\sigma^{\alpha}]\sigma^{-n}\big(\mu(B_\sigma(a)) + \mu(\tilde{B}_\sigma(a))\big)$$
$$\leqslant \exp[c\varkappa\varrho + \Lambda R^{1-\alpha}\varrho^{\alpha}]\varrho^{-n}\big(\mu(B_\varrho(a)) + \mu(\tilde{B}_\varrho(a))\big)$$
$$- c'\Bigg\{\int\limits_{B_\varrho(a)\sim B_\sigma(a)} \left|\left(\frac{x-a}{r}\right)^{\perp}\right|^2 r^{-n}\,d\mu + \int\limits_{\tilde{B}_\varrho(a)\sim \tilde{B}_\sigma(a)} \left|\left(i_{\tilde{x}}\left(\frac{\tilde{x}-a}{\tilde{r}}\right)\right)^{\perp}\right|^2 \tilde{r}^{-n}\,d\mu\Bigg\}.$$

Here c, c' *are positive constants depending only on* n, k.

3.5 REMARKS:

(i) In the situation of 3.3 (ii) we see that (15) is true with $\alpha = 1 - n/p$ and $\Lambda = (p/(p-n))\,C\Gamma R^{-n/p}$, C as in (14).

(ii) As in [SL], 17.6, we get the reverse inequalities corresponding to (16) and (17) if we replace

$$\exp[c\varkappa\sigma + \Lambda R^{1-\alpha}\sigma^{\alpha}] \quad \text{by} \quad \exp[-2c\varkappa\sigma - 2\Lambda R^{1-\alpha}\sigma^{\alpha}],$$

provided $c\varkappa \leqslant \frac{1}{2}$.

PROOF. Because of [SL], 17.6, it is sufficient to prove (17). Denote

$$M(\varrho) := \mu(B_\varrho(a)) + \mu(\tilde{B}_\varrho(a))$$

and

$$N(\varrho) := \int\limits_{B_\varrho(a)} \left|\left(\frac{x-a}{r}\right)^{\perp}\right|^2 d\mu + \int\limits_{\tilde{B}_\varrho(a)} \left|\left(i_{\tilde{x}}\left(\frac{\tilde{x}-a}{\tilde{r}}\right)\right)^{\perp}\right|^2 d\mu\,.$$

Then (10) and (15) imply together with (11)

$$\frac{d}{d\varrho}[\varrho^{-n}M(\varrho)] \geqslant \varrho^{-n}N'(\varrho) - \alpha\Lambda(\varrho/R)^{\alpha-1}\varrho^{-n}M(\varrho) - c\varkappa\varrho^{-n}M(\varrho) - c\varkappa\varrho^{1-n}M'(\varrho)$$

which implies

$$\frac{d}{d\varrho}\,[\varrho^{-n}M(\varrho)]\geqslant(1+c\varkappa\varrho)^{-1}\varrho^{-n}N'(\varrho)-\varrho^{-n}M(\varrho)\left((n+1)c\varkappa+\alpha\Lambda(\varrho/R)^{\alpha-1}\right)$$
$$\geqslant c_1\varrho^{-n}N'(\varrho)-\varrho^{-n}M(\varrho)\left(c_2\varkappa+\alpha\Lambda(\varrho/R)^{\alpha-1}\right).$$

Multiplying by $\exp[c_2\varkappa\varrho+\Lambda R^{1-\alpha}\varrho^\alpha]$ and integrating from σ to ϱ we deduce (17). □

Before concluding this section with a technical lemma concerning densities we have to make a different choice of the vectorfield X in (7).

If $h\in C^1(B_1(0))$ with $h\geqslant 0$ we use

$$X(x)=h(x)[\gamma(r)(x-a)+\gamma(\tilde r)\,i_{\tilde x}(x-a)]$$

in (7) if $\operatorname{dist}\left(\operatorname{spt}\mu\cap(B_\varrho(a)\cup\tilde B_\varrho(a)),\,\partial B_1(0)\right)>0$.

As

$$\operatorname{div}_M X=h\operatorname{div}_M[\ldots]+[\ldots]\cdot\nabla_M h,$$

we get (compare the derivation of (10))

$$\sigma^{-n}\int_{B_\sigma(a)}h\,d\mu\leqslant\varrho^{-n}\int_{B_\varrho(a)}h\,d\mu+\int_\sigma^\varrho\tau^{-n-1}\int_{B_\tau(a)}r(h|\boldsymbol H|+|\nabla_M h|)\,d\mu\,d\tau,\tag{18}$$

whenever $0<\sigma<\varrho\leqslant\varrho(a)$; while for $\varrho(a)\leqslant\sigma<\varrho$ we have

$$(1+c\varkappa\sigma)\,\sigma^{-n}\Bigg[\int_{B_\sigma(a)}h\,d\mu+\int_{\tilde B_\sigma(a)}h\,d\mu\Bigg]$$
$$\leqslant(1+c\varkappa\varrho)\,\varrho^{-n}\Bigg[\int_{B_\varrho(a)}h\,d\mu+\int_{\tilde B_\varrho(a)}h\,d\mu\Bigg]+nc\varkappa\int_\sigma^\varrho\tau^{-n}\Bigg[\int_{B_\tau(a)}h\,d\mu+\int_{\tilde B_\tau(a)}h\,d\mu\Bigg]d\tau$$
$$+\int_\sigma^\varrho\tau^{-n-1}\Bigg\{\int_{B_\tau(a)}r(h|\boldsymbol H|+|\nabla_M h|)\,d\mu+\int_{\tilde B_\tau(a)}\tilde r(h|\boldsymbol H|+|\nabla_M h|)\,d\mu\,d\tau\Bigg\}.$$

3.6. Lemma. *Suppose* $0<l\leqslant 1$, $0<\beta<\frac12$, $a\in B_1(0)$

$$\Bigg[\omega_n^{-1}\int_{B_1(0)}|\boldsymbol H|^p\,d\mu\Bigg]^{1/p}\leqslant\Gamma,\qquad\frac{\Gamma}{p-n}R^{1-n/p}\leqslant\frac12\tag{*}$$

such that dist $\big(\operatorname{spt}\mu \cap (B_R(a) \cup \tilde{B}_R(a)), \partial B_1(0)\big) > 0$, *and suppose* $y, z \in B_{\beta R}(a)$ *with* $|y-z| = \beta R$, $\tilde{\theta}^n(\mu, y)$, $\tilde{\theta}^n(\mu, z) \geqslant 1$ *and* $|q(y-z)| \geqslant l|y-z|$, *where* q *is the orthogonal projection of* $\mathbb{R}^{n+k}$ *onto* $\mathbb{R}^k$. *Then*

$$\tilde{\theta}^n(\mu, y) + \tilde{\theta}^n(\mu, z) \leqslant \{1 + c(l\beta)^{-n}[\varkappa R + \Gamma R^{1-n/p}]\} \cdot (1-\beta)^{-n}\omega_n^{-1}R^{-n}\{\mu(B_R(a)) + \mu(\tilde{B}_R(a))\} + C(l\beta)^{-n-1}R^{-n}\Big(\int_{B_R(a)} \|p_x - p\|\, d\mu + \int_{\tilde{B}_R(a)} \|p_x - p\|\, d\mu\Big).$$

where $c = c(n, k, p)$, $p = q^{\perp}$, $p_x = p_{T_xM}$.

The statement of the lemma can be illustrated in the following way: If, for simplicity, $\varkappa = \Gamma = 0$ and the mass of V in $B_R(a)$ is close to the mass of a half ball, then we cannot have points in the support of V with parallel tangent planes whose connection is almost vertical to these planes.

Later on in 4.2, when we want to represent V as a graph over a tangent plane, Lemma 3.6 will be used to control the gradient of this representation.

PROOF. In the proof, we shall apply the monotonicity formula twice. Once we use 3.4 with radii $\tau < \sigma$ and $\xi = y, z$ and let $\tau \to 0$ to get the densities on the left hand side. On the other hand, we use (19) (or (18)) with radii $\sigma < \varrho$ and a cut-off function h which equals 1 on the cylinder over $B^k_\sigma(\xi)$ and 0 outside the cylinder over $B^k_\varrho(\xi)$. The gradient of h then contributes the integrals $\int \|p_x - p\|$. A careful choice of σ and ϱ finally gives the desired inequality.

Here are the details.

Because of (*), Remarks 3.3 (ii), 3.5 (i) we may apply Theorem 3.4 with $\xi = y$ or z instead of a and with $(1-\beta)R$ instead of R. We thus get that

$$\exp[c\varkappa\sigma + c\Gamma\sigma^{1-n/p}]\,\sigma^{-n}\big(\mu(B_\sigma(\xi)) + \mu(\tilde{B}_\sigma(\xi))\big)$$

is non-decreasing on $\big(0, (1-\beta)R\big]$. Since $e^t \leqslant 1 + ct$ for $t \leqslant t_0$, $c = c(t_0)$ we get for $0 < \tau < \sigma \leqslant (1-\beta)R$

$$(**) \qquad \omega_n^{-1}\tau^{-n}\big(\mu(B_\tau(\xi)) + \mu(\tilde{B}_\tau(\xi))\big) \leqslant [1 + c(\varkappa\sigma + \Gamma\sigma^{1-n/p})]\,\omega_n^{-1}\sigma^{-n}\big(\mu(B_\sigma(\xi)) + \mu(\tilde{B}_\sigma(\xi))\big)$$

where $c = c(n, k, p)$ and $\xi = y$ or $\xi = z$.

We now combine (18) and (19) choosing $h(x) = f(|q(x-\xi)|)$, where $f \in C^1(\mathbb{R})$ with

$$f(t) \equiv 1 \quad \text{for} \quad |t| < l\beta R/4\,, \qquad f(t) \equiv 0 \quad \text{for} \quad |t| \geqslant l\beta R/2\,,$$
$$|f'(t)| \leqslant 5(l\beta R)^{-1} \quad \text{and} \quad 0 \leqslant f \leqslant 1\,.$$

Noting that for $\sigma < \varrho(\xi)$ we have $\mu(\tilde{B}_\sigma(\xi)) = 0$ and for $\sigma \geqslant \varrho(\xi)$ we have $\tilde{B}_\sigma(\xi) \subset B_{9\sigma}(\xi)$ (cf. section 2), we get for $\sigma \leqslant l\beta R/36$ and $\varrho \leqslant (1-\beta)R$, $\sigma < \varrho$

$$\begin{aligned}
\omega_n^{-1}\sigma^{-n}[\mu(B_\sigma(\xi)) + \mu(\tilde{B}_\sigma(\xi))]&\\
&\leqslant (1 + c\varkappa\varrho)\omega_n^{-1}\varrho^{-n}[\mu(B_\varrho(\xi) \cap \{x : |q(x-\xi)| < l\beta R/2\})\\
&+ \mu(\tilde{B}_\varrho(\xi) \cap \{x : |q(x-\xi)| < l\beta R/2\})]\\
&+ \omega_n^{-1} c\varrho\sigma^{-n}\Big[\int_{B_\varrho(\xi)}|\boldsymbol{H}|\,d\mu + \int_{\tilde{B}_\varrho(\xi)}|\boldsymbol{H}|\,d\mu\Big]\\
&+ \omega_n^{-1} c\varkappa\sigma^{1-n}[\mu(B_\varrho(\xi)) + \mu(\tilde{B}_\varrho(\xi))]\\
&+ \omega_n^{-1} c\varrho\sigma^{-n}(l\beta R)^{-1}\Big[\int_{B_\varrho(\xi)}\|p_x - p\|\,d\mu + \int_{\tilde{B}_\varrho(\xi)}\|p_x - p\|\,d\mu\Big]\,,
\end{aligned}$$

where we have used the fact that

$$|\nabla_j^M(q(x-\xi))| \leqslant |p_x \circ q| \equiv |(p_x - p)\circ q| \leqslant |p_x - p| \leqslant \sqrt{n+k}\;\|p_x - p\| \qquad \text{for } j = 1, \ldots, n+k\,.$$

Combining (**) with this estimate, letting $\tau \downarrow 0$ and using (14) of Remark 3.3 (ii) we get for $\sigma = l\beta R/36$ and $\varrho = (1-\beta)R$ after adding up

$$\begin{aligned}
\tilde{\theta}^n(\mu, y) + \tilde{\theta}^n(\mu, z) \leqslant [1 + c(\varkappa R + \Gamma R^{1-n/p})]&\\
&\cdot\Big\{(1 + ckR)(1-\beta)^{-n}\omega_n^{-1}R^{-n}[\mu(B_R(a)) + \mu(\tilde{B}_R(a))]\\
&+ \omega_n^{-1} c(l\beta)^{-n}\Gamma R^{1-n/p}[\mu(B_R(a)) + \mu(\tilde{B}_R(a))]\\
&+ \omega_n^{-1} c\varkappa(l\beta)^{-n} R R^{-n}[\mu(B_R(a)) + \mu(\tilde{B}_R(a))]\\
&+ c(l\beta)^{-n-1}R^{-n}\Big[\int_{B_R(a)}\|p_x - p\|\,d\mu + \int_{\tilde{B}_R(a)}\|p_x - p\|\,d\mu\Big]\Big\}\,.
\end{aligned}$$

Collecting terms we deduce the desired result. □

4. – The regularity theorem.

In addition to the assumptions made in section 3 we assume the following for the next theorem

$$(4.1)\quad \begin{cases} 0\in \operatorname{spt}\mu,\ R>0 \text{ s.t. } \operatorname{dist}\Big(\operatorname{spt}\mu\cap\big(B_R(0)\cap \tilde{B}_R(0)\big),\ \partial B_1(0)\Big)>0\,, \\ \Big(\omega_n^{-1}\int\limits_{B_1(0)}|\boldsymbol{H}|^p\,d\mu\Big)^{1/p}\leqslant \Gamma, \qquad \dfrac{\Gamma}{p-n}R^{1-n/p}\leqslant\dfrac{1}{2}, \qquad p>n\,, \\ \theta\geqslant 1,\ \mu\text{ - a.e.}, \quad \omega_n^{-1}R^{-n}\Big(\mu\big(B_R(0)\big)+\mu\big(\tilde{B}_R(0)\big)\Big)\leqslant 2(1-\alpha)\,, \end{cases}$$

where $a\in(0,1)$. Furthermore we define

$$E=R^{-n}\Big\{\int\limits_{B_R(0)}\|p_x-p\|^2\,d\mu+\int\limits_{\tilde{B}_R(0)}\|p_x-p\|^2\,d\mu\Big\}+\varkappa R+\Gamma^2R^{2(1-n/p)}\,.$$

4.2 Theorem (Lipschitz Approximation). *Suppose $\nu(0)\subset\mathbf{R}^n$. Under the assumptions of section 3 and of 4.1 there exists $\gamma=\gamma(n,\alpha,k,p)\in(0,\frac{1}{2})$ such that if $l\in(0,1]$ there exists $f=(f^1,\ldots,f^k):B^n_{\gamma R}(0)\to\mathbf{R}^k$ with*

$$(1)\qquad \operatorname{Lip} f\leqslant l\,,\qquad \sup|f|\leqslant cE^{1/2n+2}R\,,$$

$$(2)\qquad \mathcal{H}^n\Big(\big[(\operatorname{spt}\mu\sim\operatorname{graph} f)\cup(\operatorname{graph} f\sim\operatorname{spt}\mu)\big]\cap\big[B_{\gamma R}(0)\cap B_1'(0)\big]\Big)\leqslant cl^{-2n-2}ER^n\,,$$

where $c=c(n,\alpha,k,p)$.

4.3 Remark. As before $p=p_{\mathbf{R}^n}$, $q=p^\perp$, $p_x=p_{T_xM}$.

Proof. We first apply 3.6 with $a=y=0$ and $z\in B_{R/2}(0)\cap\operatorname{spt}\mu$, where we define $\beta=|z|/R$ and $h=|q(z)|/R$, so that we may take $l=h/\beta$ (we may assume β, $h\in(0,1)$). Using 4.1 and Cauchy-Schwarz we get

$$\begin{aligned}2&\leqslant 2\{1+ch^{-n}E^{\frac{1}{2}}\}(1-\beta)^{-n}(1-\alpha)+ch^{-n-1}E^{\frac{1}{2}}\\ &\leqslant 2(1-\alpha)(1-\beta)^{-n}+c\{[(1-\beta)^{-n}+1]\cdot(h^{-2n-2}E)^{\frac{1}{2}}\}\,.\end{aligned}$$

We now pick $\gamma_1=\gamma_1(n,\alpha)$ so that for $\beta\leqslant\gamma_1$ we have

$$2(1-\alpha)(1-\beta)^{-n}\leqslant 2\Big(1-\frac{\alpha}{2}\Big)\,,$$

which implies

$$2 \leqslant 2\left(1-\frac{\alpha}{2}\right)+c\{[(1-\gamma_1)^{-n}+1]\cdot(h^{-2n-2}E)^{\frac{1}{2}}\}\,.$$

Now we would get a contradiction if we had

$$Eh^{-2n-2} \leqslant \delta(c, \alpha, \gamma_1, n)$$

for some small δ. Thus for some c as in (1) we get

$$h \leqslant cE^{1/2n+2}\,,$$

which finally implies for any $z \in B_{\gamma_1 R}(0) \cap \operatorname{spt}\mu$

(3) $$|q(z)| \leqslant cE^{1/2n+2}R\,.$$

Let now δ, $l \in (0,1)$ be arbitrary. Then we may assume

(4) $$KR + \Gamma^2 R^{2(1-n/p)} \leqslant \delta l^{2n+2}\,,$$

because if $R^n E l^{-2n-2}$ is not small the claim follows trivially by setting $f \equiv 0$ and choosing c suitably, as long as $\delta = \delta(n, \alpha, k, p)$. Now let $\beta \in (0, \frac{1}{2})$ and consider the set

$$G = \left\{\xi \in \operatorname{spt}\mu \cap B_{\beta R/20}(0) : \sigma^{-n}\left\{\int_{B_\sigma(\xi)} \|p_x - p\|^2 d\mu + \int_{\tilde{B}_\sigma(\xi)} \|p_x - p\|^2 d\mu\right\} \leqslant \delta l^{2n+2},\ \forall \sigma \in (0, R/10)\right\}.$$

Next observe that by the monotonicity formula 3.4 and by 4.1 we have for any $\xi \in \operatorname{spt}\mu \cap B_{\beta R}(0)$ and any $0 < \sigma \leqslant (1-\beta)R$

$$\begin{aligned}\tfrac{1}{2} \leqslant \omega_n^{-1}\sigma^{-n}\big(\mu(B_\sigma(\xi)) + \mu(\tilde{B}_\sigma(\xi))\big)&\\ \leqslant (1+c\delta^{\frac{1}{2}})\,\omega_n^{-1}\Big[(1-\beta)^{-n}R^{-n}\big(\mu(B_{(1-\beta)R}(\xi)) + \mu(\tilde{B}_{(1-\beta)R}(\xi))\big)\Big]&\\ \leqslant (1+c\delta^{\frac{1}{2}})(1-\beta)^{-n}\omega_n^{-1}R^{-n}\big(\mu(B_R(0)) + \mu(\tilde{B}_R(0))\big)&\\ \leqslant 2(1+c\delta^{\frac{1}{2}})(1-\beta)^{-n}(1-\alpha) \leqslant 2\left(1-\frac{\alpha}{2}\right),&\end{aligned}$$

if δ, β are small enough. Note that the appropriate choice of δ indeed depends only on n, α, k, p by 4.1.

Now let y, $z \in G$. We again apply 3.6 with $a = y$ and $R' = \beta^{-1}|y - z|$ instead of R and $l/(1+l)$ instead of l, and conclude (note that $R' \leqslant R/10$

$\leqslant (1-\beta)R)$ for β, δ small enough, using an easy argument by contradiction,

(6) $$|q(y-z)| \leqslant \frac{l}{l+1}\,|y-z|$$

or

(7) $$|q(y)-q(z)| \leqslant l|p(y)-p(z)|\,.$$

By Kirszbraun's Theorem we get a function $f: B^n_{\beta R/20}(0) \to \mathbb{R}^k$ with $\operatorname{Lip} f \leqslant l$ such that

(8) $$G \subset \operatorname{graph} f\,.$$

Thus, taking (3) into account, we get (1) for any $\gamma \leqslant \min\{\gamma_1, \beta/20\}$ and it remains to prove (2).

We now want to control $\operatorname{spt}\mu \sim G$. The idea is clear: Since on the complement of G,

$$\sigma^{-n}\left\{\int_{B_\sigma(\xi)} \|p_x - p\|^2\,d\mu + \int_{\tilde{B}_\sigma(\xi)} \|p_x - p\|^2\,d\mu\right\}$$

is large, this set has to be small, or, more precisely, its measure is controlled by E.

By the definition of G we have for each $\xi \in \operatorname{spt}\mu \cap B_{\beta R/20}(0) \sim G$ a radius $\sigma(\xi) \in (0, R/10)$ such that

$$\sigma(\xi)^n \leqslant \delta^{-1} l^{-2n-2}\left\{\int_{B_{\sigma(\xi)}(\xi)} \|p_x - p\|^2\,d\mu + \int_{\tilde{B}_{\sigma(\xi)}(\xi)} \|p_x - p\|^2\,d\mu\right\}$$

and by (5) we therefore get (note that $5\sigma(\xi) < R/2 < (1-\beta)R$)

(9) $$\mu(\bar{B}_{5\sigma(\xi)}(\xi)) \leqslant c\delta^{-1} l^{-2n-2}\left\{\int_{B_{\sigma(\xi)}(\xi)} \|p_x - p\|^2\,d\mu + \int_{\tilde{B}_{\sigma(\xi)}(\xi)} \|p_x - p\|^2\,d\mu\right\}.$$

If however $\xi \in (B_{\beta R/20}(0) \sim G) \sim \operatorname{spt}\mu$, then (9) is trivially true for some small $\sigma(\xi)$.

By a well known covering argument we can now select points $\xi_1, \xi_2, \ldots \in B_{\beta R/20}(0) \sim G$ such that $\{B_{\sigma(\xi_j)}(\xi_j)\}$ is disjoint and $\{\bar{B}_{5\sigma(\xi_j)}(\xi_j)\}$ still covers $B_{\beta R/20}(0) \sim G$.

Summing over j we get

(10) $$\mu(B_{\beta R/20}(0) \sim G) \leqslant c\delta^{-1} l^{-2n-2}\left\{\int_{B_R(0)} \|p_x - p\|^2\,d\mu + \int_{\tilde{B}_R(0)} \|p_x - p\|^2\,d\mu\right\}.$$

Since $\theta^n(\mu, \xi) \geqslant \frac{1}{2}$ for $\xi \in \operatorname{spt}\mu \cap B_1(0)$ we get using (8) and incorporating δ^{-1} into the constant c

$$\mathcal{H}^n\big((\operatorname{spt}\mu \sim \operatorname{graph} f) \cap B_{\beta R/20}(0)\big) \leqslant c l^{-2n-2} R^n E\,.$$

To estimate the measure of the remaining set we take any

$$\eta \in (\operatorname{graph} f \sim \operatorname{spt}\mu) \cap B_{\beta R/100}(0) \cap \overline{B_1'(0)}$$

and pick $\sigma \in (0, \beta R/80)$ such that $B_{9\sigma/11}(\eta) \cap \operatorname{spt}\mu = \emptyset$ and $B_{10\sigma/11}(\eta) \cap \operatorname{spt}\mu \neq \emptyset$ (note that $0 \in \operatorname{spt}\mu$!). This implies $\mu(B_{\sigma/11}(\eta)) + \mu(\tilde{B}_{\sigma/11}(\eta)) = 0$ and 3.4 in connection with 3.5 (ii) yields ($F = \operatorname{graph} f$)

$$\begin{aligned}
\mu(B_\sigma(\eta)) + \mu(\tilde{B}_\sigma(\eta)) &\leqslant c\sigma^n\Bigg\{\int\limits_{B_\sigma(\eta)\sim B_{\sigma/11}(\eta)} |x-\eta|^{-n}\left|p_x^\perp\left(\frac{x-\eta}{|x-\eta|}\right)\right|^2 d\mu \\
&\quad + \int\limits_{\tilde{B}_\sigma(\eta)\sim \tilde{B}_{\sigma/11}(\eta)} |\tilde{x}-\eta|^{-n}\left|p_x^\perp\left(i_{\tilde{x}}\left(\frac{\tilde{x}-\eta}{|\tilde{x}-\eta|}\right)\right)\right|^2 d\mu\Bigg\} \\
&\leqslant c\Bigg\{\int\limits_{B_\sigma(\eta)}\left|p_x^\perp\left(\frac{x-\eta}{\sigma}\right)\right|^2 d\mu + \int\limits_{\tilde{B}_\sigma(\eta)}\left|p_x^\perp\left(i_{\tilde{x}}\left(\frac{\tilde{x}-\eta}{\sigma}\right)\right)\right|^2 d\mu\Bigg\} \\
&\leqslant c\Bigg\{\int\limits_{B_\sigma(\eta)}\left|q\left(\frac{x-\eta}{\sigma}\right)\right|^2 d\mu + \int\limits_{B_\sigma(\eta)}\|p_x - p\|^2\, d\mu \\
&\quad + \int\limits_{\tilde{B}_\sigma(\eta)}\left|q\left(i_{\tilde{x}}\left(\frac{\tilde{x}-\eta}{\sigma}\right)\right)\right|^2 d\mu + \int\limits_{\tilde{B}_\sigma(\eta)}\|p_x - p\|^2\, d\mu\Bigg\} \\
&\leqslant c\Bigg\{\int\limits_{B_\sigma(\eta)\cap F}\left|q\left(\frac{x-\eta}{\sigma}\right)\right|^2 d\mu + \mu(B_\sigma(\eta) \sim F) \\
&\quad + \int\limits_{\tilde{B}_\sigma(\eta)\cap F}\left|q\left(i_{\tilde{x}}\left(\frac{\tilde{x}-\eta}{\sigma}\right)\right)\right|^2 d\mu + \mu(\tilde{B}_\sigma(\eta) \sim F) \\
&\quad + \int\limits_{B_\sigma(\eta)}\|p_x - p\|^2\, d\mu + \int\limits_{\tilde{B}_\sigma(\eta)}\|p_x - p\|^2\, d\mu\Bigg\}.
\end{aligned}$$

Now, using the fact that $\nu(0) \subset \mathbb{R}^n$ and that $\|i_{\tilde{x}} - i_0\| \leqslant c\varkappa R$, $\operatorname{Lip} f \leqslant l$ we get

$$\begin{aligned}
\mu(B_\sigma(\eta)) + \mu(\tilde{B}_\sigma(\eta)) &\leqslant c\Big\{l^2\mu(B_\sigma(\eta)) + (l^2 + \varkappa^2R^2)\mu(\tilde{B}_\sigma(\eta)) \\
&\quad + \mu(B_\sigma(\eta) \sim F) + \mu(\tilde{B}_\sigma(\eta) \sim F)\int\limits_{B_\sigma(\eta)}\|p_x - p\|^2\, d\mu + \int\limits_{\tilde{B}_\sigma(\eta)}\|p_x - p\|^2\, d\mu\,.
\end{aligned}$$

Here we used the fact that

$$q\big(i_{\tilde{x}}(\tilde{x}-\eta)\big)=q(\tilde{x}-\eta)+q(i_{\tilde{x}}-i_0)(\tilde{x}-\eta)$$

as well as

$$q(\tilde{x}-\eta)=q(x-\eta)+q(\tilde{x}-x)=q(x-\eta)+2q\circ\big(\nu(0)-\nu(\xi(x))\big)\big(x-\xi(x)\big)\,.$$

By the properties listed in section 2 we have $\tilde{B}_\sigma(\eta)\subset B_{9\sigma}(\eta)$, $\|\nu(0)-\nu(\xi(x))\|\leqslant c\varkappa R$, and the above inequality follows easily.

Since we can assume $c(l^2+\varkappa^2R^2)\leqslant\frac{1}{2}$ we may conclude

$$\text{(11)}\quad \mu(B_\sigma(\eta))+\mu(\tilde{B}_\sigma(\eta))\leqslant c\Big\{\mu(B_\sigma(\eta)\sim F)+\mu(\tilde{B}_\sigma(\eta)\sim F)+ +\int_{B_\sigma(\eta)}\|p_x-p\|^2\,d\mu+\int_{\tilde{B}_\sigma(\eta)}\|p_x-p\|^2\,d\mu\Big\}\,.$$

Now the condition $B_{10\sigma/11}(\eta)\cap\operatorname{spt}\mu\neq\emptyset$ together with (5) ensures that

$$\text{(12)}\qquad \mu(B_\sigma(\eta))+\mu(\tilde{B}_\sigma(\eta))\geqslant c\sigma^n\,.$$

Writing $\eta'=p(\eta)$ we see that (11) and (12) imply

$$\begin{aligned}\mathfrak{L}^n(B^n_{5\sigma}(\eta'))\leqslant c\Big\{&\mu\big((B^n_\sigma(\eta')\times\mathbf{R}^k)\cap B_{\beta R/40}(0)\sim F\big)\\&+\mu\big((p(\tilde{B}_\sigma(\eta))\times\mathbf{R}^k)\cap\tilde{B}_{\beta R/40}(0)\sim F\big)\\&+\int_{(B^n_\sigma(\eta')\times\mathbf{R}^k)\cap B_{\beta R/40}(0)}\|p_x-p\|^2\,d\mu+\int_{(p(\tilde{B}_\sigma(\eta))\times\mathbf{R}^k)\cap\tilde{B}_{\beta R/40}(0)}\|p_x-p\|^2\,d\mu\Big\}\,.\end{aligned}$$

This is true for any $\eta\in(F\sim\operatorname{spt}\mu)\cap B_{\beta R/100}(0)\cap B_1'(0)$ and the covering lemma gives

$$\begin{aligned}&\mathfrak{L}^n\big(p((F\sim\operatorname{spt}\mu)\cap B_{\beta R/100}\cap\overline{B_1'(0)})\big)\\&\qquad\leqslant c\Big\{\mu(B_{\beta R/40}(0)\sim F)+\mu(\tilde{B}_{\beta R/40}(0)\sim F)\\&\qquad+\int_{B_{\beta R/40}(0)}\|p_x-p\|^2\,d\mu+\int_{\tilde{B}_{\beta R/40}(0)}\|p_x-p\|^2\,d\mu\Big\}\leqslant cl^{-2n-2}ER^n\,.\end{aligned}$$

Here (10) was used together with $\tilde{B}_{\beta R/40}(0)\sim F\subset B_{\beta R/20}(0)\sim G$. Since $\operatorname{Lip}f\leqslant1$ the theorem is proved with $\gamma=\beta/100$. □

4.4 DEFINITION. The *tilt-excess* $E(\xi, \varrho, T)$ is defined as usual by

$$E(\xi, \varrho, T) = \varrho^{-n} \int_{B_\varrho(\xi)} |p_{T_xM} - p_T|^2 \, d\mu_V ,$$

where $V = \mathbf{v}(M, \theta)$ is a rectifiable n-varifold, $\varrho > 0$, $\xi \in \mathbb{R}^{n+k}$ and T an n-dimensional subspace of $\mathbb{R}^{n+k}$.

It measure the mean L^2-deviation of the tangent planes of M from some fixed plane.

We have the following lemma concerning tilt-excess and height.

4.5 LEMMA. *Under the general assumptions of section 3 we have: If $\xi \in B_1(0)$, $\varrho < 1 - |\xi|$ and T is an n-dimensional subspace of $\mathbb{R}^{n+k}$ such that $\nu(0) \subset T$, then*

$$E(\xi, \varrho/2, T) \leqslant c\Big\{\varrho^{-n-2} \int_{B_\varrho(\xi)} \operatorname{dist}(x - \xi, T)^2 \, d\mu + \varrho^{2-n} \int_{B_\varrho(\xi)} |\boldsymbol{H}|^2 \, d\mu + \varkappa^2 \varrho^{2-n} \mu\big(B_\varrho(\xi)\big)\big(1 + \varrho^{-2}|\xi|^2\big)\Big\} ,$$

for some constant $c = c(n, k)$.

PROOF. We may assume $T = \mathbb{R}^n$ and consider the vector-field $\big(x' = (0, x^{n+1}, \ldots, x^{n+k})\big)$

$$X(x) = \zeta^2(x)\, \tau\big(\xi(x)\big)(x' - \xi') ,$$

where $\zeta \in C_c^\infty\big(B_1(0)\big)$ with $0 \leqslant \zeta \leqslant 1$, $\zeta \equiv 1$ in $B_{\varrho/2}(\xi)$, $\operatorname{spt} \zeta \subset B_\varrho(\xi)$ and $|\operatorname{grad} \zeta| \leqslant 4/\varrho$.

As $\xi(b) = b$ for $b \in B$ we see that X is admissible. By the definition of div_M and because of $\mathbb{R}^k \subset \tau(0)$ we get for μ - a.e. $x \in M$

$$\begin{aligned} \operatorname{div}_M \Big(\tau\big(\xi(x)\big)(x' - \xi')\Big) &= \operatorname{div}_M (x' - \xi') + \operatorname{div}_M \Big[\Big(\tau\big(\xi(x)\big) - \tau(0)\Big)(x' - \xi')\Big] \\ &= \sum_{i=n+1}^{n+k} e^{ii} + \sum_{i,j,l=1}^{n+k} \sum_{m=n+1}^{n+k} e^{jl} D_i \tau_{jm}\big(\xi(x)\big) D_l \xi^i(x)(x - \xi)^m \\ &+ \sum_{j=1}^{n+k} \sum_{l=m+1}^{n+k} \Big(\tau_{jl}\big(\xi(x)\big) - \tau_{jl}(0)\Big) e^{jl} . \end{aligned}$$

Here and in the following (e^{ij}) and (τ_{ij}) are the components of the matrices of p_{T_xM} and τ.

Now we observe that ((ε^{ij}) being the matrix of $p_{\mathbf{R}^n}$)

$$\sum_{i=n+1}^{n+k} e^{ii} = \tfrac{1}{2}\sum_{j,i=1}^{n+k}(e^{ij}-\varepsilon^{ij})^2 = \tfrac{1}{2}|p_{T_xM}-p_{\mathbf{R}^n}|^2 ,$$

so that by (7) of section 3 we have

$$\frac{1}{2}\int |p_x - p|^2\zeta^2\, d\mu = -\int\Big(\zeta^2\tau(\xi(x))\,(x'-\xi')\cdot \boldsymbol{H} + 2\zeta\sum_{i=n+1}^{n+k}\sum_{j=1}^{n+k} e^{ij}(x-\xi)^i D_j\zeta + 2\zeta R_1 + \zeta^2 R_2\Big)\, d\mu ,$$

where

$$R_1 = \sum_{i=n+1}^{n+k}\sum_{j,l=1}^{n+k} e^{jl} D_l\zeta\big(\tau_{ji}(\xi(x)) - \tau_{ji}(0)\big)(x-\xi)^i$$

and

$$R_2 = \sum_{i=n+1}^{n+k}\sum_{j=1}^{n+k}\big(\tau_{ji}(\xi(x)) - \tau_{ji}(0)\big)e^{ji} + \sum_{i=n+1}^{n+k}\sum_{j,l,m=1}^{n+k} e^{jl} D_m\tau_{ji}(\xi(x))\, D_l\xi^m(x)(x-\xi)^i .$$

Using the estimates for $D\tau$ from section 2 we get (13). □

The following « Tilt-excess-decay-Theorem » is the main step in the proof of the regularity theorem.

We consider the following assumptions

$$(4.6)\qquad \begin{cases} 1\leqslant\theta\leqslant 1+\varepsilon , & \mu\text{ - a.e. in } B_1(0) ,\\ \xi\in \operatorname{spt}\mu\cap B , & \overline{B_{3\varrho}}(\xi)\subset B_1(0) ,\\ \omega_n^{-1}\varrho^{-n}[\mu(B_\varrho(\xi)) + \mu(\tilde{B}_\varrho(\xi))]\leqslant 2(1-\alpha) , & \alpha > 0 ,\\ E_*(\xi,\varrho,T)\leqslant\varepsilon , \end{cases}$$

where $\nu(\xi)\subset T$, T an n-dimensional subspace of $\mathbf{R}^{n+k}$, and where $E_*(\xi,\varrho,T)$ is defined by

$$E_*(\xi,\varrho,T) = \max\Big\{E(\xi,\varrho,T),\ \varepsilon^{-1}\Big(\int_{B_\varrho(\xi)}|\boldsymbol{H}|^p\, d\mu\Big)^{2/p}\varrho^{2(1-n/p)},\ \varkappa\varrho\Big\} .$$

4.7 THEOREM. *For any $\alpha \in (0, 1)$, $p > n$ there are constants η, $\varepsilon \in (0, \frac{1}{2})$ depending only on n, p, k, α such that the assumptions* (4.6) *imply*

$$E_*(\xi, \eta\varrho, S) \leqslant \eta^{2\gamma} E_*(\xi, \varrho, T),$$

for some n-dimensional subspace $S \subset \mathbb{R}^{n+k}$ with $\nu(\xi) \subset S$, where

$$\gamma = \min\{\tfrac{1}{2}, (1 - n/p)\}.$$

4.8. REMARKS:

(i) Such S automatically satisfies

$$|p_S - p_T|^2 \leqslant c(\eta) E_*(\xi, \varrho, T),$$

as one sees from 4.7 and 3.3 (ii), using

$$(\eta\varrho)^{-n} \int_{B_{\eta\varrho}(\xi)} |p_{T_xM} - p_T|^2 \, d\mu \leqslant \eta^{-n} E(\xi, \varrho, T).$$

(ii) The condition $\theta \leqslant 1 + \varepsilon$ can probably be dropped. This was recently shown in the case of interior regularity by J. Duggan in his thesis [DJ].

PROOF OF 4.7. We may of course assume $\xi = 0$ and $T = \mathbb{R}^n$. By the Lipschitz-Approximation-Theorem 4.2 there is $\beta = \beta(n, k, \alpha, p) > 0$, and a function $f: B^n_{\beta\varrho}(0) \to \mathbb{R}^k$ satisfying

$$\operatorname{Lip} f \leqslant 1, \sup_{B^n_{\beta\varrho}(0)} |f| \leqslant c E_*^{1/(2n+2)} \varrho \leqslant c \varepsilon^{1/(2n+2)} \varrho, \tag{13}$$

and

$$\mathcal{H}^n\Big(\big((\operatorname{spt}\mu \sim \operatorname{graph} f) \cup (\operatorname{graph} f \sim \operatorname{spt}\mu)\big) \cap \big(B_{\beta\varrho}(0) \cap \overline{B_1'(0)}\big)\Big) \leqslant c E_* \varrho^n, \tag{14}$$

where $E_* = E_*(0, \varrho, \mathbb{R}^n)$.

The scheme of the proof now is as follows:

We let $H^n_r(0)$ denote the half ball $B^n_r(0) \cap \{x : x \cdot \nu(0) > 0\}$ where $\nu(0)$ is chosen to point into $B_1'(0)$. E^* then controls the integrated difference between $\nabla^M f^j$ and $\operatorname{grad} f^j$, and we derive

$$\varrho^{-n} \int_{H^n_\varrho(0)} |\operatorname{grad} f^j|^2 \leqslant c E^*$$

and that f^j is close to a harmonic function in the sense that for a test function ζ_1

$$\Big|\varrho^{-n}\int\limits_{H^n_{\beta\varrho}(0)} \operatorname{grad} f^j\cdot\operatorname{grad}\zeta_1\Big|\leqslant c\varepsilon^{\frac{1}{2}}E_*^{\frac{1}{2}}|\operatorname{grad}\zeta_1|_\infty$$

(cf. (28) and (27)). Therefore, f^j can be approximated by a harmonic function u^j with vanishing normal derivative on $\{x : x\cdot\nu(0) = 0\}$ the mean Dirichlet integral of which is likewise controlled by E_*. Standard estimates for harmonic functions then imply that on smaller half balls u^j is close to a constant. This in turn yields enough information about f to prove the claim. Note that this is the old device of De Giorgi.

The details are as follows. Because of the height estimate (3) in the proof of 4.2 we get for $j = n+1,\dots, n+k$

$$\sup_{B_{\beta\varrho}(0)\cap\operatorname{spt}\mu} |x^j|\leqslant c\varepsilon^{1/(2n+2)}\varrho\,, \tag{15}$$

so that by assuming $c\varepsilon^{1/(2n+2)}\leqslant\beta/4$ we have

$$\sup_{B_{\beta\varrho}(0)\cap\operatorname{spt}\mu} |x^j|\leqslant\beta\varrho/4\,. \tag{16}$$

This implies

$$\big(\mathbf{R}^k\times B^n_{\beta\varrho/2}(0)\big)\cap\operatorname{spt}\mu\cap\partial B_{\beta\varrho}(0)=\emptyset\,. \tag{17}$$

Now let $\zeta_1\in C^1_0\big(B^n_{\beta\varrho/2}(0)\big)$ and note that $\tilde\zeta_1(x^1,\dots,x^{n+k}) := \zeta_1(x^1,\dots,x^n)$ agrees with a function $\zeta\in C^1_0\big(B_{\beta\varrho}(0)\big)$ in a neighbourhood of $(B^n_{\beta\varrho/2}(0)\times\mathbf{R}^k)$ $\cap\operatorname{spt}\mu\cap B_{\beta\varrho}(0)$.

Consider the vector-fields

$$X(x) = \zeta(x)\,e_{n+j} \quad\text{ and }\quad \tilde X(x) = \zeta(x)\,\tau\big(\xi(x)\big)\,e_{n+j}.$$

We get

$$\int\limits_M \nabla^M_{n+j}\zeta\,d\mu = \int\limits_M \operatorname{div}_M\tilde X\,d\mu + D_j(\zeta) = -\int\limits_M \boldsymbol{H}\cdot\tilde X\,d\mu + D_j(\zeta)\,,$$

where $D_j(\zeta) = \int\limits_M \operatorname{div}_M(X-\tilde X)\,d\mu$.

Since $\nabla^M_{n+j} = e_{n+j}\cdot\nabla^M = p_x(e_{n+j})\cdot\nabla^M = (\nabla^M x_{n+j})\cdot\nabla^M$ this can be written as

$$\int\limits_M (\nabla^M x_{n+j})\cdot\nabla^M\zeta\,d\mu = -\int\limits_M \boldsymbol{H}\cdot\tilde X\,d\mu + D_j(\zeta)\,.$$

Now μ - a.e. on $\tilde{M} = M \cap \operatorname{graph} f$ we have $x^{n+j} \equiv \tilde{f}^j(x)$, where $\tilde{f}^j(x_1, \ldots, x_{n+k}) = f^j(x_1, \ldots, x_n)$, and therefore

$$\nabla^M x_{n+j} = \nabla^M \tilde{f}^j(x)\,, \quad \mu\text{ - a.a. } x \in \tilde{M}\,. \tag{18}$$

Thus we get

$$\int_{\tilde{M}} (\nabla^M \tilde{f}^j)\cdot\nabla^M \zeta \, d\mu = -\int_{M\sim\tilde{M}} (\nabla^M x^{n+j})\cdot \nabla^M \zeta \, d\mu - \int_M \boldsymbol{H}\cdot\tilde{X}\, d\mu + D_j(\zeta)\,.$$

The terms on the right hand side are now estimated as follows. By (14) we get

$$\left|\varrho^{-n}\int_{M\sim\tilde{M}} (\nabla^M x^{n+j})\cdot\nabla^M\zeta\, d\mu\right| \leqslant c \sup |\operatorname{grad}\zeta| E_* \leqslant c \sup |\operatorname{grad}\zeta| \varepsilon^{\frac{1}{2}} E_*^{\frac{1}{2}}$$

and 4.6 implies

$$\varrho^{-n}\left|\int_M \boldsymbol{H}\cdot\tilde{X}\, d\mu\right| \leqslant \sup|\zeta|\varrho^{-n}\, \mu(B_\varrho(0))^{1-1/p}\left(\int_{B_\varrho(0)} |\boldsymbol{H}|^p d\mu\right)^{1/p} \leqslant c \sup|\zeta|\varrho^{-1}\varepsilon^{\frac{1}{2}} E_*^{\frac{1}{2}}\,.$$

In order to estimate $D_j(\zeta)$ we observe that

$$|\operatorname{div}_M(X - \tilde{X})| \leqslant c\varkappa\varrho(\sup|\operatorname{grad}\zeta| + \varrho^{-1}\sup|\zeta|)\,,$$

because of $e_{n+j} = \tau(0)\, e_{n+j}$ and the usual estimates for $D\tau$. Combining these estimates we arrive at

$$\varrho^{-n}\left|\int_{\tilde{M}} (\nabla^M \tilde{f}^j)\cdot\nabla^M \zeta_1\, d\mu\right| \leqslant c\varepsilon^{\frac{1}{2}} E_*^{\frac{1}{2}} \sup|\operatorname{grad}\zeta_1|\,. \tag{19}$$

Since $|p_x - p|^2 = 2\sum_{j=1}^{k} |\nabla^M x^{n+j}|^2$ we see that (18) yields

$$\varrho^{-n}\int_{\tilde{M}\cap B_{\beta\varrho}(0)} |\nabla^M \tilde{f}^j|^2 d\mu \leqslant cE_*\,. \tag{20}$$

Using $p_{\mathbf{R}^n}\operatorname{grad}\tilde{\zeta}_1 = \operatorname{grad}\tilde{\zeta}_1$ and $p_{\mathbf{R}^n}\operatorname{grad}\tilde{f}^j = \operatorname{grad}\tilde{f}^j$ we get

$$|\nabla^M \tilde{f}^j\cdot\nabla^M\tilde{\zeta}_1 - \operatorname{grad}\tilde{f}^j\cdot\operatorname{grad}\tilde{\zeta}_1| \leqslant |p_x - p|^2|\operatorname{grad}\tilde{f}_j|\,|\operatorname{grad}\tilde{\zeta}_1|$$

and hence (19) and (13) imply

$$(21)\qquad \left|\varrho^{-n}\int_{\tilde M}\operatorname{grad}\tilde f^j\cdot\operatorname{grad}\tilde\zeta_1\,d\mu\right|\leqslant c\varepsilon^{\frac12}E_*^{\frac12}\sup|\operatorname{grad}\zeta_1|.$$

By the same argument we deduce from (20)

$$(22)\qquad \varrho^{-n}\int_{\tilde M\cap B_{\beta\varrho}(0)}|\operatorname{grad}\tilde f_j|^2\,d\mu\leqslant cE_*.$$

By the area formula we see that (21) and (22) imply in view of (13) and (14) $\left(B^{n'}_{\beta\varrho}(0):=p(\operatorname{graph}f\cap B_{\beta\varrho}(0)\cap\overline{B_1'(0)})\right)$

$$(23)\qquad \left|\varrho^{-n}\int_{B^{n'}_{\beta\varrho}(0)}\operatorname{grad}f^j\cdot\operatorname{grad}\zeta_1\theta\circ FJ(F)\,d\mathfrak{L}^n\right|\leqslant c\varepsilon^{\frac12}E_*^{\frac12}\sup|\operatorname{grad}\zeta_1|$$

as well as

$$(24)\qquad \varrho^{-n}\int_{B^{n'}_{\beta\varrho}(0)}|\operatorname{grad}f^j|^2\theta\circ FJ(F)\,d\mathfrak{L}^n\leqslant cE_*,$$

where $F(x)=(x,f(x))$, $x\in B^n_{\beta\varrho}(0)$ and $J(F)=\left[\det\left((dF_x)^*\circ dF\right)\right]^{\frac12}$. Since $1\leqslant J(F)\leqslant 1+c|\operatorname{grad}f|^2$ and $1\leqslant\theta\leqslant 1+\varepsilon$ we conclude

$$(25)\qquad \varrho^{-n}\int_{B^{n'}_{\beta\varrho}(0)}\operatorname{grad}f^j\cdot\operatorname{grad}\zeta_1\,d\mathfrak{L}^n\Bigg|$$
$$\leqslant c\left\{\varepsilon^{\frac12}E_*^{\frac12}+E_*+\varepsilon\int_{B^{n'}_{\beta\varrho}(0)}|\operatorname{grad}f^j|\,d\mathfrak{L}^n\right\}\sup|\operatorname{grad}\zeta_1|\leqslant c\varepsilon^{\frac12}E_*^{\frac12}\sup|\operatorname{grad}\zeta_1|,$$

and

$$(26)\qquad \varrho^{-n}\int_{B^{n'}_{\beta\varrho}(0)}|\operatorname{grad}f^j|^2\,d\mathfrak{L}^n\leqslant cE_*.$$

Now let $\nu(0)=\{\lambda e:\lambda\in\mathbb{R},\ |e|=1,\ e\in\mathbb{R}^n\}$, where e is chosen in such a way that it points into $B_1'(0)$, and define the half-ball $H^n_{\beta\varrho}(0):=B^n_{\beta\varrho}(0)\cap\{x:x\cdot e>0\}$. For $y\in B\cap B_{\beta\varrho}(0)$ we get

$$\operatorname{dist}\left(y,\tau(0)\right)=|\nu(0)(y-0)|\leqslant\frac{\varkappa}{2}|y|^2\leqslant\frac{\varkappa}{2}\beta^2\varrho^2$$

so that

$$\operatorname{dist}\left(B,\tau(0)\right)\leqslant\varkappa\beta^2\varrho^2/2.$$

This yields

$$B^{n'}_{\beta\varrho}(0) \sim H^n_{\beta\varrho}(0) \cap H^n_{\beta\varrho}(0) \sim B^{n'}_{\beta\varrho}(0) \subset B^n_{\beta\varrho}(0) \cap \left\{x\colon |\nu(0)\,x| \leqslant \frac{\varkappa}{2}\beta^2\varrho^2\right\}$$

which implies

$$\text{(26a)} \qquad \mathfrak{L}^n\big(B^{n'}_{\beta\varrho}(0) \sim H^n_{\beta\varrho}(0) \cup H^n_{\beta\varrho}(0) \sim B^{n'}_{\beta\varrho}(0)\big) \leqslant c\varkappa(\beta\varrho)^{n+1}\,.$$

Thus (25) and (26) respectively imply

$$\text{(27)} \qquad \Bigg|\varrho^{-n} \int\limits_{H^n_{\beta\varrho}(0)} \operatorname{grad} f^j \cdot \operatorname{grad} \zeta_1 \, d\mathfrak{L}^n\Bigg| \leqslant c\varepsilon^{\frac{1}{2}} E_*^{\frac{1}{2}} \sup |\operatorname{grad} \zeta_1|$$

and

$$\text{(28)} \qquad \varrho^{-n} \int\limits_{H^n_{\beta\varrho}(0)} |\operatorname{grad} f^j|^2 \, d\mathfrak{L}^n \leqslant cE_*\,.$$

Applying Lemma 5.1 of the appendix to $f^j E_*^{-\frac{1}{2}}$ we see that for any given $\delta > 0$ there is $\varepsilon_0 = \varepsilon_0(n, k, \delta)$ such that if 4.6 holds with $\varepsilon \leqslant \varepsilon_0$ there are harmonic functions $u^1, \dots, u^k : H^n_{\beta\varrho}(0) \to \mathbb{R}$ with the properties

$$\text{(29)} \qquad \sigma^{-n} \int\limits_{H^n_\sigma(0)} |\operatorname{grad} u^j|^2 \, d\mathfrak{L}^n \leqslant cE_*\,,$$

$$\text{(30)} \qquad \sigma^{-n-2} \int\limits_{H^n_\sigma(0)} |f^j - u^j|^2 \, d\mathfrak{L}^n \leqslant \delta E_*$$

and

$$\text{(31)} \qquad \partial_{\nu(0)} u^j = 0 \qquad \text{on } \overline{H^n_\sigma(0)} \cap \{x\colon x\cdot e = 0\}\,,$$

where $\sigma = \beta\varrho$ and $\partial_{\nu(0)}$ denotes the normal derivative. Defining l^j by $l^j(x) = u^j(0) + x\cdot\operatorname{grad} u^j(0)$ we note that (29), (31) imply for $\eta \in (0, \frac{1}{2}]$, using standard estimates for harmonic functions,

$$\text{(32)} \qquad \sup_{H^n_{\eta\sigma}(0)} |u^j - l^j|^2 \leqslant c\eta^4\sigma^2 E_*\,.$$

Thus we get by (30)

$$\text{(33)} \qquad (\eta\sigma)^{-n-2} \int\limits_{H^n_{\eta\sigma}(0)} |f^j - l^j|^2 \, d\mathfrak{L}^n \leqslant 2\eta^{-n-2}\delta E_* + c\eta^2 E_*\,.$$

Since $\sup|f|\leqslant c\varepsilon^{1/(2n+2)}\sigma$, (30) implies

$$\sum_{j=1}^{k}|l^j(0)|\leqslant c\varepsilon^{1/(2n+2)}\sigma\,. \tag{34}$$

Now let $l=(l^1,\ldots,l^k):\mathbb{R}^n\to\mathbb{R}^k$ and let S be the n-dimensional subspace graph $\big(l-l(0)\big)$. Note that (31) implies $\nu(0)\subset S$.

Note furthermore that we also get by (26a)

$$(\eta\sigma)^{-n-2}\int\limits_{B^{n'}_{\eta\sigma}(0)}|f^j-l^j|^2\,d\mathfrak{L}^n\leqslant c\eta^{-n-2}\,\delta E_*+c\eta^2E_*\,, \tag{35}$$

if we assume

$$c\varepsilon^{1/(n+1)}\leqslant\delta\,. \tag{36}$$

But (35) now implies

$$(\eta\sigma)^{-n-2}\int\limits_{B_{\eta\sigma}(\tau)}\operatorname{dist}(x-\tau,S)^2\,d\mu\leqslant c\eta^{-n-2}\,\delta E_*+c\eta^2E_*\,, \tag{37}$$

where $\tau=\big(0,l(0)\big)$.

If we assume $c\varepsilon^{1/(2n+2)}<\eta/4$ we get by (34) that $B_{\eta\sigma/4}(0)\subset B_{\eta\sigma/2}(\tau)$, so that (4.5) and (37) imply

$$\begin{aligned}E(0,\eta\sigma/4,S)&\leqslant 2^nE(\tau,\eta\sigma/2,S)\\&\leqslant c\eta^{-n-2}\,\delta E_*+c\eta^2E_*+c\eta^{2(1-n/p)}\varepsilon E_*+cE_*^2\,.\end{aligned} \tag{38}$$

To complete the proof of 4.7 we argue as follows.

Let $\eta\in(0,\frac12]$ such that $c\eta^2\leqslant\frac14(\eta\beta/4)^{2(1-n/p)}$ and $\delta>0$ such that $c\eta^{-n-2}\delta<\frac12(\eta\beta/4)^{2(1-n/p)}$ (in both cases c is the constant from (38)). Finally choose ε_0 such that for $\varepsilon\leqslant\varepsilon_0$ all the conditions required in the proof hold as well as $c\varepsilon_0\leqslant\frac14(\eta\beta/4)^{2(1-n/p)}$ (in particular ε_0 must satisfy the conditions leading to (16), (29), (30), (36), (37) and (38)). Thus we get for $\tilde\eta=\eta\beta/4$

$$E(0,\tilde\eta\varrho,S)\leqslant\tilde\eta^{2(1-n/p)}E_*\,.$$

Since we trivially have

$$\Big(\int\limits_{B_{\tilde\eta\varrho}(0)}|\boldsymbol{H}|^p\,d\mu\Big)^{1/p}(\tilde\eta\varrho)^{1-n/p}\leqslant\tilde\eta^{1-n/p}\Big(\int\limits_{B_\varrho(0)}|\boldsymbol{H}|^p\,d\mu\Big)^{1/p}\varrho^{1-n/p}$$

as well as $\varkappa\tilde{\eta}\varrho \leq (\tilde{\eta}^{\frac{1}{2}})^2 \varkappa\varrho$, we conclude

$$E_*(0, \tilde{\eta}\varrho, S) \leq \tilde{\eta}^{2\gamma} E_*(0, \varrho, T),$$

and this finishes the proof of Theorem 4.7. □

We are now in a position to prove (one version of) the Regularity Theorem. It will follow rather easily from iteration of 4.7.

4.9 THEOREM. *Suppose $\alpha \in (0, 1)$ and $p > n$ are given. There are constants $\varepsilon = \varepsilon(n, k, \alpha, p)$ and $\gamma = \gamma(n, k, \alpha, p) \in (0, 1)$ such that if (4.6) holds with $\xi = 0$, $T = \mathbb{R}^n$ as well as the general assumptions of sections 2 and 3, then there is a $C^{1,\delta}$ function $u = (u^1, \ldots, u^k) : B^n_{\gamma\varrho}(0) \to \mathbb{R}^k$ such that $u(0) = 0$, $\nu(x) \subset T_x$* graph *u for $x \in B \cap$* graph *u,*

$$\operatorname{spt}\mu \cap B_{\gamma\varrho}(0) = \operatorname{graph} u \cap B_{\gamma\varrho}(0) \cap \overline{B'_1(0)} \tag{39}$$

and $\left(D^n_{\gamma\varrho}(0) := p(\operatorname{graph} u \cap B_{\gamma\varrho}(0) \cap \overline{B'_1(0)})\right)$

$$\tag{40} \begin{cases} \varrho^{-1} \sup\limits_{D^n_{\gamma\varrho}(0)} |u| + \sup\limits_{D^n_{\gamma\varrho}(0)} |Du| \leq c\varepsilon^{1/(2n+3)}, \\ \varrho^{\delta} \sup\limits_{\substack{x, y \in D^n_{\gamma\varrho}(0) \\ x \neq y}} |x - y|^{-\delta} |Du(x) - Du(y)| \\ \qquad \leq c\left\{E^{\frac{1}{2}}(0, \varrho, \mathbb{R}^n) + \left(\int\limits_{B_\varrho(0)} |\boldsymbol{H}|^p d\mu\right)^{1/p} \varrho^{1-n/p} + \varkappa^{\frac{1}{2}} \varrho^{\frac{1}{2}}\right\}, \end{cases}$$

where $\delta = \min\{\frac{1}{2}, 1 - n/p\}$ and $c = c(n, k, \alpha, p)$.

PROOF. By the monotonicity result 3.4, Remark 3.3 (ii) we see that for $0 < \sigma < (1 - \beta)\varrho$ and for $\zeta \in \operatorname{spt}\mu \cap B_{\beta\varrho}(0)$ we get (using 4.6)

$$\begin{aligned} \omega_n^{-1}\sigma^{-n}&\left(\mu(B_\sigma(\zeta)) + \mu(\tilde{B}_\sigma(\zeta))\right) \\ &\leq (1 + c\varepsilon)\omega_n^{-1}(1 - \beta)^{-n}\varrho^{-n}\left(\mu(B_{(1-\beta)\varrho}(\zeta)) + \mu(\tilde{B}_{(1-\beta)\varrho}(\zeta))\right) \\ &\leq (1 + c\varepsilon)(1 - \beta)^{-n}\omega_n^{-1}\varrho^{-n}\left(\mu(B_\varrho(0)) + \mu(\tilde{B}_\varrho(0))\right) \\ &\leq 2(1 + c\varepsilon)(1 - \beta)^{-n}(1 - \alpha), \end{aligned}$$

so that for β, ε small enough we have

$$\omega_n^{-1}\sigma^{-n}\left(\mu(B_\sigma(\zeta)) + \mu(\tilde{B}_\sigma(\zeta))\right) \leq 2(1 - \alpha/2) \tag{41}$$

for $0 < \sigma \leq \varrho/2$ and $\zeta \in \operatorname{spt}\mu \cap B_{\beta\varrho}(0)$.

At first let us consider the case

$$(*) \qquad \zeta \in \operatorname{spt}\mu \cap B_{\beta\varrho}(0) \cap B\,.$$

By the Tilt-Excess-Decay-Theorem 4.7 and (41) (replace ϱ by σ, α by $\alpha/2$, ξ by ζ) we know that there are ε and η such that for $\sigma \leqslant \varrho/2$ and ζ as in (*)

$$(42) \qquad E_*(\zeta, \sigma, S_0) < \varepsilon \Rightarrow E_*(\zeta, \eta\sigma, S_1) < \eta^{2\delta} E_*(\zeta, \sigma, S_0)\,,$$

if $S_0 \subset \mathbb{R}^{n+k}$ is any n-dimensional subspace with $\nu(\zeta) \subset S_0$ and S_1 a suitable n-dimensional space $\subset \mathbb{R}^{n+k}$ with $\nu(\zeta) \subset S_1$. By induction we get a sequence of n-dimensional subspaces $\{S_j\}_{j\in\mathbb{N}}$, $\nu(\zeta) \subset S_j$ such that $E_*(\zeta, \sigma, S_0) < \varepsilon$ implies

$$(43) \qquad E_*(\xi, \eta^j\sigma, S_j) \leqslant \eta^{2\delta j} E_*(\zeta, \sigma, S_0)\,,$$

and by Remark 4.8

$$(44) \qquad |p_{S_j} - p_{S_{j-1}}|^2 \leqslant c E_*(\zeta, \eta^{j-1}\sigma, S_{j-1}) \leqslant c\eta^{2\delta j} E_*(\zeta, \sigma, S_0)\,.$$

Now note that for $S_0 = \nu(\zeta) + T'$, where $\mathbb{R}^n = \nu(0) \oplus T'$, and for $\zeta \in B \cap B_{\varrho/2}(0)$, we get $\nu(\zeta) \subset S_0$ as well as

$$(\varrho/2)^{-n} \int\limits_{B_{\varrho/2}(\zeta)} |p_{T_xM} - p_{S_0}|^2\, d\mu \leqslant 2^{n+1} \varrho^{-n} \int\limits_{B_\varrho(0)} |p_{T_xM} - p_{\mathbb{R}^n}|^2\, d\mu$$
$$+ c|\nu(0) - \nu(\zeta)|^2 \leqslant 2^{n+1} E(0, \varrho, \mathbb{R}^n) + c\varkappa^2 \varrho^2\,,$$

which implies for this choice of S_0 and $\sigma = \varrho/2$

$$(45) \qquad E_*(\zeta, \sigma, S_0) \leqslant c E_*(0, \varrho, \mathbb{R}^n)\,.$$

If thus 4.6 holds with $\xi = 0$, $T = \mathbb{R}^n$ and $c^{-1}\varepsilon$ in place of ε (c as in (45)), we see that (45) in connection with (43) and (44) yields

$$(46) \qquad E_*(\zeta, \eta^j \varrho/2, S_j) \leqslant c\eta^{2\delta j} E_*(0, \varrho, \mathbb{R}^n)$$

and

$$(47) \qquad |p_{S_j} - p_{S_{j-1}}|^2 \leqslant c\eta^{2\delta j} E_*(0, \varrho, \mathbb{R}^n)$$

for each $j \geqslant 1$ with $S_0 = \nu(\zeta) + T'$. Obviously (47) implies the existence of an n-dimensional subspace $S(\zeta) \subset \mathbb{R}^{n+k}$ satisfying $\nu(\zeta) \subset S(\zeta)$ such that for all $j \geqslant 0$

$$|p_{S(\zeta)} - p_{S_j}|^2 \leqslant c\eta^{2\delta j} E_*(0, \varrho, \mathbb{R}^n)\,; \tag{48}$$

in particular we get for $j = 0$

$$|p_{S(\zeta)} - p_{\mathbb{R}^n}|^2 \leqslant c E_*(0, \varrho, \mathbb{R}^n)\,. \tag{49}$$

If $r \in (0, \varrho/2)$ then (46) and (48) yield in the usual way

$$E_*\big(\zeta, r, S(\zeta)\big) \leqslant c(r/\varrho)^{2\delta} E_*(0, \varrho, \mathbb{R}^n) \tag{50}$$

for each $\zeta \in \operatorname{spt}\mu \cap B_{\beta\varrho}(0) \cap B$ and each $0 < r \leqslant \varrho/2$. Notice that (49) and (50) imply for $r \leqslant \varrho/2$

$$E_*(\zeta, r, \mathbb{R}^n) \leqslant c E_*(0, \varrho, \mathbb{R}^n) \leqslant c\varepsilon\,. \tag{51}$$

If now on the other hand

$$\zeta \in \operatorname{spt}\mu \cap B_{\beta\varrho}(0) \sim B\,, \tag{**}$$

we get (cf. [SL]) from the interior regularity $\big(d(\zeta) = \operatorname{dist}(\zeta, \operatorname{spt}\mu \cap B)\big)$

$$E_*\big(\zeta, \eta^j d(\zeta), T_j\big) \leqslant \eta^{2\delta j} E_*\big(\zeta, d(\zeta), T_0\big)\,, \tag{52}$$

if $E_*\big(\zeta, d(\zeta), T_0\big) < \varepsilon$, as well as

$$|p_{T_j} - p_{T_{j-1}}|^2 \leqslant c\eta^{2\delta j} E_*\big(\zeta, d(\zeta), T_0\big)\,. \tag{53}$$

Now let $\tilde{\zeta} \in \operatorname{spt}\mu \cap B_{\beta\varrho}(0) \cap B$ such that $|\zeta - \tilde{\zeta}| = d(\zeta)$ and choose $i \in \mathbb{N}$ such that $\eta^{i+1} < 4d(\zeta)/\varrho \leqslant \eta^i$. Then (52) and (53) respectively imply with $T_0 = \tilde{S}_i$ ($\tilde{S}_i$ as in (46) depending on $\tilde{\zeta}$) and $\tilde{S}_{i+j} = T_j$

$$E_*(\zeta, \eta^{i+j+1}\varrho/4, \tilde{S}_{i+j+1}) \leqslant c(\eta)\,\eta^{2\delta(i+j+1)} E_*(0, \varrho, \mathbb{R}^n) \tag{54}$$

and

$$|p_{\tilde{S}_{i+j+1}} - p_{\tilde{S}_{i+j}}|^2 \leqslant c(\eta)\,\eta^{2\delta(i+j+1)} E_*(0, \varrho, \mathbb{R}^n)\,. \tag{55}$$

As above this implies the existence of an n-dimensional subspace $S(\zeta) \subset \mathbb{R}^{n+k}$ such that for all $j \geqslant 0$

$$|p_{S(\zeta)} - p_{\tilde{S}_j}|^2 \leqslant c\eta^{2\delta j} E_*(0, \varrho, \mathbb{R}^n) \tag{56}$$

and

(57) $$|p_{S(\zeta)} - p_{\mathbb{R}^n}|^2 \leqslant cE_*(0, \varrho, \mathbb{R}^n)\,.$$

We also get analogous to (50) and (51) for $0 < r \leqslant \varrho/4$ and ζ as in (**)

(58) $$E_*(\zeta, r, S(\zeta)) \leqslant c(r/\varrho)^{2\delta} E_*(0, \varrho, \mathbb{R}^n)$$

and

(59) $$E_*(\zeta, r, \mathbb{R}^n) \leqslant cE_*(0, \varrho, \mathbb{R}^n) \leqslant c\varepsilon\,.$$

Thus we get for sufficiently small ε (by (51) and (59)) that if G is as in the proof of 4.2 (with $l = \varepsilon^{1/(2n+3)}$) then $\mu(B_{\beta\varrho}(0) \sim G) = 0$ (β, ε small enough): That is

(60) $$\operatorname{spt} \mu \cap B_{\beta\varrho}(0) \subset \operatorname{graph} f\,,$$

where $f : B^n_{\beta\varrho}(0) \to \mathbb{R}^k$ is Lipschitz with $f(0) = 0$ and

(61) $$\operatorname{Lip} f \leqslant \varepsilon^{1/(2n+3)}\,, \qquad \sup |f| \leqslant c\varepsilon^{1/(2n+2)} \varrho\,.$$

We shall now show that we even have

(62) $$\operatorname{spt} \mu \cap B_{\beta\varrho}(0) = \operatorname{graph} f \cap B_{\beta\varrho}(0) \cap \overline{B'_1(0)}\,.$$

Suppose there is a $\zeta = f(\bar\zeta) \in B_{\beta\varrho}(0) \cap B'_1(0) \sim \operatorname{spt} \mu$. Since $0 \in \operatorname{spt} \mu$ there is $0 < \sigma < \beta\varrho$ such that

(63) $$\begin{cases} (B^n_\sigma(\bar\zeta) \times \mathbb{R}^k) \cap B_{\beta\varrho}(0) \cap \operatorname{spt} \mu = \emptyset\,, \\ \overline{(B^n_\sigma(\bar\zeta) \times \mathbb{R}^k)} \cap B_{\beta\varrho}(0) \cap \operatorname{spt} \mu \neq \emptyset; \end{cases}$$

now take $\zeta_* \in \overline{(B^n_\sigma(\bar\zeta) \times \mathbb{R}^k)} \cap B_{\beta\varrho}(0) \cap \operatorname{spt} \mu$. If $\zeta_* \notin B$ we see that (63), (60), (61) and $\theta \leqslant 1 + \varepsilon$ imply $\theta^n(\mu, \zeta_*) < 1$ (if ε is sufficiently small) which contradicts the fact that $\theta^n(\mu, x) \geqslant 1$ for any $x \in \operatorname{spt} \mu \cap B_\varrho(0) \sim B$. Note that we make use of the upper semicontinuity of θ^n. If on the other hand $\zeta_* \in B$ we know that $\theta^n(\mu, \zeta_*) \geqslant \frac{1}{2}$. But since both ζ and ζ_* are in $\overline{B'_1(0)}$ (63) would again imply $\theta^n(\mu, \zeta_*) < \frac{1}{2}$. Thus (62) is established.

Using the area formula we see that (62) implies for any n-dimensional subspace $S = \operatorname{graph} l$, where $l : \mathbb{R}^n \to \mathbb{R}^k$ is linear with $|\operatorname{grad} l^j| \leqslant 1$, $j = 1$,

..., k, and $\sigma \in (0, \beta\varrho/2)$, $\zeta \in B_{\beta\varrho/2}(0) \cap \operatorname{spt}\mu$,

$$\sigma^{-n}\int_{B^{n'}_{\sigma/2}(\zeta)} \sum_{j=1}^{k} |\operatorname{grad} f^j(x) - \operatorname{grad} l^j|^2 \, d\mathfrak{L}^n(x) \leqslant cE(\zeta, \sigma, S);$$

by (49), (50) and (57), (58) respectively this implies that for each $\zeta \in B_{\beta\varrho/2}(0) \cap \operatorname{spt}\mu$ there is a linear function $l_\zeta : \mathbb{R}^n \to \mathbb{R}^k$ such that

$$(64) \qquad r^{-n}\int_{B^{n'}_{r}(\zeta)} \sum_{j=1}^{k} |\operatorname{grad} f^j(x) - \operatorname{grad} l^j_\zeta|^2 \, d\mathfrak{L}^n(x) \leqslant c(r/\varrho)^{2\delta} E_*(0, \varrho, \mathbb{R}^n)$$

for $0 < r < \beta\varrho/4$.

If $p(\zeta) \in \operatorname{int} p(\operatorname{graph} f \cap B_{\beta\varrho/2}(0) \cap \overline{B'_1(0)})$ then for small enough r we have $B^{n'}_r(\zeta) = B^n_r(p(\zeta))$ so that (64) yields, letting $r\downarrow 0$, $\operatorname{grad} f^j(p(\zeta)) = \operatorname{grad} l^j_\zeta$. The only other case to consider is $p(\zeta) \in p(\operatorname{graph} f \cap B_{\beta\varrho/2}(0) \cap B)$. Because of (61) and our assumptions about B we get $\liminf_{r\downarrow 0} r^{-n}\mathfrak{L}^n(B^{n'}_r(\zeta)) \geqslant \omega_n/2$, which also implies $\operatorname{grad} f^j(p(\zeta)) = \operatorname{grad} l^j_\zeta$.

Now, using (64) again, we conclude

$$(65) \qquad |\operatorname{grad} f^j(p(\zeta_1)) - \operatorname{grad} f^j(p(\zeta_2))| \leqslant c(r/\varrho)^{\delta} E_*(0, \varrho, \mathbb{R}^n)^{\frac{1}{2}}$$

if $\zeta_2 \in B_r(\zeta_1)$ and $0 < r < \beta\varrho/8$. This clearly implies

$$(66) \qquad |\operatorname{grad} f^j(x_1) - \operatorname{grad} f^j(x_2)| \leqslant c(|x_1 - x_2|/\varrho)^{\delta} E_*(0, \varrho, \mathbb{R}^n)^{\frac{1}{2}}$$

for $\mathfrak{L}^n$ - a.e. $x_1, x_2 \in p(\operatorname{graph} f \cap B_{\beta\varrho/2}(0) \cap \overline{B'_1(0)})$. The theorem now follows with $u = f$ and $\gamma = \beta/8$. □

Let us finally show that the conclusions of the Regularity-Theorem remain true if we make the following assumptions

$$(4.10) \qquad \begin{cases} 1 \leqslant \theta, \ \mu\text{ - a.e.}, \ 0 \in \operatorname{spt}\mu, \ \varrho \leqslant 1, \\ \omega_n^{-1}\varrho^{-n}\mu(B_\varrho(0)) \leqslant \frac{1}{2}(1+\delta), \\ \left(\int_{B_\varrho(0)} |\boldsymbol{H}|^p \, d\mu\right)^{1/p} \varrho^{1-n/p} \leqslant \delta \text{ for some } p > n, \ \varkappa\varrho \leqslant \delta, \end{cases}$$

where $\delta \in (0, \frac{1}{8})$ is to be specified.

Repeating the argument which led to (41) in the proof of (4.9) we get for $\zeta \in \operatorname{spt} \mu \cap B_{\beta\varrho}(0)$ and $0 < \sigma \leqslant \varrho/2$

$$(67) \qquad \begin{aligned} (1+c\delta)^{-1} &\leqslant \omega_n^{-1} \sigma^{-n} \big(\mu(B_\sigma(\zeta)) + \mu(\tilde{B}_\sigma(\zeta))\big) \\ &\leqslant (1+c\delta)(1-\tilde{\beta})^{-n} \omega_n^{-1} \tilde{\varrho}^{-n} \big(\mu(B_{(1-\tilde{\beta})\tilde{\varrho}}(\zeta)) + \mu(\tilde{B}_{(1-\tilde{\beta})\tilde{\varrho}}(\zeta))\big) \\ &\leqslant (1+c\delta)(1-\tilde{\beta})^{-n} \omega_n^{-1} \tilde{\varrho}^{-n} \big(\mu(B_{\tilde{\varrho}}(0)) + \mu(\tilde{B}_{\tilde{\varrho}}(0))\big) \\ &\leqslant (1+c\delta)(1+4\delta)^n (1-\tilde{\beta})^n 2\omega_n^{-1} \varrho^{-n} \mu(B_\varrho(0)) \\ &\leqslant 1 + c\delta , \end{aligned}$$

if $\beta \leqslant \tilde{\beta}/(1+4\delta)$ and where $\tilde{\beta}(\delta)$ is small enough ($\tilde{\varrho}$ is defined by

$$\varrho = \left[1 + \frac{2\varkappa\tilde{\varrho}}{1-\varkappa\tilde{\varrho}}\right]\tilde{\varrho} \leqslant (1+4\delta)\,\tilde{\varrho}$$

and we have used $\tilde{B}_\varrho(0) \subset B_\varrho(0)$).

Thus to show that the assumptions (4.10) imply the assumptions (4.6) we have to find a suitable subspace T such that $E(\xi, \sigma, T)$ can be made as small as we wish. In view of Lemma 4.5 it is sufficient to bound the height appropriately. This will be done in the next lemma.

Let us first make some important remarks.

4.11 REMARKS:

(i) Let us first show that the assumptions (4.10) imply

$$(*) \qquad \mu(B \cap B_{\beta\varrho}(0)) = 0$$

for some $\beta = \beta(\delta) > 0$ and $\delta > 0$ small enough. From (67) and the upper semicontinuity of $\tilde{\theta}^n(\mu, \cdot)$ (c.f. Corollary 3.2) it follows that

$$2\theta^n(\mu, \zeta) \leqslant 1 + c\delta < 2$$

for $\zeta \in \operatorname{spt} \mu \cap B \cap B_{\beta\varrho}(0)$. Since $\theta \geqslant 1$ μ - a.e. we get (*). Thus from now on we may assume

$$(**) \qquad \mu(B) = 0 .$$

(ii) We consider the total first variation of $V = \boldsymbol{v}(M, \theta)$, defined as usual as the largest Borel regular measure $\|\delta V\|$ such that for all open

$G \subset B_1(0)$

$$\|\delta V\|(G) = \sup\left\{\int \operatorname{div}_M X\, d\mu : X \in C^1_c(G, \mathbb{R}^{n+k}),\ |X| \leqslant 1\right\}.$$

We are now going to show that $\|\delta V\|$ is a Radon measure on $B_1(0)$. As in [AW2, 3.1] this will follow from a monotonicity formula for tubular neighbourhoods of the supporting surface B. We proceed as follows.

For any $\psi \in C^1_c(B_1(0))$, $\psi \geqslant 0$, we consider the vector-field

$$X(x) = \psi(x)\gamma(\varrho(x))(x - \xi(x))$$

where $\gamma \in C^1(\mathbb{R})$ is decreasing and satisfies $(0 < h \ll 1)$

$$\gamma(t) \equiv 1 \quad \text{for } t \leqslant h/2,$$

$$\gamma(t) \equiv 0 \quad \text{for } t \geqslant h.$$

Obviously X is admissible and a simple calculation yields

$$\int \psi\gamma(\varrho)\, p_{T_xM}\cdot\nu(\xi)\, d\mu + \int \psi\varrho\gamma'(\varrho) p_{T_xM}\cdot\nu(\xi)\, d\mu$$
$$= \int \psi\gamma(\varrho)[p_{T_xM}\cdot \boldsymbol{Q} - (x - \xi)\cdot \boldsymbol{H}]\, d\mu - \int \gamma(\varrho)\nabla^M \psi\cdot(x - \xi)\, d\mu .$$

Here we used the fact that $|p_{T_xM}\circ\nu(\xi(x))(\operatorname{grad} \varrho(x))|^2 = p_{T_xM}\cdot\nu(\xi(x))$.

As in section 3 we take $\phi \in C^1(\mathbb{R})$ such that $\phi(t) \equiv 1$ for $t \leqslant \frac{1}{2}$, $\phi(t) \equiv 0$ for $t \geqslant 1$, $\phi' \leqslant 0$ and set $\gamma(t) = \phi(t/h)$. Setting

$$I(h) = \int \phi(\varrho/h)\, \psi p_{T_xM}\cdot\nu(\xi)\, d\mu ,$$

$$L_1(h) = -\int \phi(\varrho/h)\, \psi[p_{T_xM}\cdot \boldsymbol{Q} - (x - \xi)\cdot \boldsymbol{H}]\, d\mu ,$$

$$L_2(h) = \int \phi(\varrho/h)\, \nabla^M \psi\cdot(x - \xi)\, d\mu$$

we get the equation

$$I(h) - hI'(h) = -\{L_1(h) + L_2(h)\}$$

which after multiplying by h^{-2} can be written as

$$\frac{d}{dh}[h^{-1}I(h)] = h^{-2}\{L_1(h) + L_2(h)\}.$$

Letting $\phi \nearrow \chi_{(-\infty,0)}$ we obtain the following *monotonicity identity for tubular neighbourhoods* of B

$$(68)\quad \frac{d}{dh} h^{-1}\int_{\{\varrho<h\}} \psi p_{T_xM}\cdot\nu(\xi)\,d\mu = h^{-2}\int_{\{\varrho<h\}} \psi[(x-\xi)\cdot \boldsymbol{H} - p_{T_xM}\cdot \boldsymbol{Q}] + \nabla^M\psi\cdot(x-\xi)\,d\mu\,.$$

Integrating (68) from h to h' $(0<h\leqslant h')$ and letting $h\downarrow 0$ we get the existence of $\Gamma(\psi)$, defined by

$$(69)\quad \Gamma(\psi) := \lim_{h\downarrow 0} h^{-1}\int_{\{\varrho<h\}} \psi p_{T_xM}\cdot\nu(\xi)\,d\mu = h'^{-1}\int_{\{\varrho<h'\}} \psi p_{T_xM}\cdot\nu(\xi)\,d\mu$$
$$-\int_{\{\varrho<h'\}} \{\psi[(x-\xi)\cdot\boldsymbol{H} - p_{T_xM}\cdot\boldsymbol{Q}] + \nabla^M\psi\cdot(x-\xi)\}\,\{\varrho^{-1}-h'^{-1}\}\,d\mu\,.$$

Note that $\|\boldsymbol{Q}\|\leqslant 2\varkappa\varrho$.

From the representation (69) we see that Γ is a distribution, but since Γ is positive it induces a Radon-measure on $B_1(0)$, i.e. it can be defined for $\psi\in C_c^0(B_1(0))$.

Now consider any $X\in C_c^1(B_1(0),\mathbb{R}^{n+k})$ and any $\varphi\in C_c^\infty(\mathbb{R})$ with $0\notin \operatorname{spt}\varphi'$. Defining X_ν and X_τ by

$$X_\nu(x) = \nu(\xi(x))\,X(x)\,,\qquad X_\tau(x) = \tau(\xi(x))\,X(x)\,,$$

we get

$$\int \operatorname{div}_M X\,d\mu = \int \operatorname{div}_M X_\tau\,d\mu + \int \operatorname{div}_M X_\nu\,d\mu = -\int X_\tau\cdot\boldsymbol{H}\,d\mu + \int\operatorname{div}_M X_\nu\,d\mu\,;$$

furthermore we have $(\varrho(x) = |x-\xi(x)|)$

$$\int\operatorname{div}_M X_\nu\,d\mu = \int \varphi\circ\varrho\,\operatorname{div}_M X_\nu\,d\mu + \int\operatorname{div}_M[(1-\varphi\circ\varrho)X_\nu]\,d\mu + \int\varphi'\circ\varrho X_\nu\cdot\nabla^M\varrho\,d\mu\,.$$

Letting $\varphi\downarrow\chi_{(-\infty,0]}$ suitably and using the fact that $\mu(B)=0$ we get

$$\int\operatorname{div}_M X\,d\mu = -\int X\cdot\boldsymbol{H}\,d\mu - \lim_{h\downarrow 0} h^{-1}\int_{\{x:\,\varrho(x)<h\}} X_\nu\cdot\nabla^M\varrho\,d\mu\,.$$

Since $\operatorname{grad}\varrho(x) = \nu(\xi(x))(\operatorname{grad}\varrho(x))$ we get

$$X_\nu(x)\cdot\nabla^M\varrho(x) = [p_{T_xM}\circ\nu(\xi(x))]X(x)\cdot[p_{T_xM}\circ\nu(\xi(x))](\operatorname{grad}\varrho(x));$$

this yields $|X_\nu(x)\cdot\nabla^M\varrho(x)|\leqslant p_{T_xM}\cdot\nu(\xi(x))\,|X(x)|\,.$

Together with the considerations made above we get that $\|\delta V\|$ is a Radon measure on $B_1(0)$ and that we have the inequality (Γ as in (69))

$$\|\delta V\| \leqslant \Gamma + \mu \llcorner |\boldsymbol{H}| .$$

By well known representation theorems there exists a $\|\delta V\|$-measurable function $\eta(V;\, \cdot): B_1(0) \to S^{n+k-1}$ such that for any $X \in C^1_c(B_1(0), \mathbb{R}^{n+k})$

$$\int \operatorname{div}_M X \, d\mu = \int X(x)\cdot \eta(V; x)\, d\|\delta V\|(x) . \tag{70}$$

(iii) In the special case where $\boldsymbol{H} \equiv 0$ and B is a hyperplane we have the following reflection principle (cf. [AW2], 3.2). First we note that (70) implies

$$\int X(x)\cdot \eta(V; x)\, d\|\delta V\|(x) = 0$$

for any $X \in C^1_c(B_1(0), \mathbb{R}^{n+k})$ such that $X(x) \in B$ if $x \in B$. Consequently we have $\operatorname{spt} \|\delta V\| \subset B$ and

$$\eta(V; x) \in B^{\perp}$$

for $\|\delta V\|$ - a.e. $x \in B$.

If we now consider the reflection $\vartheta(x) = B(x) - B^{\perp}(x)$ of $x \in \mathbb{R}^{n+k}$ across B and the reflected varifold $\vartheta_{\#} V$ we get

$$\eta(\vartheta_{\#} V; x) = \vartheta[\eta(V; x)] = -\, \eta(V; x)$$

for $\|\delta V\|$ - a.e. $x \in B$. Thus we see that $V' = V + \vartheta_{\#} V$ is stationary in B, i.e. for any $X \in C^1_c(B_1(0), \mathbb{R}^{n+k})$ we have

$$\int \operatorname{div}_{M_{V'}} X \, d\mu_{V'} = 0 .$$

We are now ready to prove the final lemma.

4.12 Lemma. *For any one-dimensional subspace $Y \subset \mathbb{R}^{n+k}$ and any $\varepsilon > 0$ there exists $\delta_0 > 0$ such that the following is true.*

If B and V satisfy the general assumptions made at the beginning of section 2 and additionally $\nu_B(0) = Y$ as well as (4.10) with $\delta = \delta_0$ then there exists an n-dimensional subspace T with $Y \subset T$ such that

$$\sup \{\operatorname{dist}(x, T) : x \in \operatorname{spt} \mu_V \cap B_{(1-2^{-1/n})\varrho/2}(0)\} \leqslant \varepsilon \varrho .$$

PROOF. We may assume $\varrho = 1$ and argue by contradiction. So suppose there is $\varepsilon > 0$ and a sequence of B_i, V_i such that for any n-space T with $Y \subset T$ we have

$$\sup\{\operatorname{dist}(x, T) : x \in \operatorname{spt}\mu_i \cap B_{\frac{1}{2}(1-2^{-1/n})}(0)\} > \varepsilon \tag{71}$$

and such that (4.10) holds with $\varrho = 1$ and $\delta_i = 1/i$. Passing to a subsequence (again denoted by i) we get the existence of a varifold C such that $\lim_{i\to\infty} V_i = C$ in the space of n-varifolds and such that the corresponding B_i converge to $B = Y^\perp$ in an obvious sense. We conclude that C is a rectifiable n-varifold in $B_1(0) \sim B$ satisfying

$$\mu_C(B_1(0)) \leqslant \tfrac{1}{2}\omega_n, \tag{72}$$

$$\delta C(X) = 0 \tag{73}$$

for any $X \in C^1_C(B_1(0), \mathbb{R}^{n+k})$ with $X_{|B} : B \to B$, because any such X can be approximated by X_i such that $\|DX_i - DX\|_\infty \to 0$ and $X_i(b) \in \tau_i(b)$ for $b \in B_i$.

From (67) we conclude that

$$0 \in \operatorname{spt}\mu_C, \tag{74}$$

and by [AW1, 5.4] we get

$$\theta^n(\mu_C, x) \geqslant 1 \tag{75}$$

for μ_C - a.e. $x \in B_1(0)$. Note that the condition $\liminf_{i\to\infty} \|\delta V_i\|(W) < \infty$ for any open $W \subset\subset B_1(0)$ easily follows from the representation of Γ in (69).

Since B is a linear subspace $r^{-n}\mu_C(B_r(x))$ is increasing on $(0, 1-R)$ if $x \in B \cap B_R(0)$ (the vectorfield used in the proof of the interior monotonicity is admissible!). Thus for any $R < 1 - 2^{-1/n} =: R^*$ and $x \in B \cap B_R(0)$

$$\theta^n(\mu_C, x) \leqslant \omega_n^{-1}(1-R)^{-n}\mu_C(B_{(1-R)}(x)) < \omega_n^{-1}2\mu_C(B_1(0)) \leqslant 1$$

because of (72). In view of (75) this implies

$$\mu_C(B \cap B_{R^*}(0)) = 0. \tag{76}$$

By the reflection principle, Remark 4.11 (iii), we see that $C' = C + \vartheta_{\#}C$ is stationary in $B_{R^*}(0)$ and that $0 \in \operatorname{spt}\mu_{C'}$. Using again the monotonicity as well as (72), (75) we get

$$R^{*-n}\mu_{C'}(B_{R^*}(0)) \leqslant 2\mu_{C'}(B_1(0)) \leqslant \omega_n \leqslant \theta^n(\mu_{C'}, x)\,\omega_n$$

for $\mu_{C'}$ - a.e. $x \in B_{R^*}(0)$. By [AW1, 5.3] this implies the existence of an n-space $T \subset \mathbb{R}^{n+k}$ such that

$$C' \llcorner B_{R^*}(0) = \theta^n(\mu_{C'}, 0)\,\boldsymbol{v}(T \cap B_{R^*}(0), 1)\,.$$

It follows that

$$1 \leqslant \theta^n(\mu_{C'}, x) = \theta^n(\mu_{C'}, 0)$$

for $\mu_{C'}$ - a.e. $x \in B_{R^*}(0)$ and since

$$\theta^n(\mu_{C'}, 0)\,\omega_n R^{*n} = \mu_{C'}(B_{R^*}(0)) \leqslant 2\mu_{C'}(B_1(0))\,R^{*n} \leqslant \omega_n R^{*n},$$

we conclude $\theta^n(\mu_{C'}, 0) = 1$ and thus

$$[C + \vartheta_{\#} C] \llcorner B_{R^*}(0) = \boldsymbol{v}(T \cap B_{R^*}(0), 1)\,. \tag{77}$$

By the definition of ϑ we now see that $Y = B^\perp \subset T$. Now (71) implies that there is a sequence $\{x_i\}$, $x_i \in \operatorname{spt} \mu_i \cap B_{R^*/2}(0)$,

$$\lim_{i\to\infty} x_i = x_0 \in \overline{B_{R^*/2}}(0) \subset B_{R^*}(0) \text{ with } |T^\perp x_0| \geqslant \varepsilon\,.$$

Since $\operatorname{spt} \mu_C \subset T$ we get for i large enough

$$0 = \mu_C(B_{\varepsilon/2}(x_0)) \geqslant \mu_{V_i}(B_{\varepsilon/4}(x_i)) - \delta \geqslant c\varepsilon^n - \delta\,,$$

where c is independent of i (use monotonicity); as $\delta > 0$ can be made arbitrarily small by choosing i large enough we get a contradiction.

This proves the claim of the lemma. □

Altogether, we have proved

4.13 THEOREM. *For any n, k, $p \in \mathbb{N}$, $p > n$, $\eta > 0$ there exist $\gamma = \gamma(n, k, p) > 0$, $\varepsilon = \varepsilon(n, k, p, \eta) > 0$ with the following property.*
If $\varrho \leqslant 1$, $B \subset \mathbb{R}^{n+k}$ is a hypersurface of class C^2 with $0 \in B$, $\bar{B} \cap B_1(0) = B$, and the curvature of B satisfies

$$\varkappa\varrho \leqslant \varepsilon^2$$

and if $V = \boldsymbol{v}(M, \theta)$ *is a rectifiable n-varifold with*

$$\operatorname{spt}\mu \subset \overline{B_1'(0)} \qquad (\mu = \mu_V)\,,$$

$$0 \in \operatorname{spt}\mu\,,$$

$$\theta \geqslant 1 \quad \mu\text{ - a.e.}\,,$$

$$\frac{1}{\omega_n \varrho^n}\mu\big(B_\varrho(0)\big) \leqslant \frac{1}{2}(1+\varepsilon)\,,$$

$$\int \operatorname{div}_M X\, d\mu = -\int X\cdot \boldsymbol{H}\, d\mu$$

for all $X \in C_c^1(B_1(0), \mathbb{R}^{n+k})$ *with* $X(b) \in \tau(b)$ *for* $b \in B$ *and*

$$\Big(\int\limits_{B_\varrho(0)} |\boldsymbol{H}|^p\, d\mu\Big)^{1/p} \varrho^{1-n/p} \leqslant \varepsilon$$

then there is a $C^{1,\delta}$*-function* $u = (u^1, \ldots, u^k) : B^n_{\gamma\varrho}(0) \to R^k$ *and an isomery* l *of* $\mathbb{R}^{n+k}$ *with*

$$u(0) = 0$$

$$\nu_{lB}(x) \subset T_x \operatorname{graph} u \qquad \text{for } x \in lB \cap \operatorname{graph} u$$

$$\operatorname{spt}\mu_{l_\# V} \cap B_{\gamma\varrho}(0) = \operatorname{graph} u \cap B_{\gamma\varrho}(0) \cap \overline{lB_1'(0)}$$

and

$$\varrho^{-1}\sup_{D^n_{\gamma\varrho}(0)} |u| + \sup_{D^n_{\gamma\varrho}(0)} |Du| + \varrho^\delta \sup_{\substack{x,y \in D^n_{\gamma\varrho}(0)\\ x\neq y}} |x-y|^{-\delta}\, |Du(x) - Du(y)| \leqslant c\eta\,,$$

$\delta = \min\{\tfrac{1}{2}, 1 - n/p\}$ *and* $D^n_r(0) = p\big(\operatorname{graph} u \cap B_r(0) \cap \overline{lB_1'(0)}\big)$.

5. – Appendix.

In the proof of Lemma 4.7—cf. the argument leading to (29) and (30)—we needed the following simple lemma concerning harmonic functions. It is an easy consequence of Rellich's Theorem.

5.1 LEMMA. *Given any* $\delta > 0$ *there is a constant* $\varepsilon(n, \delta) > 0$ *such that if* $f \in H_2^1(H)$, $H = \{x \in \mathbb{R}^n : x^n > 0\} \cap B_1(0)$, *satisfies*

$$\int\limits_H |\operatorname{grad} f|^2\, d\mathfrak{L}^n \leqslant 1$$

and

$$\left|\int_H \operatorname{grad} f \cdot \operatorname{grad} \zeta \, d\mathfrak{L}^n\right| \leqslant \varepsilon \sup |\operatorname{grad} \zeta|$$

for any $\zeta \in C_c^1(B_1(0))$, *then there is a harmonic function* u *on* H *satisfying* $\partial_{x_n} u(x) = 0$ *on* $x_n = 0$ *such that*

$$\int_H |\operatorname{grad} u|^2 \, d\mathfrak{L}^n \leqslant 1$$

and

$$\int_H |u - f|^2 \, d\mathfrak{L} \leqslant \delta .$$

PROOF. If the lemma were false there would exist $\delta > 0$ and a sequence $\{f_k\}_{k \in \mathbb{N}}$, $f_k \in H_2^1(H)$ such that for any $\zeta \in C_c^1(B_1(0))$

$$\left|\int_H \operatorname{grad} f_k \cdot \operatorname{grad} \zeta \, d\mathfrak{L}^n\right| \leqslant k^{-1} \sup |\operatorname{grad} \zeta| , \tag{1}$$

and

$$\int_H |\operatorname{grad} f_k|^2 \, d\mathfrak{L}^n \leqslant 1 , \tag{2}$$

but

$$\int_H |f_k - u|^2 \, d\mathfrak{L}^n > \delta , \tag{3}$$

whenever u is harmonic on H, $\partial_{x_n} u(x) = 0$ on $x_n = 0$ and

$$\int_H |\operatorname{grad} u|^2 \, d\mathfrak{L}^n \leqslant 1.$$

If $\lambda_k = 2\omega_n^{-1} \int_H f_k \, d\mathfrak{L}^n$ then

$$\int_H |f_k - \lambda_k|^2 \, d\mathfrak{L}^n \leqslant c \int_H |\operatorname{grad} f_k|^2 \, d\mathfrak{L}^n \leqslant c \tag{4}$$

by the Poincaré inequality. By Rellich's Theorem there is a subsequence $\{k'\} \subset \{k\}$ such that $f_{k'} - \lambda_{k'} \rightharpoonup w$ ($\rightharpoonup$ means weak convergence in H_2^1) and $f_{k'} - \lambda_{k'} \to w$ in $L^2(H)$, where $\int_H |\operatorname{grad} w|^2 \, d\mathfrak{L}^n \leqslant 1$.

By (1) we have

$$\int_H \operatorname{grad} w \cdot \operatorname{grad} \zeta \, d\mathfrak{L}^n = 0$$

for any $\zeta \in C_c^1(B_1(0))$. This implies that w is harmonic on H and $\partial_{x_n} w(x) = 0$ on $x_n = 0$. With $u_{k'} = w + \lambda_{k'}$ we get

$$\int_H |f_{k'} - u_{k'}|^2 \, d\mathfrak{L}^n \to 0 \, .$$

This contradicts (3) and the lemma is proved. □

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