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On the Boundary Regularity of Proper Mappings.

FRANC FORSTNERIČ (*)

1. – Statement of the results.

There exist well-known results on smooth extensions of proper holomorphic maps between certain classes of smoothly bounded domains in $\mathbb{C}^n$ [2, 5]. On the other hand, very little is known about proper holomorphic maps into domains in higher dimensional spaces. Suppose that $D \subset \mathbb{C}^n$ and $\Omega \subset \mathbb{C}^N$ $(N > n)$ are bounded domains and that $f\colon D \to \Omega$ is a proper holomorphic map. What can be said about the boundary regularity of the image subvariety $f(D)$ in Ω and about the boundary regularity of f in terms of the regularity of bD and $b\Omega$?

It has been proved recently that, unlike in the equidimensional case $N = n$, the map f needs not extend continuously to $\bar{D}$ even if bD and $b\Omega$ are smooth or real analytic [10]. Therefore additional hypotheses are needed. In this paper we shall prove some results under the assumption that the nontangential boundary values of f at bD, which exist almost everywhere on bD with respect to the surface measure on bD, lie in a smooth submanifold M of dimension $2n-1$ of $\mathbb{C}^N$ contained in $b\Omega$. Our first main result is the following.

1.1. THEOREM. *Let $D \subset \mathbb{C}^n$ and $\Omega \subset \mathbb{C}^N$ $(N > n)$ be bounded domains of class C^2, let $b\Omega$ be strictly pseudoconvex, and let M be a compact connected real submanifold of $\mathbb{C}^n$ of class C^r $(r \geqslant 2)$ and of dimension $2n-1$ that is contained in the boundary of Ω. If f is a proper holomorphic map of D into Ω such that for almost every point $p \in bD$ with respect to the Lebesgue measure on bD the nontangential limit $f^*(p)$ of f at p lies in M, then the following hold:*

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(i) *the closure* $\overline{V}$ *of the subvariety* $V = f(D)$ *of* Ω *is* $V \cup M$, *and the pair* (V, M) *is a local* C^r *manifold with boundary in a neighborhood of each point* $q \in M$. *In particular, the singular variety* V_{sing} *is finite*;

(ii) *the map* f *extends to a continuous map on* $\overline{D}$ *which satisfies the Hölder condition with exponent* $\frac{1}{2} - \varepsilon$ *for every* $\varepsilon > 0$;

(iii) *if* D *is also strictly pseudoconvex, then the restriction*

$$f\colon \overline{D} \setminus f^{-1}(V_{\text{sing}}) \to \overline{V} \setminus V_{\text{sing}}$$

is a finite covering projection that is Hölder-continuous with the exponent $\frac{1}{2}$.

Note that if a proper map $f\colon D \to \Omega$ exists, then D is necessarily pseudoconvex. Using a local extension theorem for biholomorphic maps due to Lempert [20, p. 467] we obtain the following corollary.

1.2. COROLLARY. *Let* $f\colon D \to \Omega$ *and* $M \subset b\Omega$ *be as in Theorem* 1.1, *and assume that both* D *and* Ω *are strictly pseudoconvex. If* bD *and* M *are of class* C^r *for some* $r \geqslant 6$, *then* f *extends to a* C^{r-4} *map on* $\overline{D}$. *In particular, if* bD *and* M *are* C^∞ *on* $\overline{D}$, *and if* bD *and* M *are real-analytic, then* f *extends holomorphically to a neighborhood of* $\overline{D}$.

NOTE. In the case when bD and M are real-analytic, Corollary 1.2 above can be considered to be a generalization of the reflection principle [21, 23, 33] to maps into higher dimensional spaces. Certain generalizations for this kind of maps have been obtained earlier by Lewy [21, p. 8] and Webster [33].

A similar result holds if M is only an immersed submanifold of $b\Omega$, provided that the set of its self-intersections is not too large. In the next theorem we assume that $D \subset \mathbf{C}^n$ and $\Omega \subset \mathbf{C}^N$, $N > n$, are bounded C^2 strictly pseudoconvex domains.

1.3. THEOREM. *Let* M^{2n-1} *be a compact connected* C^r *manifold,* $r \geqslant 2$, *and let* $i\colon M \to \mathbf{C}^N$ *be an immersion of class* C^r, *with the image* $i(M)$ *contained in* $b\Omega$. *Denote by* S *the set of points* $q \in i(M)$ *at which* $i(M)$ *is not a manifold. Assume that*

(*a*) $i(M) \setminus S$ *is connected, and*

(*b*) $\mathcal{H}^{2n-1}(S) = 0$, *where* $\mathcal{H}^k$ *denotes the* k*-dimensional Hausdorff measure.*

If $f\colon D \to \Omega$ *is a proper holomorphic map with* $f^*(p) \in i(M)$ *for almost every point* p *in* bD, *then the following hold.*

(i) *Each point* $q \in M$ *has a neighborhood* U *in* $\mathbf{C}^N$ *such that*

$$U \cap M \quad = M_1 \cup M_2 \cup \ldots \cup M_s\,,$$
$$U \cap f(D) = V_1 \cup V_2 \cup \ldots \cup V_s\,,$$

and $V_j \cup M_j$ *is a* C^k *manifold with boundary* M_j *for each* $j = 1, \ldots, s$. *In particular, the singular locus of the variety* $V = f(D)$ *is finite*;

(ii) f *extends to a Hölder continuous map on* $\bar{D}$, *and its branching locus consists of at most finitely many points of* D;

(iii) *if* $r \geqslant 6$, *then* f *extends to a* C^{r-4} *map on* $\bar{D}$.

Remark 1. Since the map f is bounded on D, the generalized theorem of Fatou [29, p. 13] asserts that there exists a set $E \subset bD$ whose complement $bD \setminus E$ has surface measure 0 such that f has a nontangential limit $f^*(p)$ at every point $p \in E$. One of our hypotheses is that this limit lies in M for almost every point $p \in E$.

Remark 2. The regularity of the subvariety $f(D)$ at the boundary of Ω can also be deduced from the work of Harvey and Lawson [14, Theorems 4.7, 4.8 and 10.3]. Their methods include the structure theorems for certain types of currents. Our proof of Theorem 1.1 is perhaps more elementary. However, the hypothesis that $b\Omega$ be strictly pseudoconvex is essential in our proof of Theorem 1.1.

Remark 3. In the case $n = 1$ our Theorem 1.1 follows from a more general result of Čirka [4, p. 293] which states that if $f\colon \Delta \to \mathbf{C}^N$ is a holomorphic map on the unit disk $\Delta \subset \mathbf{C}$ such that all of its boundary values on an open arc $\gamma \subset b\Delta$ lie in a totally real submanifold $M \subset \mathbf{C}^N$ of class C^r, $r \geqslant 2$, then f is of class $C^{r-1,\alpha}$ on $\Delta \cup \gamma$ for all $0 < \alpha < 1$. If D is a domain of class C^r in $\mathbf{C}$, then we can find for every point $p \in bD$ a simply connected domain $U \subset D$ with bU of class C^r such that $\bar{U} \cap bD$ contains an open arc γ and $p \in \gamma$. If $f\colon D \to \Omega$ is as in Theorem 1.1 above and if all boundary values of f lie in a C^r curve M contained in $b\Omega$, then the theorem of Čirka implies that f is of class $C^{r-1,\alpha}$ on $\bar{D}$. If ϱ is a strictly plurisubharmonic defining function for Ω, then $\varrho \circ f$ is a negative subharmonic function on D that vanishes on bD. The Hopf lemma implies $d(\varrho \circ f) \neq 0$ on bD. It follows that $df \neq 0$ on bD, and $f(\bar{D})$ intersects $b\Omega$ transversely. From the proof of part (i) of Theorem 1.1 we shall be able to see that the set $f(\bar{D})$ is in fact of class C^r near its boundary $f(bD) = M$.

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2. – Boundary regularity of the image variety.

In this section we shall give a self-contained proof of Theorem 1.1 in the case when $n \geqslant 2$. The first part of the proof applies also to the case $n = 1$.

By an embedding theorem of Fronæss and Khenkin [9, 17] we may assume that Ω is strictly *convex*. The maximum modulus principle for functions in $H^\infty(D)$ implies that $f(D)$ lies in the polynomially convex hull $\hat{M}$ of M. Since Ω is strictly convex, we have $\hat{M} \cap b\Omega = M$ and hence

$$\overline{f(D)} \subset f(D) \cup M ,$$

i.e., *all* limiting values of f at bD lie in M.

We shall first prove that $\overline{f(D)}$ is a C^r manifold with boundary in a small neighborhood of each point $p \in \overline{f(D)} \cap M$ at which the following condition holds:

$$T_p M \not\subset T_p^{\mathbf{C}} b\Omega . \tag{2.1}$$

Here, $T_p^{\mathbf{C}} b\Omega$ denotes the maximal complex subspace of the tangent space $T_p b\Omega$. By translating to the origin we may assume that $p = 0$. The assumption (2.1) implies that $W = T_0 M \cap T_0^{\mathbf{C}} b\Omega$ is a real $(2n-2)$-dimensional vector subspace of $\mathbf{C}^N$.

We claim that we can find a complex $(n-1)$-dimensional subspace Σ' of $\mathbf{C}^N$ such that the orthogonal projection $\pi' \colon \mathbf{C}^N \to \Sigma'$ maps W bijectively onto Σ'. This is equivalent to finding a complex subspace Σ'' of $\mathbf{C}^N$ such that $W \oplus \Sigma'' = \mathbf{C}^N$, since we may then take for Σ' the orthogonal complement of Σ'' in $\mathbf{C}^N$. If $W = \{(x, y) \in \mathbf{C}^2 \colon x, y \text{ real}\}$, we may take $\Sigma'' = \mathbf{C}\cdot(1, i)$. In general, if we choose coordinates correctly, we have

$$W = \mathbf{C}^m \oplus (\mathbb{R}^2)^l \oplus \{0\} \subset \mathbf{C}^N ,$$

where each copy of $\mathbb{R}^2$ is embedded as the standard totally real plane in $\mathbf{C}^2$, and $m + l = n - 1$. For each copy of $\mathbb{R}^2$ in the above sum we take $\Sigma_j'' = \mathbf{C}\cdot(1, i)$ as above. The complex subspace

$$\Sigma'' = \{0\} \oplus \Sigma_1'' \oplus \ldots \oplus \Sigma_l'' \oplus \mathbf{C}^{N-n+1}$$

has the required property $W \oplus \Sigma'' = \mathbf{C}^N$, and we take Σ' to be the orthogonal complement of Σ'' in $\mathbf{C}^N$.

Let Σ be the complex n-dimensional subspace of $\mathbf{C}^N$ spanned by Σ' and by the normal vector to $b\Omega$ at 0. We denote by π the orthogonal projection of $\mathbf{C}^N$ onto Σ. The restriction $\pi\colon T_0M \to \Sigma$ is one-to-one by the choice of Σ, and therefore $\pi\colon M \to \Sigma$ is a C^r embedding near 0.

We will show that $\pi(M) \subset \Sigma$ is a strictly *convex* hypersurface near the point $0 \in \Sigma$. By a unitary change of coordinates at 0 we may assume that

$$\Sigma = \{z \in \mathbf{C}^N\colon z_{n+1} = \ldots = z_N = 0\}$$

and that in some neighborhood U of 0 the domain Ω is given by

$$\Omega \cap U = \{z \in U\colon x_1 + Q(z) + o(|z|^2) < 0\},$$

where $z_j = x_j + iy_j$ and $Q(z)$ is a real positive definite quadratic form in z. Let

$$c = \tfrac{1}{2} \inf \{Q(z)\colon |z| = 1\} > 0 .$$

For all sufficiently small $\varepsilon > 0$ we have

$$\Omega \cap \{x_1 > -\varepsilon\} \subset \{z \in \mathbf{C}^N\colon x_1 + c|z|^2 < 0\} = B_c$$

and therefore

$$\pi(\Omega \cap \{x_1 > -\varepsilon\}) \subset B_c \cap \Sigma .$$

In particular, $\pi(M \cap \{x_1 > -\varepsilon\})$ is a hypersurface in the ball $B_c \cap \Sigma$ that is internally tangent to the sphere $bB_c \cap \Sigma$ at 0, and therefore $\pi(M)$ is strictly convex near 0 as claimed.

Let G be the domain in Σ bounded by $\pi(M) \cap \{x_1 > -\varepsilon\}$ and by $\{z_1 = -\varepsilon\}$. For each sufficiently small $\varepsilon > 0$ we have

$$\widehat{\pi(M)} \cap \{x_1 \geqslant -\varepsilon\} = \bar{G} ,$$

where $\widehat{\pi(M)}$ is the polynomially convex hull of $\pi(M)$. The maximum maximum modulus principle for H^∞ implies

$$(\pi \circ f)(D) \subset \widehat{\pi(M)} .$$

It follows that

$$\pi(f(D) \cap \{x_1 > -\varepsilon\}) = \pi(f(D)) \cap \{x_1 > -\varepsilon\} \subset \widehat{\pi(M)} \cap \{x_1 > -\varepsilon\} \subset \bar{G} .$$

By the maximum principle for varieties [22, p. 54] we have

$$\pi(f(D) \cap \{x_1 > -\varepsilon\}) \subset G .$$

The variety $V = f(D) \cap \{x_1 > -\varepsilon\}$ is closed in $\pi^{-1}(G)$, and the restriction $\pi|_V\colon V \to G$ maps V properly and holomorphically into G. Hence the pair $(V, \pi|_V)$ is an analytic cover [13, p. 101] of multiplicity λ for some integer λ.

We claim that $\lambda = 1$. The following is the crucial observation about V:

If $\{w_\nu\} \subset V$ is a sequence for which $\{\pi(w_\nu)\}$ converges to a point

$$q \in (M) \cap \{x_1 > -\varepsilon\}\,,$$

then $\{w_\nu\}$ converges to the unique point $\tilde{q} \in M$ for which $\pi(\tilde{q}) = q$.

Intuitively this says that all sheets of the analytic cover $\pi\colon V \to G$ are glued together along M, and will show that as a consequence there is only one sheet.

After a unitary change of coordinates $z_{n+1}, \ldots, z_N$ we can assume that for some $z \in G$ there are λ distinct points $w^{(1)}(z), \ldots, w^{(\lambda)}(z)$ in $\pi^{-1}(z) \cap V$ with distinct N-th coordinates $w_N^{(1)}(z), \ldots, w_N^{(\lambda)}(z)$. The same is then true for every point z outside a proper subvariety $L \subset G$, and each $w_N^{(j)}$ is locally a holomorphic function of z. However, these function need not be well-defined globally.

Consider the polynomial $P(t, z) \in O(G \setminus L)[t]$ in the variable t defined by

$$P(t, z) = \prod_{j=1}^{\lambda} \left(t - w_N^{(j)}(z)\right) = t^\lambda + a_1(z)\, t^{\lambda-1} + \ldots + a_\lambda(z)\,, \qquad z \in G \setminus L\,.$$

The coefficients $a_j(z)$ are elementary symmetric polynomials in the $w_N^{(j)}(z)$'s, and hence they are well-defined bounded holomorphic functions on $G \setminus L$ that extend to bounded holomorphic functions on G. The same is then true for the discriminant $\Delta(z)$ of P. By the generalized theorem of Fatou [29, p. 13] there is a set E contained in $\pi(M) \cap \{x_1 > -\varepsilon\} = S$, E being of full measure in S, such that all coefficients $a_j(z)$ and $\Delta(z)$ have nontangential limits at all points of E. Since Δ is not identically zero on G by the construction of P, the boundary uniqueness theorem [27] implies that $\Delta(e) \neq 0$ for some $e \in E$ (in fact $\Delta \neq 0$ almost everywhere on E). Hence the polynomial $P(t, e)$ has λ distinct complex roots $t_1, \ldots, t_\lambda$.

In order to reach a contradiction we assume that $\lambda > 1$, and let $t_1 \neq t_2$ be two distinct roots of $P(t, e)$. Since the roots of a polynomial depend continuously on its coefficients, we can find a sequence of points $\{z_\nu\}$ in G converging nontangentially to e, and we can find roots $t_1(z_\nu)$, $t_2(z_\nu)$ of $P(t, z_\nu)$ such that

$$\lim_{\nu \to \infty} t_1(z_\nu) = t_1 \quad \text{and} \quad \lim_{\nu \to \infty} t_2(z_\nu) = t_2\,.$$

By the definition of $P(t, z)$ there exist points $w_\nu^{(1)}$ and $w_\nu^{(2)}$ in $V \cap \pi^{-1}(z_\nu)$ with the N-th coordinates equal to $t_1(z_\nu)$ and $t_2(z_\nu)$, respectively. Clearly the sequences $\{w_\nu^{(1)}\}$ and $\{w_\nu^{(2)}\}$ cannot both converge to the same point $\tilde{e} = M \cap \pi^{-1}(e)$. This contradicts the observation about V that we have made above.

Therefore $\lambda = 1$ as claimed. Hence the map $\pi|_V \colon V \to G$ is one-to-one and therefore it is a biholomorphism of V onto G. Its inverse is of the form

$$z \to (z, \sigma(z)) , \qquad z \in G ,$$

where $\sigma \colon G \to \mathbf{C}^{N-n}$ is a holomorphic map on G. Our observation about V implies that σ extends continuous to $G \cup S$, where $S = \pi(M) \cap \{x_1 > -\varepsilon\}$, and the map $z \to (z, \sigma(z))$, $z \in S$, is the inverse of $\pi|_M$ on S. Since $\pi|_M$ is a C^r diffeomorphism onto S, $\sigma|_S$ is of class C^r by the inverse mapping theorem. The regularity theorem [14, Theorem 5.6] implies that σ is of class C^r on $G \cup S$.

This proves that $\overline{f(D)} \cap M$ is a C^r manifold with boundary near every point $p \in f(D) \cap M$ at which the condition (2.1) holds. In particular, M is maximally complex near every such point p, and a neighborhood of p in M is contained in $\overline{f(D)}$. It remains to show that (2.1) holds for every point $p \in \overline{f(D)} \cap M$. In the case $n = 1$ we refer to the theorem of Čirka [4]. (See Remark 3 in Section 1). We shall give a self-contained proof in the case $n \geqslant 2$.

Define the subsets C and E of M by

$$C = \{p \in M \mid T_p M \subset T_p^{\mathbf{C}} b\Omega\} , \tag{2.2}$$

$$E = M \cap \overline{f(D)} . \tag{2.3}$$

We have seen above that $E \setminus C$ is an open subset of $M \setminus C$, and M is maximally complex at each point of $E \setminus C$. Since E is closed, $E \setminus C$ is also closed in $M \setminus C$ and therefore it is a union of connected components of $M \setminus C$. We want to show that $C = \emptyset$ and hence $E = M$.

We will first show that the set $E \setminus C$ is not empty. Suppose on the contrary that $E \subset C$, i.e., the transversality condition (2.1) does not hold at any point of E. Extending Ω to a strictly convex domain in $\mathbf{C}^{N'}$ for a $N' \geqslant N$ we may assume that $\dim b\Omega \geqslant 2 \dim M + 1$.

The strictly pseudoconvex hypersurface $b\Omega$ is a contact manifold with the contact form $\eta = i(\bar\partial - \partial)\varrho$ whose kernel is $\ker \eta = T^{\mathbf{C}} b\Omega$, where ϱ is a defining function for Ω [31]. (For the general theory of contact manifolds see [3].) Let $\iota \colon M \hookrightarrow b\Omega$ be the inclusion of M into $b\Omega$. We have $\iota^* \eta = 0$

on the set C. By an argument of Duchamp [7] every point $p \in M$ has an open neighborhood $U \subset M$ and a C^1 embedding $\tilde{\iota}\colon U \to b\Omega$ such that $\tilde{\iota} = \iota$ on the set $C \cap U$, and $\iota^*\eta = 0$ on U. Then $\tilde{\iota}\colon U \to b\Omega$ is an *interpolation manifold* [31], and by a theorem of Rudin [26] each compact subset of $\tilde{\iota}(U)$ is a peak-interpolation set for the algebra $A(\Omega)$. It follows that E is a local peak-interpolation set and hence a peak-interpolation set [30, Chapter 4]. If $h \in A(\Omega)$ is a peak function on E, then $h \circ f$ is a nonconstant bounded holomorphic function on D whose boundary values equal 1 almost everywhere on bD. This is a contradiction which implies that $E \setminus C \neq \emptyset$.

The following lemma implies that the set C is empty, thereby concluding the proof of part (i) of Theorem 1.1.

2.1. LEMMA. *Let S be a strictly pseudoconvex hypersurface of class C^2 in $\mathbf{C}^N$ and let M be a C^2 submanifold of S of dimension $2m+1$ for some $m \geqslant 1$. If M is maximally complex at every point of an open subset $U \subset M$, then we have for every $p \in \overline{U}$*

$$T_pM \not\subset T_p^{\mathbf{C}}S\,. \tag{2.4}$$

Assume the validity of Lemma 2.1 for a moment. Let $S = b\Omega$ and $U = E \setminus C$. If $C \neq \emptyset$, then there exists a point $p \in C \cap \overline{U}$. By Lemma 2.1 the condition (2.4) holds at p which is a contradiction with the definition (2.2) of the set C. Hence $C = \emptyset$ and Theorem 1.1 is proved provided that Lemma 2.1 holds.

PROOF OF LEMMA 2.1. Let η be a contact form on S with kernel $T^{\mathbf{C}}S$. If X is a C^1 vector field on S that is tangent to $T^{\mathbf{C}}S$, then the vector field JX is also tangent to $T^{\mathbf{C}}S$. (Here J denotes the almost complex structure on $T^{\mathbf{C}}S$.) By virtue fo the strict pseudoconvexity of S we have

$$-\langle d\eta, (X + iJX) \otimes (X - iJX)\rangle_p \neq 0 \tag{2.5}$$

at every point p where $X_p \neq 0$. By the Cartan formula (2.5) is equal to

$$\begin{aligned} -\langle \eta, [X + iJX, X - iJX]\rangle_p &= -\langle \eta, -2i[X, JX]\rangle_p \\ &= 2i\langle \eta, [X, JX]\rangle_p\,. \end{aligned} \tag{2.6}$$

Hence the continuous vector field $Y = [X, JX]$ satisfies $Y_p \notin T_p^{\mathbf{C}}S$ if $X_p \neq 0$. This shows that (2.4) holds at each point $p \in U$. We need to prove that (2.4) also holds on the boundary of U.

Fix a point $p_0 \in \overline{U} \setminus U$ and choose real functions $r_1, \ldots, r_s$ of class C^2

on $\mathbf{C}^N$ such that near p_0 the manifold M is defined by the equations

$$r_1(z) = \ldots = r_s(z) = 0 .$$

Let $\theta_j = i\partial r_j$ for $1 \leqslant j \leqslant s$. Each θ_j is a complex 1-form of class C^1 which is real-valued on TM. Moreover, we have

$$T_p^{\mathbf{C}} M = \bigcap_{j=1}^{s} (\ker \theta_j)_p$$

for every p near p_0. Since M is odd dimensional, $T_{p_0}^{\mathbf{C}} M \neq T_{p_0} M$, and hence one of the forms, say θ_{j_0}, does not vanish on $T_{p_0} M$. Hence the restriction of θ_{j_0} to TM defines a C^1 distribution of codimension 1 on TM near p_0. Since M is assumed to be maximally complex at every point of U, it follows that

$$T_p M \cap (\ker \theta_{j_0})_p = T_p^{\mathbf{C}} M$$

for each $p \in U$ near p_0.

Choose a C^1 vector field X' on M near p_0, $X'_{p_0} \neq 0$, such that

$$\langle \theta_{j_0}, X' \rangle \equiv 0 .$$

Since $\eta = 0$ on $T^{\mathbf{C}} M$, we have

$$\langle \eta, X' \rangle_p = 0 \qquad \text{for } p \in U .$$

We claim that there is a C^1 vector field X on a neighborhood of p_0 in S such that

$$X_p = X'_p \qquad \text{for } p \in U \text{ and } \langle \eta, X \rangle \equiv 0 .$$

The problem is local near p_0. Choose local coordinates such that $p_0 = 0$, $M = \mathbf{R}^{2m+1}$, $S = \mathbf{R}^{2N-1}$, U is an open subset of M with $0 \in \overline{U}$,

$$\eta(x) = \sum_{j=1}^{2N-1} a_j(x)\, dx_j \quad \text{and} \quad X'(x) = \sum_{j=1}^{2N-1} b_j(x)(\partial/\partial x_j) \quad \text{for } x \in \mathbf{R}^{2m+1} .$$

One of the coefficients a_j is nonzero at 0, say $a_1(0) \neq 0$. We have

$$\langle \eta, X' \rangle_x = \sum_{j=1}^{2N-1} a_j(x)\, b_j(x) = 0$$

for $x \in U$. Rewrite this as

$$b_1(x) = -\frac{1}{a_1(x)} \sum_{j=2}^{2N-1} a_j(x) b_j(x) \tag{2.7}$$

for x near 0 in U. We extend the functions $b_2, \ldots, b_{2N-1}$ smoothly to a neighborhood of 0 in $\mathbb{R}^{2N-1}$, and we let $b_1(x)$ be defined by (2.7). This gives us a vector field $X(x) = \sum_{j=1}^{2N-1} b_j(x)(\partial/\partial x_j)$ on $\mathbb{R}^{2N-1}$ with the required properties.

The C^0 vector field $Y = [X, JX]$ defined on S near $p_0 = 0$ is tangent to M on the set U. By the continuity it follows that $Y_0 \in T_0 M$. Moreover, the strict pseudoconvexity of S implies that $\langle \eta, Y \rangle_0 \neq 0$ (see (2.5) and (2.6)). Together these imply that $T_0 M \not\subset T_0^{\mathbb{C}} S$ and Lemma 2.1 is proved.

3. – Continuous extension to the boundary.

In this section we shall conclude the proof of Theorem 1.1. Following an idea of Khenkin [16] we first prove that the map f in Theorem 1.1 extends continuously to $\bar{D}$.

3.1. LEMMA. *Let $f: D \to \Omega$ be as in Theorem 1.1. Denote by $d_D(z)$ the Euclidean distance fo a point $z \in D$ to bD, and similarly for d_Ω. Then there exist constants $c_1, c_2 > 0$ and $0 < \varepsilon < 1$ such that the inequality*

$$c_1 d_D(z) \leqslant d_\Omega(f(z)) \leqslant c_2 d_D(z)^\varepsilon \tag{3.1}$$

holds for all $z \in D$. If D is also strictly pseudoconvex, we may take $\varepsilon = 1$ in (3.1).

PROOF. Let r_D and r_Ω be C^2 defining functions for D resp. Ω. Since Ω is strictly pseudoconvex, we may take r_Ω to be plurisubharmonic on Ω. Hence $r_\Omega \circ f$ is plurisubharmonic on D, it is negative and tends to 0 as we approach bD. By the Hopf lemma [15] there is a constant $c_1 > 0$ such that

$$r_\Omega(f(z)) \leqslant c_1 r_D(z), \qquad z \in D.$$

Since the function $-r_D$ is proportional to d_D on D and similarly $-r_\Omega$ is proportional to d_Ω on Ω, the above is equivalent to the left estimate in (3.1).

To prove the right estimate in (3.1) we choose by [6] an ε in $(0, 1)$

such that the function

$$r' = -(-r_D)^\varepsilon$$

is plurisubharmonic on D. If D is strictly pseudoconvex, we may assume that r_D is plurisubharmonic and hence $\varepsilon = 1$ would do. There is a proper subvariety V' of $V = f(D)$ such that $V \setminus V'$ is regular and the restriction

$$f\colon D \setminus f^{-1}(V') \to V \setminus V'$$

is a finite unbranched covering projection. We define a function φ on V by

$$\varphi(w) = \max\{r'(z)\colon z \in D \text{ and } f(z) = w\}.$$

Locally on $V \setminus V'$ the function φ is the maximum of a finite number of plurisubharmonic functions and hence it is itself plurisubharmonic. Since φ is clearly continuous on V, it is plurisubharmonic on all of V according to [12, Satz 3]. Moreover, φ is negative on V and tends to 0 as we approach $bV = M$. Since $\overline{V}$ is transversal to $b\Omega$ by the proof of part (i) of Theorem 1.1, we have

$$d(r_\Omega|_{\overline{V}})(q) \neq 0, \qquad q \in M = \overline{V} \cap b\Omega.$$

The Hopf lemma implies

$$c_2\varphi(w) \leqslant r_\Omega(w), \qquad w \in V$$

for some constant $c_2 > 0$. Taking the absolute values we have

$$c_2|r'(z)| \geqslant |r_\Omega(f(z))|$$

for $z \in D$. By the definition of r' we have $|r'(z)| = |r_D(z)|^\varepsilon$, and hence

$$|r_\Omega(f(z))| \leqslant c_2|r_D(z)|^\varepsilon.$$

This is equivalent to the right estimate in (3.1) and Lemma 3.1 is proved.

Using Lemma 3.1 and the properties of the infinitesimal Kobayashi metric we can prove that f extends to a Hölder continuous map with the exponent $\varepsilon/2$ on $\overline{D}$, where ε is as in (3.1). The idea of this proof is due to Khenkin [16].

If N is an arbitrary complex manifold, $z \in N$ and $X \in T_z^{1,0}N$ is a complex tangent vector to N at z, the Kobayashi metric $K_N(z, X)$ is given by

$$K_N(z, X) = \inf\{\alpha > 0 \mid \text{there is a holomorphic } f\colon \Delta \to M \text{ with } f(0) = z \text{ and } f'(0) = \alpha^{-1}X\},$$
$$= \inf\{r^{-1} \mid \text{there is a holomorphic } f\colon \Delta_r \to M \text{ with } f(0) = z \text{ and } f'(0) = X\}.$$

(Here Δ_r denotes the disk of radius r centered at 0 in $\mathbf{C}$.) For further details concerning the Kobayashi metric see [18].

If $D \subset \mathbf{C}^n$ is a bounded domain, then

$$K_D(s, X) \leqslant |X|/d_D(z)\,, \tag{3.2}$$

where $|X|$ is the Euclidean length of X. If D is strictly pseudoconvex, then

$$K_D(z, X) \geqslant c|X|/d(z)^{\frac{1}{2}} \tag{3.3}$$

for some constant $c > 0$ [11]. Finally, if $f\colon D \to \Omega$ is a holomorphic map, then

$$K_\Omega(f(z), f_*X) \leqslant K_D(z, X)\,,$$

where $f_*X = df(z)X$. These properties together imply

$$c|f_*X|/d_\Omega(f(z))^{\frac{1}{2}} \leqslant K_\Omega(f(z), f_*X) \leqslant K_D(z, X) \leqslant |X|/d_D(z)\,.$$

If $X \neq 0$, Lemma 3.1 implies

$$|f_*X|/|X| \leqslant c d_D(z)^{-1+\varepsilon/2}\,, \qquad X \in T_z^{1,0}(D)\,.$$

From this it follows by a simple integration argument that f is Hölder continuous of the exponent $\varepsilon/2$ on D, and hence it extends continuously to $\bar{D}$.

Once we know that f is continuous on $\bar{D}$, we can improve our result by using the local plurisubharmonic exhaustion functions on D constructed in [6, Theorem 3]. In particular it follows that Lemma 3.1 above holds for *every* $0 < \varepsilon < 1$, and hence f is Hölder continuous on $\bar{D}$ of the exponent α for every $0 < \alpha < \frac{1}{2}$. If D is strictly pseudoconvex, we may take $\alpha = \frac{1}{2}$. This proves part (ii) of Theorem 1.1.

We shall use the idea of Pinčuk [24] to show that the map f is unbranched in a neighborhood of each point $p \in bD$ at which bD is strictly pseudoconvex. We need the following local version of the result of Pinčuk:

3.2. THEOREM. *Let D^j $(j = 1, 2)$ be bounded strictly pseudoconvex domains in $\mathbb{C}^m$ with C^2 boundaries and let $p^j \in bD^j$. Suppose that U^j is an open subset of D^j such that for some small $\varepsilon > 0$ we have*

$$D^j \cap B_\varepsilon(p^j) \subset U^j , \qquad j = 1, 2 ,$$

where $B_\varepsilon(p)$ is the ball of radius ε centered at p. Let f: $U^1 \to U^2$ be a proper holomorphic map that extends continuously to $\overline{U}^1$ and $f(p^1) = p^2$. Then the branching locus of f avoids a neighborhood of p^1 in D^1.

NOTE. The difference between Theorem 3.2 and [24] is that in our case the map f is only defined on an open subset of D^j.

PROOF. We recall the proof of Pinčuk given in [24]. Assume that there is a sequence of points $\{p_k\} \subset U^1$ converging to p^1 such that each p_k is a branch point of f. Pinčuk constructed a sequence of domains D^j_k $(k = 1, 2, \ldots)$ such that $\overline{D}^j_k$ is biholomorphically equivalent to $\overline{D}^j$ for each $k \in \mathbb{Z}_+$, the point $p_k \in D^1$ (resp. $f(p_k) \in D^2$) corresponds to the point $(0, \ldots, 0, -1) \in D^1_k$ (resp. $(0, \ldots, 0, -1) \in D^2_k$), and as $k \to \infty$ the sequence of domains D^j_k converges uniformly on compact subsets of $\mathbb{C}^m$ to the domain

$$B = \left\{z \in \mathbb{C}^m \mid 2 \operatorname{Re} z_m + \sum_{s=1}^{m-1} |z_s|^2 < 0\right\}$$

for $j = 1, 2$. The domain B is biholomorphically equivalent to the unit ball $\mathbb{B}^m$ [25, p. 31], and the map f gives rise to a proper holomorphic map F: $B \to B$ such that $F(0, \ldots, -1) = (0, \ldots, 0, -1)$, and F is branched at the point $(0, \ldots, 0, -1)$. A theorem of Alexander [1, 25, p. 316] implies that F is an automorphism of B. This contradicts the fact that F is branched at $(0, \ldots, 0, -1) \in B$, and hence the original map f is unbranched in a neighborhood of the point p^1.

To prove the local version of the theorem as stated above we perform the same construction of domains D^j_k. (See Lemma 1 in [24].) Let $U^j_k \subset D^j_k$ be the subset that corresponds to $U^j \subset D^j$ under the given biholomorphism of D^j onto D^j_k. It follows from the construction in [24] that the sequence U^j_k converges to B as $k \to \infty$ and the map F: $B \to B$ can still be constructed, thus yielding a contradiction exactly as above. For the details we refer the reader to [24].

To apply Theorem 3.2 we choose a point $p^1 \in bD$ and let $p^2 = f(p^1) \in M$. Let Σ be a complex n-plane through p^2 such that the corresponding orthogonal projection $\pi\colon \mathbf{C}^N \to \Sigma$ maps a neighborhood of p^2 in V biholomorphically onto a strictly pseudoconvex domain $D^2 \subset \Sigma$ with C^2 boundary. Let $U^2 = D^2$ and $U^1 = (\pi \circ f)^{-1}(D^2) \subset D = D^1$. By Theorem 3.2 the map $\pi \circ f$ is not branched near p^1 and hence f is not branched near p^1.

This proves that the branching locus of f stays away from the strictly pseudoconvex boundary points of D. In particular, if D is strictly pseudoconvex, then the branching locus of f is compactly contained in D and hence it is finite.

It remains to prove the part (iii) of Theorem 1.1. The restriction

$$f\colon D \setminus f^{-1}(V_{\mathrm{sing}}) \to V \setminus V_{\mathrm{sing}} \tag{3.4}$$

is a proper holomorphic map of n-dimensional complex manifolds, and hence its branching locus is either empty or else it is a subvariety of $D \setminus f^{-1}(V_{\mathrm{sing}})$ of pure dimension $n-1$. Since the second case is excluded by what we have just said above, the map (3.4) is unbranched.

Consider now the extended map

$$f\colon \overline{D} \setminus f^{-1}(V_{\mathrm{sing}}) \to \overline{V} \setminus V_{\mathrm{sing}}\,. \tag{3.5}$$

We fix a point $q \in M = \overline{V} \setminus V$ and choose a simply connected subset $V_0 \subset V \setminus V_{\mathrm{sing}}$ with C^2 strictly pseudoconvex boundary such that for some $\varepsilon > 0$ we have

$$B_\varepsilon(q) \cap V \subset V_0\,. \tag{3.6}$$

Since (3.4) is a covering projection, the inverse image $f^{-1}(V_0)$ is a disjoint union of k connected open subsets $D_1, D_2, \ldots, D_k$ of D such that the restriction of f to D_j is a biholomorphism of D_j onto V_0 for each $j = 1, \ldots, k$. Let D_0 be any of the sets D_j, and denote by $g\colon V_0 \to D_0$ the inverse of $f\colon D_0 \to V_0$. If V_0 is chosen sufficiently small, then V_0 is very close to its projection onto the complex plane $T_q\overline{V}$, and hence property (3.2) of the Kobayashi metric gives an estimate

$$K_{V_0}(w, X) \leqslant |X|/d(w, bV_0) \tag{3.7}$$

for $X \in T^{1,0}_w V_0$. Since $\overline{V}$ is transversal to $b\Omega$ at q, we have

$$d(w, bV_0) \geqslant c_1 d(w, b\Omega) \tag{3.8}$$

for each $w \in V_0$ sufficiently close to q. The estimates (3.7) and (3.8) together imply

$$K_{V_0}(w, X) \leqslant c_2|X|/d(w, b\Omega) \tag{3.9}$$

for each $w \in V_0$ close to q and $X \in \boldsymbol{T}^{1,0}_w D$. Hence

$$c_3|g_* X|/d(g(w), bD)^{\frac{1}{2}} \leqslant K_D(g(w), g_* X) \leqslant K_{V_0}(w, X) \leqslant c_2|X|/d(w, b\Omega) .$$

From this and Lemma 3.1 we obtain an estimate

$$\|dg(w)\| \leqslant c_5/d(w, b\Omega)^{\frac{1}{2}}$$

on the norm of the derivative $dg = g_*$ at the points $w \in V_0$ close to q. This implies that g is Hölder-continuous with the exponent $\frac{1}{2}$ on V_0 near q [8, p. 74] and hence it extends to a Hölder-continuous map on $\overline{V}_0$ near q.

This is true for each local inverse $g_j\colon V_0 \to D_j$. By shrinking V_0 if necessary we may assume that $g_j\colon \overline{V}_0 \to \overline{D}_j$ is a Hölder continuous map that is the inverse of $f\colon \overline{D}_j \to \overline{V}_0$.

Let $V_1 = \overline{V}_0 \cap B_\varepsilon(q)$, where ε is as in (3.6). We claim that

$$f^{-1}(V_1) = \bigcup_{j=1}^{k} g_j(V_1) . \tag{3.10}$$

To prove this, suppose that $f(z)$ lies in V_1 for some $z \in \overline{D}$. Pick a sequence $\{z_\nu\} \subset D$ such that $\lim_{\nu\to\infty} z_\nu = z$. By the continuity of f we have $\lim_{\nu\to\infty} f(z_\nu) = f(z)$. There is a ν_0 such that $f(\nu_0) \in V_0$ for each $\nu \geqslant \nu_0$. Since

$$f^{-1}(V_0) = \bigcup_{j=1}^{k} g_j(V_0) ,$$

it follows that

$$z_\nu = g_j(f(z_\nu)) \tag{3.11}$$

for some $j = j(\nu) \in \{1, \ldots, k\}$. One j has to appear infinitely many times as $\nu \to \infty$. Passing to a subsequence we may assume that (3.11) holds for all ν, with j fixed. Hence

$$z = \lim_{\nu\to\infty} z_\nu = \lim_{\nu\to\infty} g_j(f(z_\nu)) = g_j(\lim_{\nu\to\infty} f(z_\nu)) = g_j(f(z))$$

which implies $z \in g_j(V_0)$. This proves (3.10). Since q was an arbitrary point of M, it follows that (3.5) is a topological covering projection. This completes the proof of Theorem 1.1.

4. – Smooth extension to the boundary.

In this section we shall prove Corollary 1.2 and Theorem 1.3. We will use a local extension theorem for biholomorphic mappings due to Lempert [20, p. 467]:

THEOREM. *Let Ω_1 and Ω_2 de domains in $\mathbb{C}^n$, let $f\colon \Omega_1 \to \Omega_2$ be a biholomorphic map, and let p_j be a point in $b\Omega_j$ for $j = 1, 2$. Assume that*

$$\lim_{\substack{z\in\Omega_1\\ z\to p_1}} = p_2 \quad \text{and} \quad \lim_{\substack{w\in\Omega_2\\ w\to p_2}} = p_1 .$$

If the boundaries $b\Omega_j$ ($j = 1, 2$) are of class C^r and strictly pseudoconvex in some neighborhood of the points p_1 resp. p_2 and if $r \geqslant 6$, then the map f extends to a C^{r-4} map on a neighborhood of p_1 in $\bar{\Omega}_1$.

Assuming this theorem we shall now prove Corollary 1.2. Suppose that the map $f\colon D \to \Omega$ is as in Theorem 1.1. Recall that f extends continuously to $\bar{D}$ by the part (ii) of Theorem 1.1. Choose a point $p_1 \in bD$ and let $p_2 = f(p_1) \in M$. Since M is of class C^r and $f(D) \cup M$ is a C^r manifold with boundary near p_2, we can find a simply connected domain $\Omega_2 \subset f(D)$ with C^r boundary such that

$$B_\varepsilon(p_2) \cap f(D) \subset \Omega_2$$

for some small $\varepsilon > 0$. We may choose Ω_2 so small that the orthogonal projection of $\mathbb{C}^N$ onto the complex n-plane $T_{p_2}\overline{f(D)}$ maps Ω_2 onto a C^r strictly pseudoconvex domain.

We have seen in Section 3 above that the map f has a local inverse g on Ω_2 that is continuous on $\bar{\Omega}_2$ and sends p_2 to p_1. If we let $\Omega_1 = g(\Omega_2) \subset D$, then the continuity of f on $\bar{D}$ implies that

$$B_\delta(p_1) \cap D \subset \Omega_1$$

for some small $\delta > 0$. In particular, a part of $b\Omega_1$ near p_1 coincides with bD, and hence $b\Omega_1$ is of class C^r and strictly pseudoconvex near p_1. The theorem of Lempert implies that f is of class C^{r-4} on $\bar{D}$ near the point p_1. Since $p_1 \in bD$ was chosen arbitrarily, f is of class C^{r-4} on $\bar{D}$.

NOTE. The same conclusion applies to each local inverse of f near M, and hence the map

$$f\colon \bar{D}\setminus f^{-1}(V_{\text{sing}}) \to \bar{V}\setminus V_{\text{sing}}$$

is a C^{r-4} covering projection (here $V = f(D)$).

PROOF OF THEOREM 1.3. Recall that $S \subset i(M)$ is the set of non-smooth points of $i(M)$. Let $V = f(D)$. We have seen in the proof of Theorem 1.1 that $\bar{V} \subset V \cup i(M)$. We claim that $\bar{V}$ cannot be contained in $V \cup S$. Since $\mathcal{H}^{2n-1}(S) = 0$, the assumption $\bar{V} \subset V \cup S$ would imply that $\bar{V}$ is a complex subvariety of $\mathbf{C}^N$ according to a theorem of Shiffman [28, p. 11]. Since $\bar{V}$ is compact, this is a contradiction. Hence the set $\bar{V} \cap i(M)\setminus S$ is not empty, and the proof of part (i) of Theorem 1.1 shows that $V \cup i(M)$ is a local C^r manifold with boundary near each point $p \in i(M)\setminus S$. Moreover, the set $\bar{V} \cap i(M)\setminus S$ is open and closed in $i(M)\setminus S$. Since $i(M)\setminus S$ is assumed to be connected, it follows that $i(M)\setminus S \subset \bar{V}$, and the immersion i is maximally complex at each point $x \in M$ for which $i(x) \notin S$. Further, because of $\mathcal{H}^{2n-1}(S) = 0$ the set S is nowhere dense in $i(M)$, hence by Lemma 2.1 the immersion i is maximally complex on all of M and we have $\bar{V} = V \cup i(M)$.

It remains to consider the structure of $\bar{V}$ at the points of S. Fix a point $p \in S$ and choose local coordinates in $\mathbf{C}^N$ near p such that $p = 0$, $b\Omega$ is strictly convex near 0, $T_0 b\Omega = \{x_1 = 0\}$ and $\Omega \subset \{x_1 < 0\}$. If we choose a sufficiently small $\varepsilon > 0$ and let $U = \{x_1 > -\varepsilon\}$, then

$$i(M) \cap U = M_1 \cup \ldots \cup M_s\,, \tag{4.1}$$

where each M_j is a closed connected submanifold of U. Since $b\Omega$ is strictly convex and each M_j is a maximally complex submanifold of $b\Omega$, we can choose ε so small that each M_j bounds a closed irreducible complex subvariety V_j of $U \cap \Omega$, and $V_j \cup M_j$ is a C^r manifold with boundary M_j. At every point $q \in M_j\setminus S$ the manifold M_j also bounds the variety V. It follows that V_j is an irreducible component of $V \cap U$, and hence

$$V_1 \cup V_2 \cup \ldots \cup V_s \subset V\,.$$

We claim that

$$V_1 \cup V_2 \cup \ldots \cup V_s = f(D) \cap U\,. \tag{4.2}$$

Suppose that there is another irreducible component V_0 of $V \cap U$. If $\bar{V}_0 \cap U$ contains a point $q \in M_j \setminus S$ for some $j = 1, \ldots, s$, then we have $V_0 = V_j$ which is a contradiction. Hence $\bar{V}_0 \cap U$ is contained in $V_0 \cup S$. The theorem of Shiffman [28, p. 111] implies that $\bar{V}_0 \cap U$ is a closed complex subvariety of U. Since $U = \{x_1 > -\varepsilon\}$, the plurisubharmonic function x_1 assumes its maximum on $\bar{V}_0$ which is a contradiction to the maximum principle [12]. This proves (4.2) and hence part (i) of Theorem 1.3.

The proof that we have given in Section 3 above shows that f extends to a Hölder continuous map on $\bar{D}$. Fix a point $p \in bD$. We will show that f is not branched in a neighborhood of p in D. Let $q = f(p) \in M$. Choose a neighborhood U of q in $\mathbb{C}^N$ such that (4.1) and (4.2) hold. The preimage $f^{-1}(U) \subset D$ has exactly one connected component D_1 such that $B_\delta(p) \cap D \subset D_1$ for some $\delta > 0$. The restriction $f\colon D_1 \to U \cap \Omega$ is a proper map and hence (4.2) implies that $f(D_1) = V_j$ for some j. If we apply Theorem 3.2 to the proper map

$$f\colon D_1 \to V_j\,,$$

we conclude that f is not branched near the point p. This proves the part (ii) of Theorem 1.3.

If we choose the set U in (4.2) sufficiently small, then the map (4.3) is a biholomorphism, and we can see the same way as in Section 3 above that the local inverse

$$g = f^{-1}\colon V_j \to D_1$$

extends to a Hölder continuous map on $\bar{V}_j$ near q. If $r \geqslant 6$, the theorem of Lempert implies that the map (4.3) is of class C^{r-4} on a neighborhood of p in $\bar{D}$. Since the point $p \in bD$ was arbitrary, f is of class C^{r-4} on $\bar{D}$ and Theorem 1.3 is proved.

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