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On Perturbed Wave Equations with Time-Dependent Coefficients (*).

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1. – Introduction.

In this note we shall study special properties of classical solutions of perturbed wave equations by « impurities » which depend upon both position and time. More precisely we shall consider the wave equation

$$\Box u + q(x, t) u = 0 \quad \text{in } \mathbf{R}^n \times \mathbf{R} \tag{1.1}$$

($n \geqslant 3$) with C^∞ compactly supported initial data at $t = 0$. Here $\Box$ denotes the d'Alembertian operator, i.e. $\Box = \partial^2/\partial t^2 - \Delta$ where Δ is the Laplacian operator. In general, energy conservation for (1.1) is ruled out because of the time dependence of q. In fact, let us assume that q is (real-valued) smooth, and let us multiply equation (1.1) by U_t and integrate in space to obtain after integration by parts

$$\frac{d}{dt} E_\infty(t) = \int q_t u^2 dx \tag{1.2}$$

where $E_\infty(t) = \frac{1}{2}\int\left(u_t^2 + |\text{grad}\, u|^2 + qu^2\right)dx$ and q_t denotes the partial derivative of q with respect to t. From (1.2) we see that our chances to conclude that $E_\infty(t)$ is constant, are very limited. A natural question arises: Under which assumptions on q, the total energy $E_\infty^-(t)$ will be *bounded* for all time? In section 2 we shall answer this question. Now, let $R > 0$ and let us define

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the «local energy» $E_R(t)$ associated with (1.1) by

$$E_R(t) = \frac{1}{2} \int_{|x|\leq R} (u_t^2 + |\operatorname{grad} u|^2 + qu^2)\, dx\,. \tag{1.3}$$

In special cases, a way to answer the above question is to study the integrability of the local energy. In order to illustrate this statement let us consider the case in which q has compact support in x, contained in a fixed ball, say $\{|x| \leqslant M\}$ for all t. Furthermore, let us assume that q is bounded in t. From (1.2) we obtain

$$\frac{d}{dt} E_\infty(t) = \int_{|x|\leqslant M} \left[\frac{d}{dt}(qu^2) - 2quu_t\right] dx\,. \tag{1.4}$$

Let us integrate (1.4) from $t = 0$ to $t = T > 0$ to obtain

$$E_\infty(T) = E_\infty(0) + \int_{|x|\leqslant M} [q(x, T)u^2(x, T) - q(x, 0)u^2(x, 0)]dx - 2\int_0^T \int_{|x|\leqslant M} quu_t\, dx\, dt\,. \tag{1.5}$$

By using Schwarz's inequality we get

$$\int_{|X|\leqslant M} quu_t dx \leqslant 2\|q\|_\infty^{\frac{1}{2}} E_M(t) \qquad \text{where} \qquad \|q\|_\infty = \sup_{\substack{|x|\leqslant M \\ t\geqslant 0}} |q(x, t)|\,.$$

Hence, from (1, 2) we conclude that

$$E_\infty(T) \leqslant E_\infty(0) + 2E_M(T) + \int_{|x|\leqslant M} q(x, 0)u^2(x, 0)\, dx + 4\|q\|_\infty^{\frac{1}{2}} \int_0^T E_M(t)\, dt \tag{1.6}$$

which implies that $E_\infty(T)$ will be bounded for all $T \geqslant 0$ provided that $E_M(t) \in L^\infty \cap L^1[0, \infty)$. In section 2 we shall prove that the local energy $E_M(t)$ is *integrable* on $[0, \infty)$ under suitable assumptions on q. In Section 3 we consider a related problem: We prove that, in general, classical solutions of (1.1) *do not enjoy the Huygens' principle* in three-space dimensions. Unfortunately we did not succeed in extending this result to higher space dimensions because we strongly use the positivity of the Riemann function (for the free wave equation) in three-space dimensions. The results in this section are an extension of the techniques we used in a joint work with

Professor T. Schonbek in [9] for time-independent potentials. Finally in Section 4 we use our results of Section 2 to show that in the presence of symmetries, we still can prove similar results to the ones given in Section 3, for higher space dimensions.

Related problems have been considered by a number of authors. The case where q depends only on space have been considered by O. Ladyzhenskaya [4], D. Thoe [13], E. Zachmanoglou [14] among others. For the wave equation on exterior domains the asymptotic behavior of the local energy have been studied by C. Morawetz [7], P. Lax and R. Phillips [5], W. Strauss [11], C. Morawetz - W. Strauss - J. V. Ralston [8] and more recently by R. Melrose [6]. The time-dependent case was considered by C. Bloom and N. Kazarinoff [1] but their paper does not include equation (1.1). Recently H. Tamura [12] studied the exponential decay of the local energy when q has compact support in x (contained in a fixed ball), for all t. His paper follows the main ideas—for a related problem—introduced by J. Cooper and W. Strauss [2] and W. Strauss [11].

As far as we know, a rigorous proof of the non-validity of the Huygens' principle for solutions of equation (1.1) is not available in the literature. In what follows we shall use the standard notation. An integral sign to which no domain is attached will be understood to be taken over all $\mathbf{R}^m$. We shall denote by grad u the gradient of u with respect to space variables. Let $F\colon \mathbf{R}^n \to \mathbf{R}^n$ be a smooth field. We shall denote by div F the divergence of F with respect to the space variables. By $C_0^\infty(\mathbf{R}^n)$ we denote the space of all C^∞ functions defined on $\mathbf{R}^n$, with compact support. We shall always assume that q is smooth enough in order to have smooth solutions of (1.1). Since equation (1.1) is reversible in time we shall perform the estimates only for $t\geqslant 0$ and the same will be true for $t\leqslant 0$. All functions consider in this note will be real-valued. The expression dS_y will always mean the surface measure with respect to the variable y.

I would like to express my gratitude to Professor Walter A. Strauss for introducing me to this line of research. Although I did not succeed in solving the original problem he suggested-namely the existence of scattering frequencies for equation (1.1)—I think I'm very close to do it.

2. – The local and total energy.

In this section we study the asymptotic behavior of the local energy $E_R(t)$ for large values of t. We shall always assume that q is smooth and the initial data at time $t = 0$ belongs to $C_0^\infty(\mathbf{R}^n)$ $(n\geqslant 3)$.

LEMMA 1. *Let* $M(u) = (t^2 + r^2)u_t + 2rtu_r + (n-1)tu$ *then*

$$M(u)[\Box u + qu] = \frac{\partial A}{\partial t} + \operatorname{div} B + D \tag{2.1}$$

where

$$A = (t^2 + r^2)e(u) + 2tru_r u_t + (n-1)\,tuu_t - \frac{(n-1)}{2}u^2$$

$$B = 2tx(e(u) - u_t^2) + [(n-1)tu + (t^2+r^2)u_t + 2tru_r]\operatorname{grad} u$$

$$D = -u^2\left[2tq + trq_r + \frac{t^2+r^2}{2}q_t\right]$$

and $e(u) = \frac{1}{2}(u_t^2 + |\operatorname{grad} u|^2 + qu^2)$, $r = |x|$, $u_r = x/r\cdot\operatorname{grad} u$.

PROOF. It is easily verified so we omit the proof.

REMARK. By using the identity

$$\operatorname{div}\left(\frac{(n-1)}{4r^2}x(r^2+t^2)u^2\right) = \frac{(n-1)}{2}u^2 + \frac{(n-1)(t^2+r^2)}{4r^2}[(n-2)u^2 + 2ruu_r]$$

and lemma 1 the conclusion (2.1) can be written as

$$M(u)[\Box u + qu] = \frac{\partial \tilde{A}}{\partial t} + \operatorname{div}\tilde{B} + \tilde{D} \tag{2.2}$$

where

$$\tilde{A} = (t^2+r^2)e(u) + 2tru_r u_t + (n-1)tuu_t + \frac{(n-1)(t^2+r^2)}{4r^2}[(n-2)u^2 + 2ruu_r]\,\tilde{B} = B - \left[\frac{(n-1)}{4r^2}x\frac{\partial}{\partial t}(t^2+r^2)u^2\right]$$

and $\tilde{D} = D$.

LEMMA 2. *Let* $\tilde{A}$ *as in* (2.2) *and* $q \geq 0$. *Then*

a) $\tilde{A} \geq 0$

and

b) $\tilde{A} \geq \frac{t^2}{4}e(u) + \frac{(n-1)t^2}{8}\left[\operatorname{div}\left(\frac{xu^2}{2r^2}\right) + \frac{3(n-3)}{4r^2}u^2\right]$

for all $r \leq t/2$ and $n \geq 3$.

PROOF. We can write $\tilde{A}$ as

$$(2.3)\qquad \tilde{A}=\frac{1}{4r^{n-1}}\Big\{(r+t)^2\,[(r^{n-1/2}u)_r+(r^{n-1/2}u)_t]^2 + (r-t)^2[(r^{n-1/2}u)_r-(r^{n-1/2}u)_t]^2\Big\}+\frac{(t^2+r^2)}{2}\,[|\mathrm{grad}\,u|^2+qu^2-u_r^2] + \frac{(n-1)(n-3)}{8r^2}\,(t^2+r^2)u^2\,.$$

Identity (2.3) can easily be verified. Each summand in (2.3) is non-negative, thus $\tilde{A}\geqslant 0$. Identity (2.3) is a simple generalization of the one given by C. Morawetz in the Appendix of [5] for the case $n=3$. If $r\leqslant t/2$ it follows from (2.3) that

$$(2.4)\qquad \tilde{A}\geqslant\frac{t^2}{16r^{n-1}}\{[(r^{n-1/2}u)_r+(r^{n-1/2}u)_t]^2+[(r^{n-1/2}u)_r-(r^{n-1/2}u)_t]^2\} + \frac{t^2}{2}\,[|\mathrm{grad}\,u|^2+qu^2-u_r^2]+\frac{(n-1)(n-3)}{8r^2}\,t^2u^2$$
$$=\frac{t^2}{8r^{n-1}}\Big\{\Big(\frac{n-1}{2}\Big)^2 r^{n-3}u^2+r^{n-1}u_r^2+(n-1)r^{n-2}uu_r+r^{n-1}u_t^2 + \frac{t^2}{2}\,[|\mathrm{grad}\,u|^2+qu^2-u_r^2]+\frac{(n-1)(n-3)}{8r^2}\,t^2u^2\,.$$

Direct simplications in (2.4) give us for $r\leqslant t/2$ that

$$(2.5)\qquad \tilde{A}\geqslant\frac{t^2}{4}\,e(u)+\frac{(n-1)}{8}\,t^2\left[\frac{uu_r}{r}+\frac{(5n-13)}{4r^2}\,u^2\right].$$

Now, we use the identity

$$(n-1)\,\frac{uu_r}{r}=\mathrm{div}\left(\frac{(n-1)}{2r^2}\,x\,u^2\right)-\frac{(n-1)(n-2)}{2r^2}\,u^2\,.$$

Substitution in (2.5) proves item *b*).

THEOREM 1. *Let u be a solution of equation* (1.1) *with $C_0^\infty(\mathbf{R}^n)$ initial data at $t=0$. Assume that q satisfies a) $q\geqslant 0$, b) $|q_t|\leqslant\big(2rt/(t^2+r^2)\big)q$ and c) $(2+r)q+rq_r\leqslant 0$ for all $r>0$, $t\geqslant 0$.*

Then, for any $T>0$ we have

1) $E_{T/2}(T)\leqslant CT^{-2}$

and

$$2)\ \lim_{T\to\infty}\int_{|x|=T/2} u^2(x,T)\,dS = 0$$

where C is a positive constant which depends only on the initial data at $t=0$.

PROOF. Integration in space (all $\mathbb{R}^n$) of identity (2.2) gives

$$\frac{d}{dt}\int \tilde{A}\,dx = \int u^2\left[2tq + trq_r + \frac{t^2+r^2}{2}q_t\right]dx\,.$$

By using assumptions *b*) and *c*) we conclude that $d/dt\int\tilde{A}\,dx\leqslant 0$. Integration from $t=0$ to $t=T$ gives us

$$\int\tilde{A}(T)\,dx\leqslant\int\tilde{A}(0)\,dx\,. \tag{2.6}$$

The right hand side of (2.6) is a constant which depends only on the initial data. By lemma 2 we know that $\tilde{A}\geqslant 0$, thus from (2.6) we obtain

$$\int_{|x|\leqslant T/2}\tilde{A}(T)\,dx\leqslant \text{Constant}\,. \tag{2.7}$$

Both results in theorem 1 now follow from (2.7) by using part *b*) in lemma 2. Next we want to prove that, under suitable assumptions, on q, the total energy $E_\infty^-(t)$ is bounded for all time $t\geqslant 0$. We shall use the same multiplier $M(u)$ used by J. Cooper and W. Strauss in [2] for a related problem. Let $\varrho=\varrho(|x|)$ be a C^∞ function which depends only on the radius $r=|x|$ and satisfies the following conditions: 1) $\varrho''\leqslant 0$, 2) $0\leqslant r\varrho'\leqslant\varrho\leqslant\varepsilon<1$ for some $0<\varepsilon<1$.

LEMMA 3. *Let $n\geqslant 3$ and $M_\varrho(u)=u_t+\varrho u_r+\big((n-1)/2r\big)\varrho u$ then*

$$M_\varrho(u)\big(\Box u+q(x,t)u\big)=\frac{\partial A}{\partial t}+\operatorname{div}B+D$$

where

$$A=e(u)+\varrho u_t\left(u_r+\frac{n-1}{2r}u\right)$$

$$B=-\operatorname{grad}u\,M_\varrho(u)+\frac{x\varrho}{2r}\left[|\operatorname{grad}u|^2-u_t^2\right]+\frac{(n-1)}{4r^2}\left(\varrho'-\frac{\varrho}{r}\right)x\,u^2+\frac{\varrho}{2r}\,x\,qu^2$$

and

$$D = \varrho' e(u) + \left(\frac{\varrho}{r} - \varrho'\right)[|\text{grad}\, u|^2 - u_r^2]$$
$$+ \left[\left(\frac{\varrho}{r} - \varrho'\right)\frac{(n-1)(n-3)}{4r^2} - \varrho'' \frac{n-1}{4r} - \frac{1}{2} q_t - q\varrho' - \frac{1}{2}\varrho q_r\right] u^2$$

where $e(u) = \frac{1}{2}(u_t^2 + |\text{grad}\, u|^2 + qu^2)$.

The proof is a direct calculation.

THEOREM 2. (Energy boundedness) *Let* $q: \mathbf{R}^{n+1} \to \mathbf{R}$ *satisfy the assumptions 1)* $q \geqslant 0$ *and 2)* $q_t + \varepsilon q_r \leqslant \varepsilon c_n / r^3$ *for some* $0 < \varepsilon < 1$ *where* $c_n = (n-1)(n-3)/8$. *Let* u *be a solution of equation* (1.1) *with* $C_0^\infty(\mathbf{R}^n)$ *initial data at* $t = 0$. *Then the total energy* $E_\infty(t) = \int e(u)dx$ *is bounded for all time* $t \geqslant 0$.

PROOF. We use lemma 3 with $\varrho \equiv \varepsilon$. We can write A in lemma 3 as

$$A = (1-\varepsilon)e(u) + \frac{\varepsilon c_n}{4r^2} u^2 + \text{div}\left(-\frac{(n-1)}{4r^2}\varepsilon x u^2\right)$$
$$+ \varepsilon\left[e(u) + \frac{n-1}{2r} uu_r + \frac{(n-1)^2}{8r^2} u^2 + \left(u_r + \frac{n-1}{2r} u\right) u_t\right]. \tag{2.8}$$

Since

$$\pm\left(u_r + \frac{n-1}{2r} u\right) u_t \leqslant \frac{1}{2}\left|u_r + \frac{n-1}{2r} u\right|^2 + \frac{1}{2} u_t^2$$

and

$$\frac{1}{2}\left|u_r + \frac{n-1}{2r} u\right|^2 = \frac{1}{2}\left|\frac{x}{r}\cdot \text{grad}\, u\right|^2 + \frac{(n-1)^2}{8r^2} u^2 + \frac{(n-1)}{2r} uu_r$$

then it follows that

$$e(u) + \frac{(n-1)^2}{8r^2} u^2 + \frac{(n-1)}{2r} uu_r \pm \left(u_r + \frac{n-1}{2r} u\right) u_t \geqslant 0$$

because $q \geqslant 0$. Thus the last term in (2.8) is non-negative. Integration of (2.8) in space gives us (for $n > 3$ and $T > 0$)

$$\int A(T)dx = (1-\varepsilon)E_\infty(T) + \frac{\varepsilon c_n}{4}\int \frac{u^2(x,T)}{r^2} dx$$
$$+ \varepsilon \int\left[e(u) + \frac{n-1}{2r} uu_r + \frac{(n-1)^2}{8r^2} u^2 + \left(u_r + \frac{(n-1)u}{2r}\right) u_t\right]_{t=T} dx. \tag{2.9}$$

Each term in (2.9) is non-negative, thus we conclude that

$$\int A(T)\,dx \geqslant (1-\varepsilon)E_\infty(T)\,. \tag{2.10}$$

Similarly, we can write A in lemma 3 as

$$A = (1+\varepsilon)e(u) - \frac{\varepsilon_1 c_n}{4r^2}u^2 + \operatorname{div}\left(\frac{(n-1)}{4r^2}\varepsilon \times u^2\right) - \varepsilon\left[e(u) + \frac{(n-1)}{2r}uu_r + \frac{(n-1)^2}{8r^2}u^2 - \left(u_r + \frac{n-1}{2r}u\right)u_t\right]. \tag{2.11}$$

By the same reason we explained above, the last term in (2.11) is $\leqslant 0$. Thus, integration in space of (2.11) at time $t=0$, yields (for $n>3$)

$$\int A(0)\,dx \leqslant (1+\varepsilon)E_\infty(0)\,. \tag{2.12}$$

Now we use lemma 3 and we obtain after integrating in space

$$0 = \frac{d}{dt}\int A\,dx + \int D\,dx \geqslant \frac{d}{dt}\int A\,dx \tag{2.13}$$

because $D \geqslant 0$ due to our assumption 2) on q. By integrating (2.13) from $t=0$ to $t=T$ and using (2.10) together with (2.12) we obtain

$$E_\infty(T) \leqslant \left(\frac{1+\varepsilon}{1-\varepsilon}\right)E_\infty(0)$$

for any $T>0$, which proves the theorem. In the case $n=3$ we have in (2.13) an additional term which is $2\pi\varepsilon u^2(0,t)$. Since it is non-negative the same conclusion holds. Next we want to study the integrability of the local energy $E_R(t)$ on $(0,\infty)$.

LEMMA 4. *Let A and D be as in lemma 3. Let ϱ be chosen as we did before lemma 3. Assume that $q\colon \mathbb{R}^{n+1} \to \mathbb{R}$ satisfies*

1) $q \geqslant 0$

2) $q_t + 2\varrho' q + q_r \varrho + \varrho'' \dfrac{(n-1)}{2r} \leqslant \left(\dfrac{\varrho}{r} - \varrho'\right)\dfrac{c_n}{r^2}$

where $c_n = (n-1)(n-3)/2$. Let u be a solution of equation (1.1) with $C_0^\infty(\mathbb{R}^n)$ *initial data at* $t = 0$. *Let* $R > 0$ *and assume* $\varrho'(R) \neq 0$, *then*

a) $\int A(T)\,dx \geqslant 0$,

b) $\int A(0)\,dx \leqslant \int (1+\varrho)e(u)|_{t=0}dx$

and

c) $\int_0^\infty E_R(t)\,dt < +\infty$

where $E_R(t)$ *is given by* (1.3).

PROOF. Clearly $D \geqslant 0$ because of assumptions 1) and 2) on q. Observe that A can be written as

$$A = (1-\varrho)e(u) + \varrho\frac{c_n}{4r^2}u^2 + \varrho'\frac{(n-1)}{4r}u^2 + \operatorname{div}\left(-\frac{(n-1)}{4r^2}\varrho\, xu^2\right) + \varrho\left[e(u) + \frac{n-1}{2r}uu_r + \frac{(n-1)^2}{8r^2}u^2 + \left(u_r + \frac{n-1}{2r}u\right)u_t\right]. \tag{2.14}$$

By observing the proof of theorem 2 we see that the same idea applies here to conclude that the last term in (2.14) is non-negative. Thus, if we consider $A(T)$ for $T > 0$ and integrate (2.14) in space we conclude *a*). Similarly, we can write A as

$$A = (1+\varrho)e(u) - \frac{\varrho c_n u^2}{4r^2} - \varrho'\frac{(n-1)}{4r}u^2 + \operatorname{div}\left(\frac{n-1}{4r^2}\varrho\, xu^2\right) - \varrho\left[e(u) + \frac{(n-1)}{2r}uu_r + \frac{(n-1)^2}{8r^2}u^2 - \left(u_r + \frac{n-1}{2r}u\right)u_t\right]. \tag{2.15}$$

Now we consider $A(0)$ and integrate (2.15) in space to conclude the validity of *b*) because the last term in (2.15) is $\leqslant 0$. In the case $n = 3$, the same comments as in theorem 2 apply. Integration of the main identity in lemma 3 gives us

$$\int D\,dx + \frac{d}{dt}\int A\,dx = 0\,. \tag{2.16}$$

Now, let $T > 0$ and let us integrate (2.16) from $t = 0$ to $t = T$. After using part b) we obtain

$$\int_0^T\int D\,dx\,dt + \int A(T)\,dx = \int A(0)\,dx \leqslant \int (1+\varrho)e(u)\Big|_{t=0} dx \leqslant (1+\varepsilon)E_\infty(0)\,.$$

By part a) we know that $\int A(T)\,dx \geqslant 0$, thus

$$\int_0^T\int D\,dx\,dt \leqslant (1+\varepsilon)E_\infty(0)\,.$$

Since $D \geqslant 0$ we conclude that

$$\int_0^T\int_{|x|\leqslant R} D\,dx\,dt \leqslant (1+\varepsilon)E_\infty(0)\,. \tag{2.17}$$

Since each term in D is non-negative then from (2.17) we obtain

$$\varrho'(R)\int_0^T E_R(t)\,dt \leqslant \int_0^T dt \int_{|x|\leqslant R} \varrho' e(u)\,dx \leqslant (1+\varepsilon)E_\infty(0)\,. \tag{2.18}$$

Letting $T \to +\infty$ in (2.18) we obtain the desired result.

3. – Nonvalidity of Huygens' principle for (1.1). The case $n = 3$.

In this section we shall assume that $q(x, t)$ satisfies the following assumptions

1) $q\colon \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$, q is smooth in x and t.

2) For each t, $q = q(x, t)$ has compact support in x and its support is contained in a fixed ball $\{x \in \mathbb{R}^3,\ |x| \leqslant \mathbb{R}\}$.

3) There exists $\varepsilon > 0$ ($\varepsilon < R$) such that $q \geqslant 0$ for all $|x| \geqslant R - \varepsilon$ and all t (We shall assume that $q > 0$ for all $R - \varepsilon < |x| < R$).

DEFINITION 1. *Let $f\colon \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ be a continuous function. We define $[f]$ as $[f] = \operatorname{Inf}\{a \in \mathbb{R}$, such that $f(x, t) = 0$ for all (x, t), $|x| \leqslant t - a$, $t \geqslant a\}$.*

REMARK. Clearly $[f] < +\infty$ if and only if f vanishes in a forward cone.

DEFINITION 2. *Let* $f\colon \mathbb{R}^3\times\mathbb{R}\to\mathbb{R}$ *be a continuous function. Let* $b,\ \delta>0$ *we say that* $f\in\mathcal{F}(b,\delta)$ *if*

1) $[f]\leqslant b$

and

2) $f(x,t)\geqslant 0$ *for all* (x,t) *such that* $|x|\leqslant t-b+\delta$. (But not identically zero there).

DEFINITION 3. *Let* $f\colon \mathbb{R}^3\times\mathbb{R}\to\mathbb{R}$ *be a continuous function and* $b\in\mathbb{R}^+$. *We say that* $f\in\mathcal{F}^+(b)$ *if*

1) $f\in\mathcal{F}(b,2\varepsilon)$, *where* $\varepsilon>0$ *is the number satisfying assumption* 3) *on* q.

2) $f(x,t)\neq 0$, (x,t) *in the region*

$$\Omega_{\varepsilon,b}=\{(x,t),\ R-\varepsilon<|x|<R,\ -|x|+b+2(R-\varepsilon)<t<|x|+b\}\,.$$

LEMMA 5. *If* $f\in\mathcal{F}^+(b)$ *for some* $b\in\mathbb{R}^+$ *then* $[f]=b$.

PROOF. Clearly $[f]\leqslant b$ by definition of $\mathcal{F}^+(b)$. Since $f(x,t)=0$ if $|x|\leqslant t-[f]$ and $f(x,t)\neq 0$ for $(x,t)\in\Omega_{\varepsilon,b}$ it follows that for $(x,t)\in\Omega_{\varepsilon,b}$ we have $|x|>t-[f]$. Let $|x|\to R$ and $t\to R+b$ to get $R\geqslant R+b-[f]$ which implies that $[f]\geqslant b$ which proves the lemma.

LEMMA 6. *Let* $f\colon \mathbb{R}^3\times\mathbb{R}\to\mathbb{R}$ *be a continuous function, let* $b\in\mathbb{R}^+$, R *and* ε *be as in the hypotheses on* q. *We have*

1) *If* $[f]\leqslant b$ *then* $[L_q(f)]\leqslant b+2R$.

2) *Let* $\delta>0$. *If* $f\in\mathcal{F}(b,\delta)$ *then*

$$L_q(f)\in\mathcal{F}\big((b+2R),\delta_1\big)$$

where $\delta_1=\min\{\delta,2\varepsilon\}$.

3) *If* $f\in\mathcal{F}^+(b)$ *then* $L_q(f)\in\mathcal{F}^+(b+2R)$.

In particular $[L_q(f)]=b+2R$. *Here* $L_q(f)$ *is given by*

$$L_q(f)(x,t)=\int\limits_{|y|\leqslant t}|y|^{-1}q\big(x+y,t-|y|\big)f\big(x+y,t-|y|\big)dy\,. \tag{3.1}$$

PROOF. 1) Let $[f]\leqslant b$ and (x,t) be such that

$$|x|\leqslant t-(b+2R)\,. \tag{3.2}$$

Let $y\in\mathbb{R}^3$ such that $|y|\leqslant t$. If $f(x+y,t-|y|)\neq 0$ for such y, then by the definition of $[f]$ we have $|x+y|\geqslant t-|y|-[f]$. Thus,

$$|x+y|\geqslant t-|y|-[f]\geqslant t-|y|-b\geqslant 2R+|x|-|y|\geqslant 2R-|x+y|$$

because (x,t) satisfy (3.2). Hence $|x+y|\geqslant R$, thus $q(x+y,t-|y|)=0$. This implies that $L_q(f)(x,t)=0$ for all such (x,t) and the proof of 1) is complete.

2) Let (x,t) such that $|x|\leqslant t-(b+2R)+\delta_1$. We want to show that $L_q(f)(x,t)\geqslant 0$ (but not identically zero there). Let $y\in\mathbb{R}^3$, $|y|\leqslant t$. Assume $q(x+y,t-|y|)f(x+y,t-|y|)\neq 0$. Thus x, y and t have to satisfy:

$$|x+y|\leqslant R\,,\qquad |x+y|\geqslant t-|y|-b\qquad\text{and}\qquad |x|\leqslant t-(b+2R)+\delta_1\,. \tag{3.3}$$

From these inequalities we can deduce that

$$\begin{aligned}|x+y|\geqslant t-|y|-b\geqslant 2R+|x|-|y|-\delta_1\geqslant 2R-|x+y|-\delta_1\\ \geqslant 2R-|x+y|-2\varepsilon\end{aligned}$$

i.e. $|x+y|\geqslant R-\varepsilon$, hence $q(x+y,t-|y|)\geqslant 0$. Let x, y and t satisfy (3.3). Then we clearly obtain the inequalities

$$2|x+y|\leqslant 2R\leqslant t-b-|x|+\delta_1\leqslant(t-b+\delta_1)+|x+y|-|y|\,.$$

Thus $|x+y|\leqslant(t-b+\delta_1)-|y|$. In particular, $|x+y|\leqslant(t-|y|)+\delta-b$. Since $f\in\mathcal{F}(b,\delta)$ then $f(x+y,t-|y|)\geqslant 0$ for such x, y and t. This proves $qf\geqslant 0$ for such x, y and t's which concludes the proof of 2).

3) By 2) we know that $L_q(f)\in\mathcal{F}(b+2R,2\varepsilon)$. It remains only to show that $L_q(f)(x,t)\neq 0$ for all (x,t) in $\Omega_{\varepsilon,b+2R}$. For such (x,t) we have, in particular that $|x|\leqslant t-(b+2R)+2\varepsilon$, so by the proof of part 2) we know that $L_q(f)(x,t)\geqslant 0$. In order to finish the proof we only have to show that there exist $y_0\in\mathbb{R}^3$, $|y_0|\leqslant t$ such that $q(x+y_0,t-|y_0|)\neq 0$ and $f(x+y_0,t-|y_0|)\neq 0$. Let $(x,t)\in\Omega_{\varepsilon,b+2R}$ and let us choose $\lambda>0$ such that *a)* $0<\lambda<R-|x|$ and *b)* $2\lambda<2R+b-|x|-t$. Define y_0 to be $y_0=(R-\lambda)(x/|x|)-x$. Clearly $|y_0|\leqslant t$ and since $R>R-\lambda=|x+y_0|$

$> R - \varepsilon$ then it follows that $q(x + y_0, t - |y_0|) \neq 0$. By using our assumptions on f and our choice of y_0 and λ we can easily check that $f(x + y_0, t - |y_0|) \neq 0$. This proves 3).

LEMMA 7. *Let* $f \in \mathcal{F}^+(b)$ *for some* $b \geq 0$ *then* $[L_q^m(f)] = b + 2mR$, $m = 0, 1, 2, \ldots$.

PROOF. By lemma 5 it is enough to prove that $L_q^m(f) \in \mathcal{F}^+(b + 2mR)$. This can be done by induction. If $m = 0$ it is true because $f \in \mathcal{F}^+(b)$. Assume $L_q^m(f) \in \mathcal{F}^+(b + 2mR)$. By part 3) of lemma 6 we know that $L_q(L_q^m(f)) \in \mathcal{F}^+(b + 2Rm + 2R)$, so $L_q^{m+1}(f) \in \mathcal{F}^+(b + 2(m+1)R)$, which completes the proof.

LEMMA 8. *Let* $g = g(x) \in C_0^\infty(\mathbb{R}^3)$, $g \geq 0$ *and* $v = v(x, t)$ *be the solution of the Cauchy problem* $\Box v = 0$ *in* $R^3 \times \mathbb{R}$, $v(x, 0) = 0$, $v_t(x, 0) = g(x)$; *then* $v \in \mathcal{F}^+(b)$ *for some* b.

PROOF. Since v is given by $v(x, t) = t/4\pi \int_{|y|=1} g(x + ty)\, dS_y$ then clearly we can choose $b = R$ and $\varepsilon = R/2$ to verify the conclusion of the lemma.

THEOREM 3. *Let* $q = q(x, t)$ *satisfy all of the hypothesis given in the beginning of this section. Let* $g \in C_0^\infty(\mathbb{R}^3)$ *and* $g \geq 0$. *Let* u *be the solution of*

$$\Box u + zq(x, t)u = 0 \quad \text{in } \mathbb{R}^3 \times \mathbb{R} \tag{3.4}$$

$$u(x, 0) = 0 \qquad u_t(x, 0) = g(x)\,;$$

then there exists a countable set $\mathcal{O}$ *of the complex plane such that if* $z \notin \mathcal{O}$ *then the solution* u *of* (3.4) *does not vanish identically in any forward cone. In particular the Huygens' principle does not hold for* (3.4) *if* $z \notin \mathcal{O}$.

PROOF. Let $v = v(x, t)$ be the solution of $\Box v = 0$ in $\mathbb{R}^3 \times \mathbb{R}$, $v(x, 0) = 0$, $v_t(x, 0) = g(x)$. By lemma 8 we know that $v \in \mathcal{F}^+(b)$ for some $b \in \mathbb{R}^+$. The candidate to be the set $\mathcal{O}$ is

$$\mathcal{O} = \{z \in \mathbf{C}, [u] < +\infty\} = \bigcup_{m=1}^{\infty} \{z \in \mathbf{C}, [u] \leq m\}.$$

To conclude the proof it only remains to prove that $\mathcal{O}$ is countable. Suppose it is not. Then there exists m_0 such that the set

$$\mathcal{O}_{m_0} = \{z \in \mathbf{C}, [u] \leq m_0\}$$

is uncountable. Let (x, t) be any point such that $|x| < t - m_0$. If $z \in \mathcal{O}_{m_0}$ then clearly $|x| \leqslant t - m_0 \leqslant t - [u]$, thus by the definition of $[u]$ we have that $u = 0$ in any (x, t, z) as above.

In other words, the map

$$\mathbb{C} \xrightarrow{\psi} \mathbb{C}$$

$$z \longrightarrow u(x, t, z)$$

vanishes identically in $\mathcal{O}_{m_0}$ (for each (x, t)) such that $|x| \leqslant t - m_0$: Since ψ is an entire function then we conclude that $\psi \equiv 0$. By observing that, the solution of (3.4) is given by the series

$$u = \sum_{m=0}^{\infty} (-1)^m z^m L_q^m(v) = \psi(z) \equiv 0 .$$

It follows that $L_q^m(v)(x, t) = 0$ for all $m = 0, 1, \ldots$ and any such (x, t). This means that $[L_q^m(v)] \leqslant m_0$ which is a contradiction because $[L_q^m(v)] = b + 2mR$ for all m. This proves the theorem.

4. – The case $n > 3$ ($n =$ odd).

In this section we shall prove the non-validity of Huygens' principle for equation (1.1) when $n > 3$ and q depends only on $|x|$ and t.

Let $u = u(|x|, t)$ be a smooth solution of

$$\square u = f(|x|, t) \tag{4.1}$$

with zero initial conditions at $t = 0$.

We can write (4.1) as (here $r = |x|$)

$$u_{tt} - u_{rr} - \frac{n-1}{r} u_r = f(r, t) . \tag{4.2}$$

It is well known that the solution of (4.2) is given by

$$u = \frac{1}{(n-2)!} \sum_{j=1}^{k} \int_0^t a_j (t-s)^j \frac{\partial^{j-1}}{\partial t^{j-1}} F(x, t-s, s)\, ds \tag{4.3}$$

where $F(x, \lambda, s) = c \int_{|y|=1} f(x + \lambda y, s)\, dSy$. Here $k = (n-1)/2$, a_j and c are constants which depend only on the dimension n. (See [3], page 692 or [10]). Using the fact that f depends only on $|x|$ (and t) then we can deduce from (4.3) that u can be written as

$$u = \frac{c}{|x|^{n-2}} \int_0^t d\tau \sum_{j=1}^{k} \int_{||x|-t+\tau|}^{|x|+t-\tau} f(s, \tau)\, sP_j(|x|, t-\tau, s)\, ds$$

where P_j are functions which vary according to the dimension n. Here $k = (n-1)/2$.

EXAMPLE. Let $n = 7$. In this case (4.3) reads

$$u = \frac{1}{5!} \int_0^t \left[120\,(t-\tau)F + 72\,(t-\tau)^2 \frac{\partial F}{\partial t} + 8(t-\tau)^3 \frac{\partial^2 F}{\partial t^2}\right] d\tau \tag{4.4}$$

where

$$F = F(x, \lambda, \tau) = c\big(\lambda|x|\big)^{-5} \int_{||x|-\lambda|}^{|x|+\lambda} f(s, \tau) s[s^2 - (|x|-\lambda)^2]^2[(|x|+\lambda)^2 - s^2]^2 ds\,.$$

Substitution of F, $\partial F/\partial t$ and $\partial^2 F/\partial t^2$ in (4.4) yields

$$u = c|x|^{-5}\int_0^t dt \int_{||x|-t+\tau|}^{|x|+t-\tau}$$
$$\cdot f(s, \tau)s\Big[\big(|x|^2 + s^2 - (t-\tau)^2\big)^2 - \big(s^2 - (|x|-t+\tau)^2\big)\big(|x|+t-\tau)^2 - s^2\big)\Big]\, ds\,.$$

Given a continuous function $f\colon \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$ ($n = \text{odd} > 3$) such that f depends only on the radius $r = |x|$ (and t) i.e. $f = f(|x|, t)$, we define the operator L as

$$(Lf)(r, t) = cr^{2-n}\int_0^t d\tau \sum_{j=1}^{k} \int_{|r-t+\tau|}^{r+t-\tau} f(s, \tau)sP_j(r, t-\tau, s)\, ds \tag{4.4}$$

where the functions P_j ($j = 1, 2, \ldots, k$) are as above and $k = (n-1)/2$.

LEMMA 9. *Let* $q\colon \mathbf{R}^n \times R \to \mathbf{R}$ ($n > 3$) *be such that*

1) $q = q(|x|, t) \geqslant 0$;

2) $q_t + \varepsilon q_r \leqslant \dfrac{\varepsilon c_n}{r^3}$

for some $0 < \varepsilon < 1$ where $c_n = \dfrac{(n-1)(n-3)}{2}$

and

3) $Q_j(\lambda, s) = \int_0^s d\tau \int_0^{\lambda+s-\tau} qP_j^2\mu^{3-n}d\mu < +\infty$

for all $j = 1, \ldots, k$ and $s \geqslant 0$, $\lambda > 0$.

4) $R_j(r, t) = \int_0^t ds \int_{\cdot - +s}^{r+t-s} qQ_j(\lambda, s)\lambda^{3-m}|P_j|d\lambda < +\infty$

for all $j = 1, \ldots, k$ and $r > 0$ and $t \geqslant 0$.

Let u be the solution of $\Box u + \delta q(r, t)u = 0$ then

a) $E_\infty(t) = \int_0^\infty (u_t^2 + u_r^2 + \delta q(r, t)u^2)r^{n-1}dr$ *is bounded.*

b) $\sqrt{\delta}|\lambda^{n-2}L(qu)(\lambda, s)| \leqslant Q(\lambda, s)$ *for all* $\lambda > 0$, $s \geqslant 0$.

c) $\sqrt{\delta}|r^{n-2}L(qL(qu)((r, t)| \leqslant R(r, t)$ *for all* $r > 0$, $t \geqslant 0$.

Here

$$Q(\lambda, s) = \text{Constant} \sum_{j=1}^k Q_j(\lambda, s)$$

and

$$R(r, t) = \text{Constant} \sum_{j=1}^k \int_0^{tr} ds \int_{|r-t+s|}^{+t-s} q\lambda^{3-n}Q|P_j| .$$

PROOF. Because of our hypothesis on q we can apply Theorem 2 of section 2 to conclude *a*). By definition of L and Hölder's inequality we get

$$|\lambda^{n-2}L(\delta qu)(\lambda, s)| = \left|c\int_0^s d\tau \sum_{j=1}^k \int_{|\lambda-s+\tau|}^{\lambda+s-\tau} \delta qu\mu P_j d\mu\right|$$
$$\leqslant |c| \int_0^s d\tau \sum_{j=1}^k \left(\int_0^\infty \delta qu^2\mu^{n-1}d\mu\right)\left(\int_{|\lambda-s+\tau|}^{\lambda+s-\tau} \delta qP_j^2\mu^{3-n}d\mu\right).$$

Using part *a*) we conclude

$$|\lambda^{n-2}L(\delta qu)(\lambda, s)| \leqslant \text{constat}\sqrt{\delta}\int_0^s d\tau \sum_{j=1}^k \left(\int_0^{\lambda+s-\tau} qP_j^2\mu^{3-n}d\mu\right)^{\frac{1}{2}}$$
$$= \text{constat}\sqrt{\delta}\sum_{j=1}^k Q_j(\lambda, s) \equiv \sqrt{\delta}Q(\lambda, s) .$$

Using the definition of L and part b) above we obtain

$$|r^{n-2}L(\delta qL(\delta qu))(r,t)| \leqslant \text{Cons.}\, \delta^{\frac{3}{2}} \int_0^t ds \sum_j^k \int_{|r-t+s|}^{r+t-s} q(\lambda, s)\lambda^{3-n}Q(\lambda, s)|P_j| ds \equiv \delta^{\frac{3}{2}} R(r, t)$$

which proves c).

THEOREM 4. *Let* $q\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ *with all the assumptions as in the above lemma. Let* u *be the solution of*

$$\Box u + \delta q(|x|, t) u = 0\,, \qquad u(x, 0) = 0\,, \qquad u_t(x, 0) = g(|x|) \tag{4.5}$$

where $\delta > 0$, g *is a radial function, non-negative and belongs to* $C_0^\infty(R^n)$. *Given* $\lambda > 0$, *there exists* $\delta_0 > 0$ *such that if* $0 < \delta < \delta_0$ *then the solution* u *of* (4.5) *cannot be identically zero in the set*

$$\Omega_\lambda = \{(x, t),\ |x| \leqslant t - \lambda,\ t \geqslant \lambda\}\,.$$

In particular, under the above assumption on q, *the Huygens' principle does not hold for equation* (4.5).

PROOF. By using the formula of variation of parameters we can write the solution of (4.5) as

$$u = u_0 - L(\delta qu) \tag{4.6}$$

where L is given by (4.4) and u_0 is the solution of $\Box u_0 = 0$ with the same initial data as u at $t = 0$. Let us iterate (4.6) to obtain

$$r^{n-2}L(\delta qu_0) = r^{n-2}[u_0 - u] + r^{n-2}L\big(\delta qL(\delta qu)\big)\,. \tag{4.7}$$

If the conclusion of theorem 4 was not true then there exists a solution of (4.5) which vanishes in Ω_λ for some $\lambda > 0$. We can assume without loss of generality that $\lambda \geqslant R$ where R is the radius of a bll containing the support of g. Since the Huygens' principle holds for u_0 we conclude from (4.7) that for any $(r, t) \in \Omega_\lambda$ we have

$$|r^{n-2}L(qu_0)| \leqslant r^{n-2}\delta\big|L\big(qL(qu)\big)\big| \leqslant r^{n-2}\sqrt{\delta}\, R(r, t)\,. \tag{4.8}$$

Because of our assumptions on q and g the left hand side of (4.8) cannot be zero identically on Ω_λ. Letting $\delta \to 0$ in (4.8) we get a contradiction which proves the theorem.