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On Relatively Bounded Perturbations of Linear $C_0$-Semigroups.

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In recent years we see an increasing interest and literature devoted to various system-theoretical investigations of systems of the form

$$\frac{d}{dt} x(t) = Ax(t) + Du(t), \quad t > 0,$$

$$x(0) = x_0.$$

Here $A$ is the infinitesimal generator of a linear $C_0$-semigroup $T(\cdot)$ on a Banach space $X$ and $D$ is a continuous linear operator from the Banach-space of control parameters into $X$. Of particular interest are problems concerning controllability, observability, boundary control etc. If the control is implemented through a feedback relation and we deal with the realistic case of having only a finite number of controls available, we face the following problem raised for instance in [11], p. 105:

Let $A$ be the infinitesimal generator of a $C_0$-semigroup $T(\cdot)$ on $X$ and let $B$ be a linear operator in $X$ satisfying

(i) Range $(B)$ is finite-dimensional and

(ii) $B$ is $A$-bounded, i.e. $D(B) \supset D(A)$ and there are nonnegative constants $a$ and $b$ such that $\|Bx\| \leq a\|Ax\| + b\|x\|$ for all $x \in D(A)$.

Under which assumptions is $(A + B)$ the infinitesimal generator of a $C_0$-semigroup on $X$?

If $X$ is reflexive, then Hess proved in [6] that (i)-(ii) imply that the $A$-bound of $B$ is zero and hence we can apply a general perturbation result

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(Kato [8], p. 499) to conclude that if $A$ generates an analytic semigroup so does $(A + B)$. (See also Zabczyk [12]).

A similar problem arises in the context of the semigroup approach to functional differential equations in the state space

$$X = \mathbb{R}^n \times L^p(-r, 0; \mathbb{R}^n) \quad \text{with} \quad 0 < r < \infty \text{ and } 1 \leq p < \infty.$$

Given a linear map $L : X \supset D(L) \to \mathbb{R}^n$ we consider the Cauchy-problem

$$\frac{d}{dt} x(t) = L(x(t), x_t), \quad t > 0,$$
$$x(0) = \eta, \quad x(s) = \phi(s) \quad \text{a.e. on} \quad -r < s < 0.$$

The history $\eta$ is given by $x_t(s) = x(s + t), \ t \geq 0, \ s \in [-r, 0]$.

If $x(t, \eta, \phi)$ denotes the solution of (1) we define the associated solution semigroup $T(\cdot)$ by $T(t) = (x(t), x_t)$. In [7] it was shown that the infinitesimal generator $A$ of this semigroup is given by

$$D(A) = \{ (\phi(0), \phi) \mid \phi \in W^{1,p}(-r, 0; \mathbb{R}^n) \},$$
$$A(\phi(0), \phi) = (L(\phi(0), \phi), \phi).$$

Obviously, $A$ can be split up as $A = A_0 + B$ where

$$A_0(\phi(0), \phi) = (0, \phi), \quad B(\phi(0), \phi) = (L(\phi(0), \phi), 0).$$

Delfour [5] proved that this operator $A$ is the infinitesimal generator of a $C_0$-semigroup ifff $L$ is a continuous map $D(A_0) \to \mathbb{R}^n$, i.e. (i) and (ii) hold. Thus it seems to be attractive to conjecture that (i) and (ii) imply that $(A + B)$ is the infinitesimal generator also for non-analytic semigroups.

It is the objective of this paper to show that the above conjecture is false even if the unperturbed semigroup is ultimately compact, differentiable or a $C_0$-group in a Hilbert-space!

On the other hand, we verify that if $B$ satisfies an additional continuity assumption then $(A + B)$ is the infinitesimal generator of a $C_0$-semigroup on $X$ without any restriction on the range of $B$.

**Some counterexamples.**

To begin with, we provide some counterexamples to the above mentioned conjecture. Let $X = l^2$. Given a sequence $(\lambda_n)$ of complex numbers so that $\Re \lambda_n < 0$ for all $n$ it is clear that the linear operator $A = \text{diag} (\lambda_n)$ is the infinitesimal generator of a $C_0$-semigroup $S(\cdot)$ given by $S(t) = \text{diag} (\exp [\lambda_n t])$. 
Next, define a linear operator $B$ in $X$ by

$$(Bx)_n = \alpha_n \sum_{j=1}^{\infty} \alpha_j \lambda_j x_j$$

with $\alpha = (\alpha_j) \in l^1(\mathbb{R})$ is chosen so that $B$ is $A$-bounded and $\limsup_{n \to \infty} \alpha_n |\lambda_n| \cdot \exp[\text{Re} \lambda_n] = \infty$. The specific choice of $(\lambda_n)$ and $(\alpha_n)$ is still at our disposal.

We claim that the operator $A = \begin{pmatrix} A & B \\ 0 & A \end{pmatrix}$ cannot be the infinitesimal generator of a $C_0$-semigroup on $X \times X$. In fact, if $A$ were the infinitesimal generator of a $C_0$-semigroup $G(\cdot)$ on $X \times X$ we consider the elements $y_m = (\delta_{m,k})_{k=1}^{\infty} \in X$. As $y_m \in D(A^\alpha)$ we infer that $t \mapsto S(t)y_m \in C^\alpha(0, \infty; D(A))$ and hence $BS(t)y_m \in C^\alpha(0, \infty; X)$.

Consequently,

$$\tilde{x}(t) = \begin{pmatrix} \int_{0}^{t} S(t-s)BS(s)y_m ds \\ S(t)y_m \end{pmatrix}$$

would be a strong solution of the Cauchy-problem $(d/dt)\tilde{x}(t) = A\tilde{x}(t)$.

Being a $C_0$-semigroup there must exist a constant $M$ so that $\|G(t)\| < M$ for $0 < t < 1$. In particular we thus would expect that $\|G(1) \begin{pmatrix} 0 \\ y_m \end{pmatrix} \| < M$, and in particular

$$\left| \left( \frac{1}{0} \int_{0}^{t} S(1-s)BS(s)y_m ds \right) \right| < M.$$  \hspace{1cm} (2)

The left hand side of this inequality can be rewritten as

$$\int_{0}^{1} \exp \left[ (1-s)\lambda_m \right] x_m^t \lambda_m \alpha_m \exp \left[ s \lambda_m \right] ds = \exp \left[ \lambda_m \right] x_m^t \lambda_m$$

and as by assumption $\limsup_{n \to \infty} \exp[\text{Re} \lambda_m] = \infty$ we see that we cannot have an estimate of the form (2), i.e. $G(\cdot)$ is not a $C_0$-semigroup.

We now specify the $(\lambda_m)$ and $(\alpha_m)$:

1. **The case of $C_0$-group.**

   Let $\lambda_m = im$, $m = 1, 2, \ldots$. Then $A = -A^*$ and so $\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$ generate a
Co-group on $X$. Putting

$$\alpha_m = \begin{cases} m^i & \text{if } m^i \text{ is an integer} \\ 0 & \text{otherwise} \end{cases}$$

we see that $\alpha = (\alpha_m) \in l^2(\mathbb{R})$.

Moreover,

$$\alpha_m^2 |\lambda_m| \exp[\Re \lambda_m] = \begin{cases} m^i & \text{if } m^i \text{ is an integer} \\ 0 & \text{otherwise} \end{cases}$$

and hence $\limsup_{m \to \infty} \alpha_m^2 |\lambda_m| \exp[\Re \lambda_m] = \infty$.

The associated operator $B$ clearly satisfies (i), (ii) but according to the above consideration $A - \sim - B$ is not an infinitesimal generator.

2. The case of an ultimately compact and differentiable semigroup.

Let $\lambda_m = -m + i \exp[4m]$ and set $\alpha_m = \exp[-m]$. Then we obtain

$$\alpha_m^2 |\lambda_m| \exp[\Re \lambda_m] \exp[m] \to \infty \quad \text{as } m \to \infty$$

and so again the associated operator $\begin{pmatrix} A & B \\ 0 & \lambda \end{pmatrix}$ cannot be an infinitesimal generator. In order to show that $S(t)$ is differentiable for $t > 4$ it is sufficient to verify that $\lambda_m \exp[\lambda_m t]$ is bounded for $t > 4$. This follows from

$$|\lambda_m \exp[\lambda_m t]| = | -m + i \exp[4m] \exp[-mt] \exp[-mt] + \exp[m(4-t)] |.$$

It is also obvious that $S(t)$ is compact for $t > 4$.

Some generation results.

As already pointed out in the introduction, we present a general generation result that seems to be very useful in applications. Throughout this section, we assume that $(X, \| \cdot \|)$ is a Banach-space. If $A$ is a closed linear operator in $X$ we let $X_A$ stand for the Banach space $(\mathcal{D}(A), \| \cdot \|_A)$, with $\| x \|_A = \| x \| + \| Ax \|$.

**Theorem.** Let $A$ be the infinitesimal generator of a $C_0$-semigroup $T(t)$ on $X$. Let $(Z, \| \cdot \|_Z)$ be a Banach space such that

(Z1) $Z$ is continuously embedded in $X$,
(Z2) there is a $t_0 > 0$ so that for all continuous functions

$$\phi: [0, t_0] \to Z \quad \text{we have} \quad \int_0^t T(t-s)\phi(s)\,ds \in D(A) \quad \text{for all} \quad t \in [0, t_0]$$

(3) there is an increasing continuous function $\gamma: [0, t_0] \to [0, \infty)$ satisfying $\gamma(0) = 0$ and

$$\left| \int_0^t T(t-s)\phi(s)\,ds \right|_A < \gamma(t) \sup_{0 \leq s \leq t} |\phi(s)|_A.$$

Then for any continuous linear operator $B: X \to Y$, $(A + B)$ is the infinitesimal generator of a $C_0$-semigroup on $X$.

**Proof.** To begin with, we verify that under the above assumptions the map $t \mapsto \int_0^t T(t-s)\phi(s)\,ds$ is continuous from $[0, t_0]$ into $X_A$: In fact, given any continuous $\phi: [0, t_0] \to Z$ and $t \in [0, t_0]$, we define for all $0 < h < t$

$$\phi_h(s) = \begin{cases} 
\phi(s - h) & \text{for } h < s < t + h \\
0 & \text{otherwise}.
\end{cases}$$

Then we obtain for all $t \in [0, t_0 - h]$

$$\left| \int_0^{t+h} T(t+h-s)\phi(s)\,ds - \int_0^t T(t-s)\phi(s)\,ds \right|_A$$

$$= \left| \int_0^{t+h} T(t+h-s)(\phi(s) - \phi_h(s))\,ds \right|_A$$

$$< \left| \int_0^h T(t+h-s)(\phi(s) - \phi_h(s))\,ds \right|_A + \int_0^h T(t+h-s)(\phi(s) - \phi_h(s))\,ds$$

$$< \gamma(h) \sup_{0 \leq s \leq h} |\phi(s) - \phi_h(s)|_A.$$

As the right side converges to 0 for $h \to 0$, the claim follows.

In the next step we verify that $(A + B)$ is a closed linear operator. To this end, we first estimate $(\lambda I - A)^{-1}B$ as an operator from $X_A$ into $X_A$. Let $N$ and $\omega$ be constants such that

$$\|T(t)\| < N \exp[\omega t] \quad \text{for all} \quad t > 0.$$
Then we obtain for all sufficiently large \( \lambda \) and all \( x \in D(A) \)

\[
|\lambda I - A|^{-1} B x|_A
\]

\[
= \left| \int_0^1 \exp \left[ -\lambda s \right] T(s) B x \, ds + \exp \left[ -\lambda t \right] T(t) \int_0^\infty \exp \left[ -\lambda s \right] T(s) B x \, ds \right|_A
\]

\[
< \exp \left[ -\lambda t \right] \left| \int T(t-s) \exp \left[ \lambda s \right] B x \, ds \right|_A + N \exp \left[ (\omega - \lambda) t \right] |(\lambda I - A)^{-1} B x|_A
\]

\[
< \gamma(t) \sup |\exp \left[ \lambda s \right] B x|_A + N \exp \left[ (\omega - \lambda) t \right] |(\lambda I - A)^{-1} B x|_A
\]

Putting

\[
t(\lambda) = \frac{\ln N + \ln 2}{\lambda - \omega}
\]

we obtain

\[
|\lambda I - A|^{-1} B x|_A < 2 \gamma(t(\lambda)) \beta |x|_A,
\]

where \( \beta \) denotes the norm of \( B \) regarded as an operator from \( X_A \) into \( Z \), and as \( t(\lambda) \to 0 \) as \( \lambda \to \infty \) we deduce that

\[
K(\lambda) = 2 \gamma(t(\lambda)) \quad \text{converges to} \quad 0 \quad \text{as} \quad \lambda \to \infty.
\]

In order to prove that \((A + B)\) is closed, let \((x_n)\) be a sequence in \( X_A \) such that \( x_n \to x \) (in \( X \)) and \( y_n := (A + B)x_n \) converges to \( y \in X \). We have to show that \((x_n)\) is a Cauchy sequence in \( X_A \): Fix some \( \lambda > 0 \) with \( K(\lambda) < 1 \). Then we have

\[
|x_n - x_m|_A = |\lambda(\lambda I - A)^{-1}(x_n - x_m) - (\lambda I - A)^{-1} A(x_n - x_m)|_A
\]

\[
= |\lambda(\lambda I - A)^{-1}(x_n - x_m) - (\lambda I - A)^{-1} (y_n - y_m) + (\lambda I - A)^{-1} B(x_n - x_m)|_A
\]

\[
< q \lambda \|x_n - x_m\| + q \|y_n - y_m\| + K(\lambda)\|x_n - x_m\|,
\]

where \( q \) denotes the norm of \((\lambda I - A)^{-1}\) regarded as an operator from \( X \) into \( X_A \).

Consequently, we obtain

\[
|x_n - x_m|_A < \frac{1}{1 - K(\lambda)} (\lambda q \|x_n - x_m\| + q \|y_n - y_m\|).
\]

This implies that \( y_n := (A + B)x_n \) converges to \((A + B)x\) so that \((A + B)x = y\).

The proof that \((A + B)\) is in fact the infinitesimal generator of a \( C_0\)-semi-
group on $X$ is performed by making use of Ball's Theorem ([1]). Roughly speaking, we are to show that for any $x \in X$ the Cauchy problem

$$\frac{d}{dt} x(t) = (A + B)x(t), \quad t > 0$$
$$x(0) = x,$$

admits a unique weak solution $x(t)$ on $[0, \infty)$, i.e. for all $x^* \in D((A + B)^*)$ and $t > 0$

$$\langle x(t), x^* \rangle = \langle x, x^* \rangle + \int_0^t \langle x(s), (A + B)^* x^* \rangle ds.$$

So, fix $x \in X$ and $t > 0$. Given any continuous function $z : [0, t] \to X_A$ we put

$$(\mathcal{C}z)(t) = \int_0^t T(t - s)Bz(s)ds + \int_0^t T(t - s)xds, \quad 0 < t \leq t.$$

By the above considerations $(\mathcal{C}z)(\cdot)$ is a continuous function $[0, t] \to X_A$. If $\beta$ denotes again the norm of $B$ regarded as an operator $X_A \to Z$, we obtain for all $0 < t \leq t$:

$$|\mathcal{C}z_1(t) - \mathcal{C}z_2(t)|_A \leq \beta \gamma(t) \sup |z_1(s) - z_2(s)|_A.$$

Choosing $\delta$ sufficiently small we conclude that there exists a unique fixed point of $\mathcal{C}$. As $\beta \gamma(t)$ does not depend on $x$, we may continue this procedure and obtain a continuous function $y : [0, \infty) \to X_A$ satisfying

$$y(t) = \int_0^t T(t - s)By(s)ds + \int_0^t T(t - s)xds.$$

Putting $x(t) = (A + B)y(t) + x$, it is clear that $x(0) = x$ and $x$ is continuous $[0, \infty) \to X$. Moreover, we have for all $h > 0$

$$\frac{1}{h} (y(t + h) - y(t)) = \frac{1}{h} \left( \int_0^{t+h} T(t + h - s)(x + By(s))ds - \int_0^t T(t - s)(x + By(s))ds \right)$$
$$= \frac{1}{h} \left( T(h) - 1 \right) \int_0^t T(t - s)(x + By(s))ds + \frac{1}{h} \int_t^{t+h} T(t + h - s)(x + By(s))ds.$$
and as \( y(t) \in D(A) \), the right hand side converges to \( Ay(t) + x + By(t) \) as \( h \to 0^+ \). Therefore, we have

\[
\frac{d^+}{dt} y(t) = (A + B)y(t) + x = x(t) .
\]

For any \( x^* \in D((A + B)^*) \) we obtain

\[
\langle x(t), x^* \rangle = \langle x, x^* \rangle + \langle y(t), (A + B)^*x^* \rangle
\]

and hence \( x(t) \) is a weak solution of (3).

In order to verify uniqueness of this weak solution, let \( x(\cdot) \) be any weak solution of (3) with \( x(0) = 0 \). Putting

\[
y(t) = \int_0^t x(s) \, ds ,
\]

we get for all \( x^* \in D((A + B)^*) \)

\[
\langle x(t), x^* \rangle = \int_0^t \langle x(s), (A + B)^*x^* \rangle \, ds = \langle y(t), (A + B)^*x^* \rangle .
\]

As \( (A + B) \) is closed, this implies that \( y(\cdot) \in D(A) \) and hence

\[
x(t) = \frac{d}{dt} y(t) = (A + B)y(t) .
\]

From the variation of constants formula for \( T(\cdot) \) we get

\[
y(t) = \int_0^t T(t-s)By(s) \, ds .
\]

The unique solution of this integral equation is \( y = 0 \) and hence \( x = 0 \).

Hence (3) admits a unique weak solution for all \( x \in X \) showing that \( (A + B) \) is the infinitesimal generator of a \( C_0 \)-semigroup on \( X \).

The particular choice of \( Z \) depends of course, heavily on the problem under consideration and it may vary through a large class of different spaces and is illustrated by the following examples.
1. \( Z = X_a \).

A particular interesting case of \( Z \) is provided by putting \( Z = X_a \). Clearly, assumptions (Z1)-(Z3) are satisfied (with \( \gamma(t) = tM \exp\{\cot\} \)), and hence we deduce that for any continuous linear operator \( B : X_a \to X_a \) the operator \( (A + B) \) is the infinitesimal generator of a \( C_0 \)-semigroup on \( X \).

2. Delay equations on product spaces.

Let \( Y \) be a real Banach space and put \( X = Y \times L^p(-r, 0; Y) \), \( 1 < p < \infty \), \( 0 < r < \infty \). Let \( T(\cdot) \) denote the solution semigroup of the unperturbed equation given by

\[
T(t)(\eta, \varphi) = (\eta, \varphi),
\]

\[
\varphi(s) = \begin{cases} 
\eta & \text{if } s + t > 0 \\
\varphi(s + t) & \text{if } s + t < 0.
\end{cases}
\]

As already mentioned in the introduction its infinitesimal generator \( A \) is given by

\[
D(A) = \{ (u(0), u) \mid u \in W^{1,p}(-r, 0; Y) \},
\]

\[
A(u(0), u) = (0, \dot{u}).
\]

Then the assumptions of the generation theorem are satisfied with \( Z = Y \times \{0\} \). Therefore, for any linear operator \( B \) that maps \( W^{1,p}(-r, 0; Y) \) continuously into \( Y \) the operator \( (A + B) \) is an infinitesimal generator of a \( C_0 \)-semigroup on \( X \).

**Proof.** For any \( t > 0 \) let \( \varphi \) be a continuous function \([0, t] \to Y\). Define \( \varphi \) by

\[
\varphi(s)(\theta) = \begin{cases} 
\varphi(s) & \text{if } \theta > s - t, \\
0 & \text{otherwise}.
\end{cases}
\]

Then for all \( \theta \in [-r, 0] \) we have

\[
\int_0^t \varphi(s)(\theta) ds = \int_0^{\theta + t} \varphi(s) ds,
\]

\[
\left( \int_0^t \varphi(s) ds, \int_0^t \varphi(s) ds \right) = \int_0^t T(t - s)(\varphi(s), \varphi(s)) ds \in D(A)
\]

and since

\[
\left( \int_{-r}^0 |\varphi(s + t)|^p ds \right)^{1/p} < t^{1/p} \sup |\varphi(\tau)|
\]
where
\[ \tilde{\phi}(s) = \begin{cases} \varphi(s) & \text{for } s > 0, \\ 0 & \text{otherwise} \end{cases} \]
also (Z3) is satisfied.

Thus it is not the finite-dimensional range property of the perturbation that ensures the generation of a $C_0$-semigroup. Of course, the above condition is too restrictive for partial-differential equations involving delay terms, although they are not far away from being necessary ([9]).

We next turn to partial differential equations: To begin with we provide

3. The Favard-class of $T(\cdot)$.

Let $A$ be the infinitesimal generator of a $C_0$-semigroup $T(\cdot)$ on $X$. Then the assumptions of the theorem are satisfied for $Z$ being the Favard class of $T(\cdot)$, i.e.

\[ Z = \left\{ x \in X \left| \limsup_{t \to 0^+} \frac{1}{t} \| T(t)x - x \| \text{ is finite} \right. \right\}, \]

\[ |x|_Z = \| x \| + \limsup_{t \to 0^+} \frac{1}{t} \| T(t)x - x \|. \]

**Proof.** Let $\varphi$ be a continuous function $[0, t] \to Z$. Then there is a sequence $(\varphi_n)$ of continuously differentiable functions such that $\varphi_n \to \varphi$ as $n \to \infty$. Clearly, \[ \int_0^t T(t-s)\varphi_n(s)ds \in D(A), \] and

\[ \left\| A \int_0^t T(t-s)\varphi_n(s)ds \right\| \leq \limsup_{h \to 0^+} \frac{1}{h} \left\| (T(h) - I) T(t-s)\varphi_n(s)ds \right\| \]

\[ \leq M \exp[\omega t] \int_0^t |\varphi_n(s)|_z ds < tM \exp[\omega t] \sup |\varphi_n|_z. \]

(Here $M$ and $\omega$ are constants such that $\| T(t) \| \leq M \exp[\omega t]$ for all $t > 0$).

A closedness argument now shows that the same estimate is also valid for $\varphi$.

An interesting application of this result is

4. Integrodifferential equations.

Let $Y$ be a Banach space and consider the Cauchy problem

\[ \frac{d}{dt} u(t) = Lu(t) + \int_0^t C(t-s)u(s)ds + f(t), \quad t > 0, \]

(4)

\[ u(0) = u_0. \]
Here $L$ is the infinitesimal generator of a $C_0$-semigroup $S(t)$ on $Y$ and $\{C(t); t > 0\}$ is a family of continuous linear operators $X_L \to Y$ such that for each $x \in X_L$ the map $Cx$ given by $(Cx)(t) = C(t)x$ belongs to $L^1(0, \infty; Y)$.

Following the notation used in [3], we say that (4) is uniformly well-posed if for each $u_0 \in D(L)$ and each $f \in W^{1,1}(0, \infty; Y)$ then exists a unique strong solution $u(t, u_0; f)$ of (4) which depends continuously on $u_0$ (with respect to the $Y$-norm) and $f$ (with respect to the $L^1$-norm), uniformly for $t$ in compact intervals.

There is a large number of papers in which uniform well-posedness of (4) is proven under various additional smoothness assumptions on $C(\cdot)$. Our approach allows the following very general result:

**Theorem.** (4) is uniformly well-posed if $Cx$ is of bounded variation for each $x \in D(L)$.

The underlying basic idea that was introduced in [10] and carried out in a more general framework in [3] is to associate to (4) a differential equation in a larger Banach space.

To this end, let $T(\cdot)$ denote the shift semigroup on $L^1(0, \infty; Y)$ defined by $(T(t)\phi)(s) = \phi(s + t), s > 0, t > 0$. Moreover, let $D_s$ denote its infinitesimal generator and let $\delta$ be the operator $W^{1,1}(0, \infty; Y) \to Y$ given by $(\delta\phi = \phi(0))$.

In [3] it is shown that (4) is uniformly well-posed if and only if the following abstract Cauchy problem is uniformly well-posed in $X = Y \times Y \times L^1(0, \infty; Y)$:

\[
\frac{d}{dt} x(t) = Ax(t), \quad t > 0,
\]

\[
x(0) = x_0,
\]

where $A$ is given by

\[
D(A) = Y \times D(L) \times W^{1,1}(0, \infty; Y),
\]

\[
A = \begin{pmatrix}
0 & L & 0 \\
0 & L & \delta \\
0 & C & D_s
\end{pmatrix}.
\]

So, we have to show that $A$ is the infinitesimal generator of a $C_0$-semigroup on $X$. To prove this claim, we split up $A$ as

\[
A = A_0 + B
\]

where

\[
A_0 = \begin{pmatrix}
0 & 0 & 0 \\
0 & L & \delta \\
0 & 0 & D_s
\end{pmatrix}
\]

and

\[
B = \begin{pmatrix}
0 & L & 0 \\
0 & 0 & 0 \\
0 & C & 0
\end{pmatrix}.
\]
It is an elementary calculation to verify that \( \mathcal{A}_a \) generates a semigroup \( \mathcal{G}(\cdot) \) given by

\[
\mathcal{G}(t) \begin{pmatrix} x \\ y \\ f \end{pmatrix} = \begin{pmatrix} x \\ S(t)y + \int_0^t S(t-s)f(s)\,ds \\ T(f) \end{pmatrix}.
\]

The range of \( \mathcal{B} \) consists of all vectors \( \begin{pmatrix} x \\ f \end{pmatrix} \) where \( f \) is of bounded variation. If we can verify that these vectors belong to the Favard class of \( \mathcal{G}(\cdot) \). Example 2 implies that \( \mathcal{A} \) generates a \( C_0 \)-semigroup on \( X \) and hence (4) is uniformly well-posed.

As

\[
\frac{1}{t} \left( \mathcal{G}(t) \begin{pmatrix} x \\ 0 \\ f \end{pmatrix} - \begin{pmatrix} x \\ 0 \\ f \end{pmatrix} \right) = \begin{pmatrix} 0 \\ \frac{1}{t} \int_0^t S(t-s)f(s)\,ds \\ \frac{1}{t} (T(t)f-f) \end{pmatrix},
\]

and

\[
\frac{1}{t} \left\| \int_0^t S(t-s)f(s)\,ds \right\| \leq \sup_{0 \leq s \leq t} \| S(t-s) \| \| f(s) \|
\]

is bounded as \( f \) has bounded variation, and

\[
\frac{1}{t} \left\| T(t)f-f \right\| = \frac{1}{t} \int_0^\infty \| f(s+t) - f(s) \| \,ds \leq \text{Var} \, (f, (0, \infty))
\]

([2], Appendix), we conclude that

\[
\limsup_{t \to 0^+} \frac{1}{t} \left\| \mathcal{G}(t) \begin{pmatrix} x \\ 0 \\ f \end{pmatrix} - \begin{pmatrix} x \\ 0 \\ f \end{pmatrix} \right\| \text{ is finite.}
\]

5. **Interpolation spaces.**

Let \( A \) be the infinitesimal generator of an analytic semigroup \( T(\cdot) \). Then we can take \( Z = (D(A), X)_I \), an interpolation space between \( D(A) \) and \( X \).

The proof follows from the general properties of interpolation spaces (see [4]).
If we are dealing with nonlinear perturbations then the situation is much more complicated as is seen by the following example. Roughly speaking, we show that even for a one-dimensional, $C^\infty$-nonlinear perturbation $B$ that maps $X_A$ into itself the operator $(A + B)$ is not a generator of a nonlinear semigroup on $X$.

**Example.** Let $X = \mathbb{R} \times C_{u,b}(\mathbb{R}; \mathbb{R})$, where $C_{u,b}(\mathbb{R}; \mathbb{R})$ denotes the usual Banach space of all uniformly continuous, bounded functions $\mathbb{R} \rightarrow \mathbb{R}$.

Let $T(\cdot)$ be the linear semigroup given by $T(t)\begin{pmatrix} x \\ \varphi \end{pmatrix} = \begin{pmatrix} x \\ \varphi(t + \cdot) \end{pmatrix}$. An easy calculation shows that its infinitesimal generator $A$ is given by

$$ D(A) = \left\{ \begin{pmatrix} x \\ \varphi \end{pmatrix} \left| \varphi' \in C_{u,b}(\mathbb{R}; \mathbb{R}) \right. \right\}$$

$$ A \begin{pmatrix} x \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi' \end{pmatrix}. $$

Let $\psi$ be a $C^\infty$-function $\mathbb{R} \rightarrow \mathbb{R}$ such that

$$ \psi(0) = 0, \quad \psi(s) > \inf(|s|, s^2), $$

and $\psi$ is Lipschitzian with constant $< 2$.

We define $B$ by $B\begin{pmatrix} x \\ \varphi \end{pmatrix} = \begin{pmatrix} \psi(\varphi'(0)) \\ 0 \end{pmatrix}$.

For $\begin{pmatrix} 0 \\ \zeta \end{pmatrix} \in D(A)$, the Cauchy problem

$$ \frac{d}{dt} \begin{pmatrix} x \\ \varphi \end{pmatrix} = A \begin{pmatrix} x \\ \varphi \end{pmatrix} + B \begin{pmatrix} x \\ \varphi \end{pmatrix}, \quad \begin{pmatrix} x \\ \varphi \end{pmatrix}(0) = \begin{pmatrix} 0 \\ \zeta \end{pmatrix}, $$

has a unique strong solution given by

$$ \begin{pmatrix} x \\ \varphi \end{pmatrix}(t) = \begin{pmatrix} \int_0^t \psi(\zeta'(s)) \, ds, \zeta(t + \cdot) \end{pmatrix} $$

Given $t \in (0, 1]$ we choose a $\zeta \in C_{u,b}(\mathbb{R}; \mathbb{R})$ such that the restriction of $\zeta$ to $(0, t]$ does not have a bounded variation. Let $(\zeta_n)$ be a sequence in $C_{u,b}(\mathbb{R}; \mathbb{R})$ so that $\zeta_n \rightarrow \zeta$ and $(0, \zeta_n)^2 \in D(A)$.

If the solutions $\begin{pmatrix} x_n(\cdot) \\ \varphi_n(\cdot) \end{pmatrix}$ with initial values $\begin{pmatrix} 0 \\ \zeta_n \end{pmatrix}$ were convergent to, say $\begin{pmatrix} \eta(t) \\ \omega(t) \end{pmatrix}$, then we clearly have $\omega(t) = \zeta(t + \cdot)$. 

Let $0 < s_1 < s_2 < t$. We choose measurable sets $E_1$ and $E_2$ such that

$$[s_1, s_2] = E_1 \cup E_2$$

and

$$\psi(\zeta_n'(s)) > |\zeta_n'(s)|$$
on $E_1$

Then

$$\int_{s_1}^{s_2} \psi(\zeta_n'(s)) ds > \int_{E_1} |\zeta_n'(s)| ds + \int_{E_2} (|\zeta_n'(s)|^2 ds > \int_{E_1} |\zeta_n'(s)| ds + \left( \int_{E_2} |\zeta_n'(s)| ds \right)^2 \cdot \frac{1}{\text{meas}(E_2)}.$$

Consequently,

$$\int_{s_1}^{s_2} |\zeta_n'(s)| ds < (s_2 - s_1) + \int_{s_1}^{s_2} \psi(\zeta_n'(s)) ds.$$

Taking the limit $n \to \infty$ we thus obtain

$$|\zeta(s_2) - \zeta(s_1)| < \gamma(s_2) - \gamma(s_1) + s_2 - s_1,$$

where $\gamma$ is the limit of the sequence of monotone functions $\int_0^t \psi(\zeta_n'(s)) ds$ and hence $\gamma$ is itself monotone. Therefore $\zeta$ must be of bounded variation which contradicts the assumptions. As a consequence we deduce that the solution operators cannot be continuous which is a standing hypothesis for all nonlinear semigroups.

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ON RELATIVELY BOUNDED PERTURBATIONS OF LINEAR $C_0$-SEMIGROUPS


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