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On Relatively Bounded Perturbations of Linear C_0-Semigroups.

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In recent years we see an increasing interest and literature devoted to various system-theoretical investigations of systems of the form

$$\frac{d}{dt}x(t) = Ax(t) + Du(t)\,, \qquad t > 0\,,$$
$$x(0) = x_0\,.$$

Here A is the infinitesimal generator of a linear C_0-semigroup $T(\cdot)$ on a Banach space X and D is a continuous linear operator from the Banach-space of control parameters into X. Of particular interest are problems concerning controllability, observability, boundary control etc. If the control is implemented through a feedback relation and we deal with the realistic case of having only a finite number of controls available, we face the following problem raised for instance in [11], p. 105:

Let A be the infinitesimal generator of a C_0-semigroup $T(\cdot)$ on X and let B be a linear operator in X satisfying

(i) Range (B) is finite-dimensional and

(ii) B is A-bounded, i.e. $D(B) \supset D(A)$ and there are nonnegative constants a and b such that $\|Bx\| \leqslant a\|Ax\| + b\|x\|$ for all $x \in D(A)$.

Under which assumptions is $(A + B)$ the infinitesimal generator of a C_0-semigroup on X?

If X is reflexive, then Hess proved in [6] that (i)-(ii) imply that the A-bound of B is zero and hence we can apply a general perturbation result

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(Kato [8], p. 499) to conclude that if A generates an analytic semigroup so does $(A+B)$. (See also Zabczyk [12]).

A similar problem arises in the context of the semigroup approach to functional differential equations in the state space

$$X = \mathbb{R}^n \times L^p(-r, 0; \mathbb{R}^n) \quad \text{with } 0 < r < \infty \text{ and } 1 \leqslant p < \infty .$$

Given a linear map $L\colon X \supset D(L) \to \mathbb{R}^n$ we consider the Cauchy-problem

$$\begin{aligned} &\frac{d}{dt} x(t) = L\big(x(t), x_t\big), \qquad t > 0, \\ &x(0) = \eta, \quad x(s) = \phi(s) \qquad \text{a.e. on } -r \leqslant s < 0 . \end{aligned} \tag{1}$$

The «history» x_t is given by $x_t(s) = x(s+t)$, $t \geqslant 0$, $s \in [-r, 0]$.

If $x(t, \eta, \phi)$ denotes the solution of (1) we define the associated solution semigroup $T(\cdot)$ by $T(t)(\eta, \phi) = \big(x(t), x_t\big)$. In [7] it was shown that the infinitesimal generator A of this semigroup is given by

$$\begin{aligned} &D(A) = \{(\phi(0), \phi)\,|\,\phi \in W^{1,p}(-r, 0, \mathbb{R}^n)\}, \\ &A(\phi(0), \phi) = \big(L(\phi(0), \phi), \dot{\phi}\big) . \end{aligned}$$

Obviously, A can be split up as $A = A_0 + B$ where

$$A_0(\phi(0), \phi) = (0, \dot{\phi}), \qquad B(\phi(0), \phi) = \big(L(\phi(0), \phi), 0\big) .$$

Delfour [5] proved that this operator A is the infinitesimal generator of a C_0-semigroup iff L is a continuous map $D(A_0) \to \mathbb{R}^n$, i.e. (i) and (ii) hold. Thus it seems to be attractive to conjecture that (i) and (ii) imply that $(A+B)$ is the infinitesimal generator also for non-analytic semigroups.

It is the objective of this paper to show that the above conjecture is false even if the unperturbed semigroup is ultimately compact, differentiable or a C_0-group in a Hilbert-space!

On the other hand, we verify that if B satisfies an additional continuity assumption then $(A+B)$ is the infinitesimal generator of a C_0-semigroup on X without any restriction on the range of B.

Some counterexamples.

To begin with, we provide some counterexamples to the above mentioned conjecture. Let $X = l^2$. Given a sequence (λ_n) of complex numbers so that $\operatorname{Re} \lambda_n \leqslant 0$ for all n it is clear that the linear operator $A = \operatorname{diag}(\lambda_n)$ is the infinitesimal generator of a C_0-semigroup $S(\cdot)$ given by $S(t) = \operatorname{diag}\big(\exp[\lambda_n t]\big)$.

Next, define a linear operator B in X by

$$(Bx)_n = \alpha_n \sum_{j=1}^{\infty} \alpha_j \lambda_j x_j$$

with $\alpha = (\alpha_j) \in l^2(\mathbf{R})$ is chosen so that B is A-bounded and $\limsup_{n\to\infty} \alpha_n^2 |\lambda_n| \cdot \exp[\operatorname{Re} \lambda_n] = \infty$. The specific choice of (λ_n) and (α_n) is still at our disposal.

We claim that the operator $\mathcal{A} = \begin{pmatrix} A & B \\ 0 & A \end{pmatrix}$ cannot be the infinitesimal generator of a C_0-semigroup on $X \times X$. Infact, if $\mathcal{A}$ were the infinitesimal generator of a C_0-semigroup $\mathcal{C}(\cdot)$ on $X \times X$ we consider the elements $y_m = (\delta_{m,k})_{k=1,\dots} \in X$. As $y_m \in D(A^\infty)$ we infer that $t \to S(t) y_m \in C^\infty(0, \infty; D(A))$ and hence $BS(t) y_m \in C^\infty(0, \infty; X)$.

Consequently,

$$\tilde{x}(t) = \begin{pmatrix} \int_0^t S(t-s) BS(s)\, y_m\, ds \\ S(t)\, y_m \end{pmatrix}$$

would be a strong solution of the Cauchy-problem $(d/dt)\tilde{x}(t) = \mathcal{A}\tilde{x}(t)$.

Being a C_0-semigroup there must exist a constant M so that $\|\mathcal{C}(t)\| \leqslant M$ for $0 \leqslant t \leqslant 1$. In particular we thus would expect that $\left\| \mathcal{C}(1) \begin{pmatrix} 0 \\ y_m \end{pmatrix} \right\| \leqslant M$, and in particular

$$\left| \left(\int_0^1 S(1-s) BS(s) y_m ds \right)_m \right| \leqslant M\,. \tag{2}$$

The left hand side of this inequality can be rewritten as

$$\int_0^1 \exp[(1-s)\lambda_m] \alpha_m \lambda_m \alpha_m \exp[s\lambda_m] ds = \exp[\lambda_m] \alpha_m^2 \lambda_m$$

and as by assumption $\limsup_{m\to\infty} [\operatorname{Re} \lambda_m] \alpha_m^2 |\lambda_m| = \infty$ we see that we cannot have an estimate of the form (2), i.e. $\mathcal{C}(\cdot)$ is not a C_0-semigroup.

We now specify the (λ_m) and (α_m):

1. *The case of C_0-group.*

Let $\lambda_m = im$, $m = 1, 2, \dots$. Then $A = -A^*$ and so $\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$ generate a

C_0-group on X. Putting

$$\alpha_m = \begin{cases} m^{-\frac{1}{4}} & \text{if } m^{\frac{1}{4}} \text{ is an integer} \\ 0 & \text{otherwise} \end{cases}$$

we see that $\alpha = (\alpha_m) \in l^2(\mathbb{R})$.

Moreover,

$$\alpha_m^2 |\lambda_m| \exp [\operatorname{Re} \lambda_m] = \begin{cases} m^{\frac{1}{2}} & \text{if } m^{\frac{1}{4}} \text{ is an integer} \\ 0 & \text{otherwise} \end{cases}$$

and hence $\limsup_{m\to\infty} \alpha_m^2 |\lambda_m| \exp [\operatorname{Re} \lambda_m] = \infty$.

The associated operator B clearly satisfies (i), (ii) but according to the above consideration $\mathcal{A} + B$ is not an infinitesimal generator.

2. *The case of an ultimately compact and differentiable semigroup.*

Let $\lambda_m = -m + i \exp [4m]$ and set $\alpha_m = \exp [-m]$. Then we obtain $\alpha_m^2 |\lambda_m| \exp [\operatorname{Re} \lambda_m] \geqslant \exp [m] \to \infty$ as $m \to \infty$ and so again the associated operator $\begin{pmatrix} A & B \\ 0 & A \end{pmatrix}$ cannot be an infinitesimal generator. In order to show that $S(\cdot)$ is differentiable for $t \geqslant 4$ it is sufficient to verify that $\lambda_m \exp [\lambda_m t]$ is bounded for $t \geqslant 4$. This follows from

$$|\lambda_m \exp [\lambda_m t]| = |-m + i \exp [4m]| \exp [-mt] \leqslant m \exp [-mt] + \exp [m(4-t)]\,|.$$

It is also obvious that $S(\cdot)$ is compact for $t \geqslant 4$.

Some generation results.

As already pointed out in the introduction, we present a general generation result that seems to be very useful in applications. Throughout this section, we assume that $(X, \|\cdot\|)$ is a Banach-space. If A is a closed linear operator in X we let X_A stand for the Banach space $(D(A), |\cdot|_A)$, with $|x|_A = \|x\| + \|Ax\|$.

THEOREM. Let A be the infinitesimal generator of a C_0-semigroup $T(\cdot)$ on X. Let $(Z, |\cdot|_Z)$ be a Banach space such that

(Z1) Z is continuously embedded in X, (Z2) there is a $t_0 > 0$ so that for all continuous functions

$$\phi\colon [0, t_0] \to Z \quad \text{we have} \int_0^t T(t-s)\phi(s)\,ds \in D(A) \quad \text{for all} \quad t \in [0, t_0]$$

(Z3) there is an increasing continuous function $\gamma\colon [0, t_0] \to [0, \infty)$ satisfying $\gamma(0) = 0$ and

$$\left|\int_0^t T(t-s)\phi(s)\,ds\right|_A \leqslant \gamma(t) \sup_{0\leqslant s\leqslant t} |\phi(s)|_Z .$$

Then for any continuous linear operator $B\colon X_A \to Z$, $(A + B)$ is the infinitesimal generator of a C_0-semigroup on X.

PROOF. To begin with, we verify that under the above assumptions the map $t \to \int_0^t T(t-s)\phi(s)\,ds$ is continuous from $[0, t_0]$ into X_A: In fact, given any continuous $\phi\colon [0, t_0] \to Z$ and $t \in [0, t_0]$, we define for all $0 \leqslant h \leqslant t$

$$\phi_h(s) = \begin{cases} \phi(s-h) & \text{for } h \leqslant s \leqslant t+h \\ 0 & \text{otherwise}. \end{cases}$$

Then we obtain for all $t \in [0, t_0 - h]$

$$\begin{aligned}
&\left|\int_0^{t+h} T(t+h-s)\phi(s)\,ds - \int_0^t T(t-s)\phi(s)\,ds\right|_A \\
&\quad = \left|\int_0^{t+h} T(t+h-s)\big(\phi(s)-\phi_h(s)\big)\,ds\right|_A \\
&\quad \leqslant \left|\int_h^{t+h} T(t+h-s)\big(\phi(s)-\phi_h(s)\,ds\right|_A + \left|\int_0^h T(t+h-s)\big(\phi(s)-\phi_h(s)\big)\,ds\right|_A \\
&\quad \leqslant \gamma(t) \sup |\phi(s+h)-\phi(s)|_Z + |T(t)|_A \gamma(h) \sup |\phi(s)-\phi_h(s)|_A .
\end{aligned}$$

As the right side converges to 0 for $h \to 0$, the claim follows.

In the next step we verify that $(A + B)$ is a closed linear operator. To this end, we first estimate $(\lambda I - A)^{-1}B$ as an operator from X_A into X_A. Let N and ω be constants such that

$$\|T(t)\| \leqslant N \exp[\omega t] \quad \text{for all } t \geqslant 0 .$$

Then we obtain for all sufficiently large λ and all $x \in D(A)$

$$\begin{aligned}
&|(\lambda I - A)^{-1} Bx|_A \\
&\quad = \left| \int_0^t \exp[-\lambda s] T(s) Bx\, ds + \exp[-\lambda t] T(t) \int_0^\infty \exp[-\lambda s] T(s) Bx\, ds \right|_A \\
&\quad \leqslant \exp[-\lambda t] \left| \int_0^t T(t-s)(\exp[\lambda s] Bx)\, ds \right|_A + N \exp[(\omega - \lambda)t] |(\lambda I - A)^{-1} Bx|_A \\
&\quad \leqslant \gamma(t) \sup |\exp[\lambda s] Bx|_Z + N \exp[(\omega - \lambda)t] |(\lambda I - A)^{-1} Bx|_A .
\end{aligned}$$

Putting

$$t(\lambda) = \frac{\ln N + \ln 2}{\lambda - \omega}$$

we obtain

$$|(\lambda I - A)^{-1} Bx|_A \leqslant 2\gamma(t(\lambda)) \beta |x|_A ,$$

where β denotes the norm of B regarded as an operator from X_A into Z, and as $t(\lambda) \to 0$ as $\lambda \to \infty$ we deduce that

$$K(\lambda) = 2\gamma(t(\lambda)) \quad \text{converges to } 0 \quad \text{as } \lambda \to \infty .$$

In order to prove that $(A + B)$ is closed, let (x_n) be a sequence in X_A such that $x_n \to x$ (in X) and $y_n := (A + B)x_n$ converges to $y \in X$. We have to show that (x_n) is a Cauchy sequence in X_A: Fix some $\lambda > 0$ with $K(\lambda) < 1$. Then we have

$$\begin{aligned}
|x_n - x_m|_A &= |\lambda(\lambda I - A)^{-1}(x_n - x_m) - (\lambda I - A)^{-1} A(x_n - x_m)|_A \\
&= |\lambda(\lambda I - A)^{-1}(x_n - x_m) - (\lambda I - A)^{-1}(y_n - y_m) + (\lambda I - A)^{-1} B(x_n - x_m)|_A \\
&\leqslant q\lambda \|x_n - x_m\| + q\|y_n - y_m\| + K(\lambda)|x_n - x_m|_A ,
\end{aligned}$$

where q denotes the norm of $(\lambda I - A)^{-1}$ regarded as an operator from X into X_A.

Consequently, we obtain

$$|x_n - x_m|_A \leqslant \frac{1}{1 - K(\lambda)} \left(\lambda q \|x_n - x_m\| + q' \|y_n - y_m\| \right) .$$

This implies that $y_n = (A + B)x_n$ converges to $(A + B)x$ so that $(A + B)x = y$. The proof that $(A + B)$ is infact the infinitesimal generator of a C_0-semigroup on X is performed by making use of Ball's Theorem ([1]). Roughly speaking, we are to show that for any $x \in X$ the Cauchy problem

$$\begin{aligned} &\frac{d}{dt}x(t) = (A + B)x(t)\,, \qquad t > 0 \\ &x(0) = x\,, \end{aligned} \tag{3}$$

admits a unique weak solution $x(t)$ on $[0, \infty)$, *i.e.* for all $x^* \in D\big((A + B)^*\big)$ and $t \geq 0$

$$\langle x(t), x^*\rangle = \langle x, x^*\rangle + \int_0^t \langle x(s), (A + B)^* x^*\rangle\, ds\,.$$

So, fix $x \in X$ and $\hat{t} > 0$. Given any continuous function $z\colon [0, \hat{t}] \to X_A$ we put

$$(\mathcal{T}z)(t) = \int_0^t T(t - s)Bz(s)\,ds + \int_0^t T(t-s)x\,ds\,, \qquad 0 \leq t \leq \hat{t}\,.$$

By the above considerations $(\mathcal{T}z)(\cdot)$ is a continuous function $[0, \hat{t}] \to X_A$. If β denotes again the norm of B regarded as an operator $X_A \to Z$, we obtain for all $0 \leq t < \hat{t}$:

$$|\mathcal{T}z_1(t) - \mathcal{T}z_2(t)|_A \leq \beta\gamma(t) \sup |z_1(s) - z_2(s)|_A\,.$$

Choosing $\hat{t}$ sufficiently small we conclude that there exists a unique fixed point of $\mathcal{T}$. As $\beta\gamma(t)$ does not depend on x, we may continue this procedure and obtain a continuous function $y\colon [0, \infty) \to X_A$ satisfying

$$y(t) = \int_0^t T(t-s)By(s)\,da + \int_0^t T(t-s)x\,ds\,.$$

Putting $x(t) = (A + B)y(t) + x$, it is clear that $x(0) = x$ and x is continuous $[0, \infty) \to X$. Moreover, we have for all $h > 0$

$$\begin{aligned} \frac{1}{h}\big(y(t + h) - y(t)\big) &= \frac{1}{h}\Big(\int_0^{t+h} T(t + h - s)\big(x + By(s)\big)\,ds - \int_0^t T(t-s)\big(x + By(s)\big)\,ds\Big) \\ &= \frac{1}{h}\big(T(h) - I\big)\int_0^t T(t-s)\big(x + By(s)\big)\,ds + \frac{1}{h}\int_t^{t+h} T(t + h - s)\big(x + By(s)\big)\,ds\,. \end{aligned}$$

and as $y(t) \in D(A)$, the right hand side converges to $Ay(t) + x + By(t)$ as $h \to 0^+$. Therefore, we have

$$\frac{d^+}{dt} y(t) = (A + B)y(t) + x = x(t) .$$

For any $x^* \in D((A + B)^*)$ we obtain

$$\langle x(t), x^* \rangle = \langle x, x^* \rangle + \langle y(t), (A + B)^* x^* \rangle$$

and hence $x(t)$ is a weak solution of (3).

In order to verify uniqueness of this weak solution, let $x(\cdot)$ be any weak solution of (3) with $x(0) = 0$. Putting

$$y(t) = \int_0^t x(s)\, ds ,$$

we get for all $x^* \in D((A + B)^*)$

$$\langle x(t), x^* \rangle = \int_0^t \langle x(s), (A + B)^* x^* \rangle ds = \langle y(t), (A + B)^* x^* \rangle .$$

As $(A + B)$ is closed, this implies that $y(\cdot) \in D(A)$ and hence

$$x(t) = \frac{d}{dt} y(t) = (A + B) y(t) .$$

From the variation of constants formula for $T(\cdot)$ we get

$$y(t) = \int_0^t T(t-s) B y(s)\, ds .$$

The unique solution of this integral equation is $y = 0$ and hence $x = 0$.

Hence (3) admits a unique weak solution for all $x \in X$ showing that $(A + B)$ is the infinitesimal generator of a C_0-semigroup on X.

The particular choice of Z depends of course, heavily on the problem under consideration and it may vary through a large class of different spaces and is illustrated by the following examples.

1. $Z = X_A$.

A particular interesting case of Z is provided by putting $Z = X_A$. Clearly, assumptions (Z1)-(Z3) are satisfied (with $\gamma(t) = tM \exp[\omega t]$), and hence we deduce that for any continuous linear operator $B\colon X_A \to X_A$ the operator $(A + B)$ is the infinitesimal generator of a C_0-semigroup on X.

2. *Delay equations on product spaces.*

Let Y be a real Banach space and put $X = Y \times L^p(-r, 0; Y)$, $1 \leqslant p < \infty$, $0 < r \leqslant \infty$. Let $T(\cdot)$ denote the solution semigroup of the unperturbed equation given by

$$T(t)(\eta, \varphi) = (\eta, \psi)$$

$$\psi(s) = \begin{cases} \eta & \text{if } s + t \geqslant 0 \\ \varphi(s + t) & \text{if } s + t < 0 . \end{cases}$$

As already mentioned in the introduction its infinitesimal generator A is given by

$$D(A) = \{(u(0), u) \mid u \in W^{1,p}(-r, 0; Y)\},$$

$$A\big(u(0), u\big) = (0, \dot{u}).$$

Then the assumptions of the generation theorem are satisfied with $Z = Y \times \{0\}$. Therefore, for any linear operator B that maps $W^{1,p}(-r, 0; Y)$ continuously into Y the operator $(A + B)$ is an infinitesimal generator of a C_0-semigroup on X.

PROOF. For any $t > 0$ let φ be a continuous function $[0, t] \to Y$. Define ψ by

$$\psi(s)(\theta) = \begin{cases} \varphi(s) & \text{if } \theta \geqslant s - t, \\ 0 & \text{otherwise}. \end{cases}$$

Then for all $\theta \in [-r, 0]$ we have

$$\int_0^t \psi(s)(\theta)\, ds = \int_0^{t+\theta} \varphi(s)\, ds,$$

$$\Big(\int_0^t \psi(s)\, ds, \int_0^t \varphi(s)\, ds\Big) = \int_0^t T(t-s)\big(\psi(s), \varphi(s)\big)\, ds \in D(A)$$

and since

$$\Big(\int_{-r}^0 |\tilde{\varphi}(s + t)|^p\, ds\Big)^{1/p} \leqslant t^{1/p} \sup |\varphi(\tau)|$$

where

$$\tilde{\varphi}(s) = \begin{cases} \varphi(s) & \text{for } s \geqslant 0\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

also (Z3) is satisfied.

Thus it is not the finite-dimensional range property of the perturbation that ensures the generation of a C_0-semigroup. Of course, the above condition is too restrictive for partial-differential equations involving delay terms, although they are not far away from being necessary ([9]).

We next turn to partial differential equations: To begin with we provide

3. *The Favard-class of* $T(\cdot)$.

Let A be the infinitesimal generator of a C_0-semigroup $T(\cdot)$ on X. Then the assumptions of the theorem are satisfied for Z being the Favard class of $T(\cdot)$, i.e.

$$Z = \left\{ x \in X \,\middle|\, \limsup_{t \to 0^+} \frac{1}{t} \|T(t)x - x\| \quad \text{is finite} \right\},$$

$$|x|_Z = \|x\| + \limsup_{t \to 0^+} \frac{1}{t} \|T(t)x - x\|\,.$$

PROOF. Let φ be a continuous function $[0, t] \to Z$. Then there is a sequence (φ_n) of continuously differentiable functions such that $\varphi_n \to \varphi$ as $n \to \infty$. Clearly, $\int\limits_0^t T(t-s)\varphi_n(s)\,ds \in D(A)$, and

$$\left\| A \int_0^t T(t-s)\varphi_n(s)\,ds \right\| \leqslant \limsup_{h \to 0^+} \int_0^t \frac{1}{h} \|(T(h) - I)\, T(t-s)\varphi_n(s)\,ds\|$$

$$\leqslant M \exp[\omega t] \int_0^t |\varphi_n(s)|_Z\,ds \leqslant tM \exp[\omega t] \sup |\varphi_n|_Z\,.$$

(Here M and ω are constants such that $\|T(t)\| \leqslant M \exp[\omega t]$ for all $t \geqslant 0$). A closedness argument now shows that the same estimate is also valid for φ.

An interesting application of this result is

4. *Integrodifferential equations.*

Let Y be a Banach space and consider the Cauchy problem

$$\text{(4)} \qquad \begin{aligned} &\frac{d}{dt} u(t) = Lu(t) + \int_0^t C(t-s)\,u(s)\,ds + f(t)\,, \quad t \geqslant 0\,, \\ &u(0) = u_0\,. \end{aligned}$$

Here L is the infinitesimal generator of a C_0-semigroup $S(\cdot)$ on Y and $\{C(t); t \geqslant 0\}$ is a family of continuous linear operators $X_L \to Y$ such that for each $x \in X_L$ the map $\mathcal{C}x$ given by $(\mathcal{C}x)(t) = C(t)x$ belongs to $L^1(0, \infty; Y)$.

Following the notation used in [3], we say that (4) is uniformly well posed if for each $u_0 \in D(L)$ and each $f \in W^{1,1}(0, \infty; Y)$ then exists a unique strong solution $u(t, u_0; f)$ of (4) which depends continuously on u_0 (with respect to the Y-norm) and f (with respect to the L^1-norm), uniformly for t in compact intervals.

There is a large number of papers in which uniform well-posedness of (4) is proven under various additional smoothness assumptions on $\mathcal{C}(\cdot)$. Our approach allows the following very general result:

THEOREM. (4) is uniformly well-posed if $\mathcal{C}x$ is of bounded variation for each $x \in D(L)$.

The underlying basic idea that was introduced in [10] and carried out in a more general framework in [3] is to associate to (4) a differential equation in a larger Banach space.

To this end, let $T(\cdot)$ denote the shift semigroup on $L^1(0, \infty; Y)$ defined by $\big(T(t)\phi\big)(s) = \phi(s + t)$, $s \geqslant 0$, $t \geqslant 0$. Moreover, let D_s denote its infinitesimal generator and let δ be the operator $W^{1,1}(0, \infty; Y) \to Y$ given by $\big(\delta\phi = \phi(0)\big)$.

In [3] it is shown that (4) is uniformly well-posed if and only if the following abstract Cauchy problem is uniformly well-posed in $X = Y \times Y \times \times L^1(0, \infty; Y)$

$$\frac{d}{dt} x(t) = \mathcal{A}x(t)\,, \qquad t \geqslant 0\,, \qquad x(0) = x_0\,, \tag{5}$$

where $\mathcal{A}$ is given by

$$D(\mathcal{A}) = Y \times D(L) \times W^{1,1}(0, \infty; Y)\,,$$

$$\mathcal{A} = \begin{pmatrix} 0 & L & 0 \\ 0 & L & \delta \\ 0 & \mathcal{C} & D_s \end{pmatrix}.$$

So, we have to show that $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup on X. To prove this claim, we split up $\mathcal{A}$ as

$$\mathcal{A} = \mathcal{A}_0 = \mathcal{B}$$

where

$$\mathcal{A}_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & L & \delta \\ 0 & 0 & D_s \end{pmatrix} \quad \text{and} \quad \mathcal{B} = \begin{pmatrix} 0 & L & 0 \\ 0 & 0 & 0 \\ 0 & \mathcal{C} & 0 \end{pmatrix}.$$

It is an elementary calculation to verify that $\mathcal{A}_0$ generates a semigroup $\mathcal{T}(\cdot)$ given by

$$\mathcal{T}(t)\begin{pmatrix}x\\y\\f\end{pmatrix}=\begin{pmatrix}x\\S(t)y+\int_0^t S(t-s)f(s)\,ds\\T(f)t\end{pmatrix}.$$

The range of $\mathcal{B}$ consists of all vectors $\begin{pmatrix}x\\0\\f\end{pmatrix}$ where f is of bounded variation. If we can verify that these vectors belong to the Favard class of $\mathcal{T}(\cdot)$. Example 2 implies that $\mathcal{A}$ generates a C_0-semigroup on X and hence (4) is uniformly well-posed.

As

$$\frac{1}{t}\left(\mathcal{T}(t)\begin{pmatrix}x\\0\\f\end{pmatrix}-\begin{pmatrix}x\\0\\f\end{pmatrix}\right)=\left(0,\frac{1}{t}\int_0^t S(t-s)f(s)\,ds\,,\,\frac{1}{t}\left(T(t)f-f\right)\right)^T$$

and

$$\frac{1}{t}\left\|\int_0^t S(t-s)f(s)\,ds\right\|\leqslant\sup_{0\leqslant s\leqslant t}\|S(t-s)\|\,\|f(s)\|$$

is bounded as f has bounded variation, and

$$\frac{1}{t}\|T(t)f-f\|=\frac{1}{t}\int_0^\infty\|f(s+t)-f(s)\|\,ds\leqslant\operatorname{Var}\left(f,(0,\infty)\right)$$

([2], Appendix), we conclude that

$$\limsup_{t\to 0^+}\frac{1}{t}\left\|\mathcal{T}(t)\begin{pmatrix}x\\0\\f\end{pmatrix}-\begin{pmatrix}x\\0\\f\end{pmatrix}\right\|\quad\text{is finite.}$$

5. *Interpolation spaces.*

Let A be the infinitesimal generator of an analytic semigroup $T(\cdot)$. Then we can take $Z=(D(A),X)_I$, an interpolation space between $D(A)$ and X.

The proof follows from the general properties of interpolation spaces (see [4]).

If we are dealing with nonlinear perturbations then the situation is much more complicated as is seen by the following example. Roughly speaking, we show that even for a one-dimensional, C^∞-nonlinear perturbation B that maps X_A into itself the operator $(A + B)$ is not a generator of a nonlinear semigroup on X.

EXAMPLE. Let $X = \mathbf{R}\times C_{u,b}(\mathbf{R};\mathbf{R})$, where $C_{u,b}(\mathbf{R};\mathbf{R})$ denotes the usual Banach space of all uniformly continuous, bounded functions $\mathbf{R}\to\mathbf{R}$.

Let $T(\cdot)$ be the linear semigroup given by $T(t)\begin{pmatrix}x\\ \varphi\end{pmatrix} = \begin{pmatrix}x\\ \varphi(t+\cdot)\end{pmatrix}$. An easy calculation shows that its infinitesimal generator A is given by

$$D(A) = \left\{\begin{pmatrix}x\\ \varphi\end{pmatrix}\,\middle|\,\varphi' \in C_{u,b}(\mathbf{R};\mathbf{R})\right\}$$

$$A\begin{pmatrix}\varphi\\ x\end{pmatrix} = \begin{pmatrix}0\\ \varphi'\end{pmatrix}.$$

Let ψ be a C^∞-function $\mathbf{R}\to\mathbf{R}$ such that

$$\psi(0) = 0\,, \qquad \psi(s)\geqslant \inf\left(|s|, s^2\right)\,,$$

and ψ is Lipschitzian with constant $\leqslant 2$.

We define B by $B\begin{pmatrix}x\\ \varphi\end{pmatrix} = \begin{pmatrix}\psi(\varphi'(0))\\ 0\end{pmatrix}$.

For $\begin{pmatrix}0\\ \zeta\end{pmatrix}\in D(A)$, the Cauchy problem

$$\frac{d}{dt}\begin{pmatrix}x\\ \varphi\end{pmatrix} = A\begin{pmatrix}x\\ \varphi\end{pmatrix} + B\begin{pmatrix}x\\ \varphi\end{pmatrix}, \quad \begin{pmatrix}x\\ \varphi\end{pmatrix}(0) = \begin{pmatrix}0\\ \zeta\end{pmatrix},$$

has a unique strong solution given by

$$\begin{pmatrix}x\\ \varphi\end{pmatrix}(t) = \left(\int^t \psi(\zeta'(s))\,ds,\ \zeta(t+.)\right)$$

Given $t\in(0,1]$ we choose a $\zeta\in C_{u,b}(\mathbf{R};\mathbf{R})$ such that the restriction of ζ to $(0,t]$ does not have a bounded variation. Let (ζ_n) be a sequence in $C_{u,b}(\mathbf{R};\mathbf{R})$ so that $\zeta_n\to\zeta$ and $(0,\zeta_n)^T\in D(A)$.

If the solutions $\begin{pmatrix}x_n(\cdot)\\ \varphi_n(\cdot)\end{pmatrix}$ with initial values $\begin{pmatrix}0\\ \zeta_n\end{pmatrix}$ were convergent to, say $\begin{pmatrix}\eta(t)\\ \omega(t)\end{pmatrix}$, then we clearly have $\omega(t) = \zeta(t+\cdot)$.

Let $0 \leqslant s_1 \leqslant s_2 \leqslant t$. We choose measurable sets E_1 and E_2 such that

$$[s_1, s_2] = E_1 \cup E_2 \qquad \text{and} \qquad \psi(\zeta_n'(s)) \geqslant |\zeta_n'(s)| \qquad \text{on } E_1$$

and

$$\psi(\zeta_n'(s)) \geqslant (\zeta_n'(s))^2 \quad \text{on } E_2 .$$

Then

$$\int_{s_1}^{s_2} \psi(\zeta_n'(s))\, ds \geqslant \int_{E_1} |\zeta_n'(s)|\, ds + \int_{E_2} (\zeta_n'(s))^2 ds \geqslant \int_{E_1} |\zeta_n'(s)|\, ds + \left(\int_{E_2} |\zeta_n'(s)|\, ds \right)^2 \cdot \frac{1}{\operatorname{meas}(E_2)} .$$

Consequently,

$$\int_{s_1}^{s_2} |\zeta_n'(s)|\, ds \leqslant (s_2 - s_1) + \int_{s_1}^{s_2} \psi(\zeta_n'(s))\, ds .$$

Taking the limit $n \to \infty$ we thus obtain

$$|\zeta(s_2) - \zeta(s_1)| \leqslant \gamma(s_2) - \gamma(s_1) + s_2 - s_1 ,$$

where γ is the limit of the sequence of monotone functions $\int_0^t \psi(\zeta_n'(s))\, ds$ and hence γ is itself monotone. Therefore ζ must be of bounded variation which contradicts the assumptions. As a consequence we deduce that the solution operators cannot be continuous which is a standing hypothesis for all nonlinear semigroups.

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