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On the Hull of Holomorphy of an n-Manifold in $\mathbf{C}^n$.

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0. – Introduction.

In this paper we consider the local properties of a real n-dimensional submanifold M in the complex space $\mathbf{C}^n$. Generically, such a manifold is totally real and basically has the characteristics of the standard $\mathbf{R}^n \subseteq \mathbf{C}^n$. The nature of M near a complex tangent can be much more complicated. Thus far, only one dimensional complex tangents which are sufficiently non-degenerate have been studied. This study was initiated by E. Bishop [1], who attached an invariant $\gamma \geqslant 0$ to each point having such a tangent. In the elliptic case, $0 \leqslant \gamma < \frac{1}{2}$, he showed the existence of a one-parameter family of analytic discs with boundaries on M and shrinking down to the point. The nature of the set $\tilde{M}$ swept out by these discs was further studied by Hunt [2]. In [3] we made a fairly complete study of the local properties of a smooth surface near an elliptic point in $\mathbf{C}^2$. We were able to show that $\tilde{M}$ is a smooth manifold-with-boundary. In [4] the real analytic case was studied by completely different methods. One of the results there is that $\tilde{M}$ is a real analytic manifold-with-boundary if $0 < \gamma < \frac{1}{2}$.

The results of the present paper yield the following theorem.

THEOREM. *Let M be a C^∞-smooth real n-manifold in $\mathbf{C}^n$ with an elliptic nondegenerate complex tangent at a point p. Then, for each $l > 0$ there exists an $(m-1)$-parameter family of discs bounding on M and sweeping out a manifold-with-boundary $\tilde{M}_l$ of differentiability class C^l. $\partial\tilde{M}_l$ contains a neighborhood of p in M.*

However, we prove more than what is stated in this theorem. The locus of complex tangents to M near p forms a smooth $(n-2)$-dimensional mani-

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fold N. A fixed neighborhood of p in N is contained in the boundary of every $\tilde{M}_t$. Also, $\tilde{M}_t - N$ is a C^∞ manifold-with-boundary.

Although our present arguments parallel those in [3], several features of the problem make the case $n > 2$ much more difficult than the case $n = 2$. First, the fact that the complex tangents are not isolated complicates the construction, given in section 1, of an approximating family of analytic discs. This in turn requires further modification of the analysis in section 2 of the Hilbert transform. The most delicate problem is the solution of the functional equation in section 3, which produces a perturbation of the approximating family so that the boundaries lie on M. Considerably more work is required to invert the linearized equations and to prove the regularity of the solution up to the boundary M.

One would like to say that the local hull of holomorphy of M is precisely a $C^\infty(n+1)$–manifold-with-boundary. This would follow immediately from the results of the present paper if one could show that each $\tilde{M}_t$ is holomorphically convex. This was essentially the argument in [3] for the case $n = 2$.

1. – An approximating family of analytic discs.

Given a smooth, real n-dimensional submanifold M in $\mathbf{C}^n$ and a point p in M at which M has an elliptic non-degenerate complex tangent, we shall construct an $(n-1)$-parameter family of analytic discs with boundaries close to M. Relative to suitable local holomorphic coordinates $z = (z^1, \dots, z^n)$ on $\mathbf{C}^n$, M is given locally near p (in column n-vector notation) by equations

$$(1.1)\qquad M\colon R(z) = 0\,, \qquad R = \bar{R} = (R^1, \dots, R^n)^t\,, \qquad dR^1\wedge\dots\wedge dR^n \neq 0\,.$$

The function R is smooth and satisfies

$$(1.2)\qquad \operatorname{rank}\frac{\partial(R^1, \dots, R^n)}{\partial(z^1, \dots, z^n)}(p) = n-1\,.$$

By [1] or [4] we may further assume, when the invariant $\gamma \neq \frac{1}{2}$, that $p = 0$ and R has the form

$$(1.3)\qquad \begin{cases} R^1 + iR^n = z^n - (h+ik)(z^1, x)\,, & x = (x^2, \dots, x^{n-1})\,,\\ R^\alpha = y^\alpha - h^\alpha(z^1, x)\,, & 2 \leqslant \alpha \leqslant n-1\,, \end{cases}$$

where $z^j = x^j + iy^j$, $1 \leqslant j \leqslant n$. (Generally Greek indices will range from 2 to $n-1$, small latin indices from 1 to n, and the summation convention will be employed, except where indicated otherwise.) The real functions h, k, h^α satisfy

$$(1.4)\qquad \begin{cases} h = |z^1|^2 + \gamma(z^1)^2 + \gamma(\bar{z}^1)^2 + O(|z|^3), & 0 \leqslant \gamma \\ h^\alpha = O(|z|^3), \quad k = O(|z|^3) \end{cases}$$

as z tends to 0. The locus of points p at which M has a complex tangent is a smooth manifold N of codimension 2 in M, given by an additional equation

$$(1.5)\qquad N: z^1 = P(x), \quad P(0) = 0,$$

obtained by setting the determinant of (1.2) equal to zero.

We introduce new parameters $t \in \mathbf{C}$, $u \in \mathbf{R}$, $s \in \mathbf{R}^{n-2}$ and define $z^*(s) \in N$ by

$$(1.6)\qquad R(z_*(s)) = 0, \quad z_*^1 = P(s), \quad x_*^\alpha = s^\alpha.$$

For each integer $l \geqslant 3$ we shall construct a hypersurface M_0, partially bounding a domain D_0 in (t, u, s)-space, of the form

$$(1.7)\qquad M_0: u = (q + f)(t, s), \quad D_0: u \geqslant (q + f)(t, s).$$

The functions f and q are to have the form

$$(1.8)\qquad \begin{cases} q(t, s) = t\bar{t} + \gamma(s)(t^2 + \bar{t}^2), & 0 \leqslant \gamma(s) < \frac{1}{2},\ \gamma(0) = \gamma, \\ f = \bar{f} = \sum_{j=3}^{l} f_j(t, s), & f_j = \sum_{0 \leqslant i \leqslant j/2} c_{ji}(s) t^{j-i} \bar{t}^i. \end{cases}$$

Also, we shall construct a smooth mapping T from D_0 into $\mathbf{C}^n$ of the form

$$(1.9)\qquad \begin{cases} T: z = z^*(s) + B(t, u, s), \quad \partial_{\bar{t}} B = 0, \\ B = \sum_{j=1}^{l} B_j, \quad B_j = \sum_{0 \leqslant i \leqslant j/2} B_{ji}(s) t^{j-2i} u^i. \end{cases}$$

Here $B = (B^1, \ldots, B^n)^t$ is a complex column n-vector, which will be chosen to satisfy

$$(1.10)\qquad \begin{cases} B^1(t, u, s) = \mu(s) t, \quad \mu\bar{\mu} = 1, \quad \mu(0) = 1, \\ \operatorname{Re} B^\alpha(0, u, s) = 0, \\ \operatorname{Re} B^n(0, u, s) = \beta(s) u, \quad \beta(0) = 1 \\ \operatorname{Im} \partial_u B^n(0, 0, 0) = 0. \end{cases}$$

The coefficients $c_{ji}(s)$, $B_{ji}(s)$, $\gamma(s)$, $\mu(s)$, $\beta(s)$ will be smooth functions on $|s| < \varepsilon_0$, for some $\varepsilon_0 > 0$.

PROPOSITION (1.1). *Let M be as above. There exists $\varepsilon_0 > 0$, independent of l, and unique functions $\gamma(s)$, f, B as above so that, as $|t| \to 0$,*

$$R(z(t, u, s)) = O(|t|^{l+1}), \tag{1.11}$$

when $u = (q + f)(t, s)$, uniformly for $|s| < \varepsilon_0$.

We begin the proof by determining $B_1 = B_{10}(s)t$, provisionally, as the unique solution of

$$R_j(z^*(s))B_{10}^j(s) = 0, \quad B_{10}^1(s) \equiv 1, \quad (R_j \equiv \partial R/\partial z^j). \tag{1.12}$$

The rank condition (1.2) permits this for $|s| < \varepsilon_0$, some $\varepsilon_0 > 0$. By (1.3)

$$B_{10}(0) = (1, 0, \dots, 0)^t. \tag{1.13}$$

By Taylor's formula we have

$$\begin{cases} R(z_* + B) = R(z^*) + 2\operatorname{Re}\{R_i(z^*)B^i\} + \sum_{j=2}^{l} \frac{1}{j!} R_{a_1\dots a_j}(z^*)B^{a_1}\dots B^{a_j} + E, \\ E = \frac{1}{l!}\int_0^1 R_{a_1\dots a_{l+1}}(z^* + \sigma B)\,d\sigma B^{a_1}\dots B^{a_{l+1}}, \end{cases} \tag{1.14}$$

where the a_i's are summed from 1 to $2n$ and

$$B^a = \bar{B}^{a-n}, \quad R_a = \partial R/\partial \bar{z}^{a-n}, \quad \text{if } a > n.$$

Thus far, we have $R(z^*(s) + B_1(s, t)) = O(|t|^2)$, for $|s| < \varepsilon_0$. (In this and the following equations the constant in O is independent of s.)

We next determine $\gamma(s)$, $\mu(s)$, and

$$B_2 = B_{20}(s)t^2 + B_{21}(s)u.$$

By (1.12) and (1.14) we have

$$R(z_* + \mu B_1 + B_2) = 2\operatorname{Re} R_j B_2^j + \operatorname{Re}(\mu^2 R_{ij} B_1^i B_1^j) + R_{i\bar{j}} B_1^i \bar{B}_1^j + \dots .$$

In this equation we substitute $u = q(t, s)$ (see 1.8) to get

$$R(z_* + \mu B_1 + B_2) = a_0(s)t^2 + b_0(s)t\bar{t} + \bar{a}_0(s)\bar{t}^2 + O(|t|^3),$$

for $|s| < \varepsilon_0$, where

$$(1.15)\quad \begin{cases} b_0 = 2\,\mathrm{Re}\,(R_j B^j_{21}) + b\,, & b(s) \equiv R_{i\bar{j}} B^i_{10} \bar{B}^j_{10}\,, \\ a_0 = R_j B^j_{20} + 2\gamma(s)\,\mathrm{Re}\,(R_j B^j_{21}) + \mu(s)^2 a\,, & a(s) \equiv \tfrac{1}{2} R_{ij} B^i_{10} B^j_{10}\,. \end{cases}$$

We first choose $B_{21}(s)$ so that $b_0(s) = 0$. To this end it suffices to show that the real linear transformation

$$(1.16)\qquad B \mapsto 2\,\mathrm{Re}\,\big(R_j(z^*(s))B^j\big)$$

is invertible when restricted to a suitable real n-dimensional subspace of $\mathbf{C}^n$. We check this first at $s = 0$, where $z^*(s) = 0$, and by (1.3)

$$R_1(0) = 0\,,\quad R_\alpha(0) = \big(0, 0, (-i/2)\delta^\beta_\alpha\big)^t\,,\quad R_n(0) = (\tfrac{1}{2}, -i/2, 0)^t\,.$$

Thus,

$$(1.17)\qquad R_j(0)B^j = \tfrac{1}{2}(B^n, -iB^n, -iB^\alpha)^t\,,$$

and

$$(1.18)\qquad 2\,\mathrm{Re}\,R_j(0)B^j = (\mathrm{Re}\,B^n, \mathrm{Im}\,B^n, \mathrm{Im}\,B^\alpha)^t\,.$$

Therefore, for $s = 0$, we can make $b_0(s) = 0$ with a unique $B_{21}(s)$ satisfying

$$(1.19)\qquad B^1_{21}(s) = 0\,,\quad \mathrm{Re}\,B^\alpha_{21}(s) = 0\,.$$

By continuity the same holds for $|s| < \varepsilon_0$, shrinking ε_0 if necessary. By (1.13), (1.15) and (1.3), (1.4) we have

$$(1.20)\qquad b(0) = R_{1\bar{1}}(0) = (-1, 0, \ldots, 0)^t\,,$$

so that the first equations of (1.15) and (1.18) give

$$(1.21)\qquad \mathrm{Re}\,B^n_{21}(0) = \beta(0) = 1\,,\quad \mathrm{Im}\,B^\alpha_{21}(0) = 0\,,\quad \mathrm{Im}\,B^n_{21}(0) = 0\,.$$

The last equation in (1.10) is now satisfied. Since the operator $B \mapsto R_j(z_*(s))B^j$ has (complex) rank $n-1$, we must show that $\mu(s)$, $\gamma(s)$ can be chosen uniquely so that the vector $a(s) - \gamma(s)\mu(s)^{-2}b(s)$ lies in its range, in order to make $a_0 = 0$ in (1.15). For a unique solution to $a_0 = 0$, we restrict to $B^1 = 0$, which we may do by the form of (1.17). By (1.17) and (1.20) $b(s)$ is not in the range of this operator for $s = 0$, and hence for $|s| < \varepsilon_0$ (we shrink ε_0 a second time if necessary). It follows that $\gamma(s) \geq 0$, $\mu(s) = \bar{\mu}(s)^{-1}$, $\mathrm{Re}\,\mu(s) > 0$, and $B_{20}(s)$ can be chosen uniquely to make $a_0(s) = 0$. By (1.3), (1.4), (1.20) we obtain

$$a(0) = \tfrac{1}{2}R_{11}(0) = (-\gamma, 0, \dots, 0)^t . \tag{1.22}$$

Substituting (1.22), (1.20), and (1.17) into (1.15), we see that

$$\mu(0) = 1 , \quad \gamma(0) = \gamma , \quad B_{20}(0) = 0 ,$$

give the (unique) solution of $a_0(0) = 0$. We replace B_1 by μB_1. Thus far, we have achieved (1.11) with $l = 2$, $f = 0$, making use only of the fact that $\gamma \neq \frac{1}{2}$. Further normalization requires $0 \leqslant \gamma < \frac{1}{2}$.

We assume inductively that $B_3, \dots, B_{l-1}, f_3, \dots, f_{l-1}$, have been uniquely determined to satisfy the proposition with l replaced by $l-1$. We shall show that (f_l, B_l) can be uniquely chosen to satisfy the proposition. Only the terms of degree l in (1.11) have to be considered. By (1.14), (f_l, B_l) occurs only in the operator (1.16), and in the form $B_{21}(s) f_l + B_l(t, q, s)$. The induction step will be completed if we can show that (1.16) is invertible for $|s| < \varepsilon_0$, ε_0 independent of l. Since (1.16) is independent of l, it will suffice to check this at $s = 0$. (ε_0 may have to be shrunk a final time.)

By (1.18) and (1.10) we must solve

$$\begin{cases} f_l(t, 0) + \operatorname{Re} B_l^n(t, u, 0)|_{q=u} = S_l^1 , \\ \operatorname{Im} B_l^n(t, u, 0)|_{q=n} = S_l^n , \\ \operatorname{Im} B_l^\alpha(t, u, 0)|_{q=u} = S_l^\alpha , \end{cases} \tag{1.23}$$

for certain real expressions S_l, homogeneous of degree l in $(t, \bar{t})$. The left hand side defines a real linear transformation from the vector space of normalized (f_l, B_l) into the vector space of S_l's. It will suffice to show that these two spaces have the same (real) dimension, and that $S_l = 0$ implies $f_l = 0$, $B_l = 0$. Suppose that $S_l^\alpha = 0$ in (1.23), so that the imaginary part of the holomorphic polynomial function $t \mapsto B_l^\alpha(t, u, 0)$ vanishes on the curve $q(t, 0) = u > 0$. Since this is an ellipse when $0 \leqslant \gamma < \frac{1}{2}$, it must vanish identically by the maximum principle. For $j \geqslant 3$, the second two conditions of (1.10) give

$$\operatorname{Re} B_l^\alpha(0, u, s) = \operatorname{Re} B_l^n(0, u, s) = 0 . \tag{1.24}$$

It follows that $B_l^\alpha(t, u, 0) = 0$. Likewise, $B_l^n(t, u, 0) = 0$, if $S_l^n = 0$. If also $S_l^1 = 0$, then $f_l(t, 0) = 0$ on $q(t, 0) = u > 0$. Since $f_l(t, 0)$ is homogeneous of degree l, it also must vanish identically. The condition (1.24) is vacuous if l is odd and means that the coefficient of $u^{l/2}$ is purely imaginary if l is even. In either case the real and the imaginary parts of the coefficients of $B_l^j(t, u, 0)$ comprise an $l+1$ dimensional space. Thus (1.23) is a bijection between two spaces of dimension $n(l+1)$.

2. – The Hilbert transform on a variable curve.

Let f and q be as in (1.8) and the proposition of section 1. We introduce the parameter $r > 0$, $r^2 = u$, and the family of closed curves

$$\gamma_{r,s} = \{t \in \mathbf{C} \colon q(t, s) + f(t, s) = r^2\}. \tag{2.1}$$

We assume that r and s are small enough so that $\theta = \arg t$ can be used as a parameter on all $\gamma_{r,s}$. For a function $\varphi = \varphi(\theta)$ on $\gamma_{r,s}$ the Hilbert transform $H_{r,s}[\varphi]$ is such that $\varphi + iH_{r,s}[\varphi]$ is the boundary value of a function holomorphic inside $\gamma_{r,s}$ and real at the origin. To study H as a function of φ, and s, we shall make use of the explicit description of H and the techniques and estimates given in section 2 of [3]. Thus, (see (2.6-9) of [3])

$$H_{r,s}[\varphi] = \operatorname{Re} C_{r,s}[\xi] - \operatorname{Re} M_{r,s}[\xi], \qquad \xi = \xi(r, s, \theta), \tag{2.2}$$

where

$$C_{r,s}[\xi](\theta) = \int_0^{2\pi} (\xi(\sigma) - \xi(\theta)) c_{r,s}(\sigma, \theta)\, d\sigma, \quad c_{r,s}(\sigma, \theta) = \frac{t_\sigma(\sigma, r, s)}{t(\sigma, r, s) - t(\theta, r, s)}, \tag{2.3}$$

$$M_{r,s}[\xi] = \int_0^{2\pi} \xi(\sigma) t_\sigma(\sigma, r, s) t(\sigma, r, s)^{-1}\, d\sigma, \tag{2.4}$$

and $t(\theta, r, s)$ is the parameterization of $\gamma_{r,s}$. ξ is defined by

$$\xi(r, s, \theta) = \int_0^\theta \mu(r, s, \sigma)\, d\sigma, \tag{2.5}$$

where the function μ is the solution of integral equation

$$S_{r,s}[\mu](\theta) \equiv \mu(r, s, \theta) + \frac{1}{2\pi} \operatorname{Im}\left\{t_\theta(r, s, \theta)\, C_{r,s}\left[\frac{\mu(r, s, \sigma)}{t_\sigma(r, s, \sigma)}\right]\right\} = -\frac{1}{2\pi} \varphi_\theta(\theta). \tag{2.6}$$

We must study the Cauchy transform $C_{r,s}$ as r tends to zero. To this purpose we compare $\gamma_{r,s}$ with the ellipse $\gamma^0_{r,s}$: $r^2 = q(t, s)$, which has the parametrization

$$t^0(\theta, r, s) \equiv rt^1(\theta, s) \equiv r\left(\frac{\cos\theta}{\sqrt{1+2\gamma}} + i\frac{\sin\theta}{\sqrt{1-2\gamma}}\right). \tag{2.7}$$

We denote the Cauchy kernel and transform on $\gamma^0_{r,s}$, which are independent of r, by $c_{0,s}(\sigma, \theta)$ and $C_{0,s}[\xi]$, respectively. As in [3] we define w, b, P, Q by

$$w(\theta, r, s) = \eta(\theta, r, s)e^{i\theta} = t(\theta, r, s) - t^0(\theta, r, s), \tag{2.8}$$

$$c_{r,s}(\sigma, \theta) - c_{0,s}(\sigma, \theta) = b(r, s, \sigma, \theta)c_{0,s}(\sigma, \theta), \tag{2.9}$$

$$b = \frac{P-Q}{1+Q}, \quad P = \frac{w_\sigma(\sigma, r, s)}{t^0_\sigma(\sigma, r, s)}, \quad Q = \frac{w(\sigma, r, s) - w(\theta, r, s)}{t^0(\sigma, r, s) - t^0(\theta, r, s)}. \tag{2.10}$$

(2.9) is the kernel of $C_{r,s} - C_{0,s}$, which we must analyse along with $C_{r,s}$. These are operators of the form

$$C^b[\xi](\theta) = \int_0^{2\pi} \frac{\xi(\sigma) - \xi(\theta)}{t(\sigma) - t(\theta)} b(\sigma, \theta) t_\sigma(\sigma)\, d\sigma \tag{2.11}$$

where $\theta \mapsto t(\theta)$ parameterizes a fixed closed curve. The following is proved in [3].

THEOREM 2.1. *Let $\theta \mapsto t(\theta)$ be a smooth regular simple closed curve parameterized by the polar angle θ. If $b \in C^{j,\nu}(\sigma, \theta)$ and $\xi \in C^{j,\nu}(\theta)$, $j \geq 0$, $0 < \nu < 1$, then*

$$\|C^b[\xi]\|_{C^{j,\nu}(\theta)} \leq N_{j,\nu} \|b\|_{C^{j,\nu}(\sigma,\theta)} \|\xi\|_{C^{j,\nu}(\theta)}, \tag{2.12}$$

where $N_{j,\nu}$ depends only on (j, ν) and on the C^∞ seminorms of $t(\sigma)$ and $t_\sigma(\sigma)^{-1}$.

The behavior of the family of curves $\gamma_{r,s}$ which is relevant to our study is given in the following lemma.

LEMMA 2.2. *Let w be given by (2.8). Then, as $r \to 0$,*

a) $\|r^{-1}w\| = O(r)$,

b) $\|\partial_r^k(r^{-1}w)\| = O(1)$,

where $k \geq 0$, the norms are in $C^j(\theta, s)$, $|s| < \varepsilon_0$, $j \geq 0$, and the constants in O depend on j and k.

PROOF. We set $\tilde{\eta} = \eta/r$ and rewrite (2.1) as

$$\tilde{\eta} = G_{r,s}(\tilde{\eta}) \equiv -\left(q(e^{i\theta}, s)\tilde{\eta}^2 + r^{-2}f(r(t^0 + \tilde{\eta}e^{i\theta}), s)\right)\cdot(2\,\mathrm{Re}\, q_t(t^0, s)e^{i\theta})^{-1}. \tag{2.13}$$

Since ∂_θ and ∂_s do not change the order of vanishing at $r = 0$, we have

$$\|G_{r,s}(\tilde{\eta})\| = O(\|\tilde{\eta}\|^2 + r),$$

if $\|\tilde{\eta}\| \leqslant 1$. We also have

$$\|G_{r,s}(\tilde{\eta}_1) - G_{r,s}(\tilde{\eta}_2)\| = O(\|\tilde{\eta}_1\| + \|\tilde{\eta}_2\| + r)\|\tilde{\eta}_2 - \tilde{\eta}_1\|.$$

By the contraction mapping principle (2.13) has a unique solution $\tilde{\eta}_{r,s}$ in $C^j(\theta, |s| < \varepsilon_0)$ with $\|\tilde{\eta}_{r,s}\| \leqslant r$, if r is sufficiently small. This proves part a. Part b is obtained by putting the expression (1.8) for f into (2.13) and differentiating with respect to (r, s, θ). $\square$

From (2.10) we have

$$\|P\|_{C^{j,\nu}(\sigma,\theta,s)} \leqslant N\|r^{-1}w\|_{C^{j+1,\nu}(\theta,s)},$$

$$\|Q\|_{C^{j,\nu}(\sigma,\theta,s)} \leqslant N\|r^{-1}w\|_{C^{j+1,\nu}(\theta,s)},$$

where N is independent of r, and $|s| < \varepsilon_0$. By lemma (2.2) $1 + Q \neq 0$ and

$$\|b\|_{C^{j,\nu}(\sigma,\theta,s)} = O(r), \tag{2.14}$$

$$\|\partial_r^k b\|_{C^{j,\nu}(\sigma,\theta,s)} = O(1), \tag{2.15}$$

where $0 < \nu < 1$, as $r \to 0$. We apply Theorem 2.1 to $C_s^b \equiv C_{r,s} - C_{0,s}$, together with the estimates

$$\|t_\sigma^1(\sigma, s)^{-1}\partial_s^\beta t_\sigma^1(\sigma, s)\|_{C^{j,\nu}(\sigma,\theta)} = O(1),$$

$$\|(t^1(\sigma, s) - t^1(\theta, s))\partial_s^\beta(t^1(\sigma, s) - t^1(\theta, s))^{-1}\|_{C^{j,\nu}(\sigma,\theta)} = O(1),$$

where $O(1)$ depends only on (j, ν), and (2.14), (2.15). This gives

COROLLARY 2.3. *For $|s| < \varepsilon_0$ and $r \to 0$,*

a) $\|C_{r,s} - C_{0,s}\| = O(r), \quad \|C_{r,s}\| = O(1),$

b) $\|\partial_r^k\partial_s^\beta C_{r,s}\| = O(1),$

where the norms are in $C^{j,\nu}(\theta)$, $j \geqslant 0$, $0 < \nu < 1$, and O depends only on (j, ν, β, k).

Lemma 2.2 gives

$$|\partial_r^k \partial_s^\beta M_{r,s}[\xi]| \leqslant N \|\xi\|_\infty , \tag{2.16}$$

$$\|M_{r,s} - M_{0,s}\|_\infty = O(r) , \qquad \text{as } r \to 0 . \tag{2.17}$$

As on p. 10 of [3], the operator $S_{r,s}$ in (2.6) has an inverse which is bounded for each (r, s) on the space of functions with mean value zero in $C^{j,\nu}(\theta)$. Lemma 2.2*a* and corollary 2.3*a*) give

LEMMA 2.4. *For* $|s| < \varepsilon_0$, *and r sufficiently small*

$$\|S_{r,s}^{-1}\|_{C^{j,\nu}(\theta)} \leqslant N$$

where the constant N is independent of r and s.

The essential properties of H are given in the following theorem.

THEOREM 2.5. *As r tends to zero and* $|s| < \varepsilon_0$

a) $\|H_{r,s}\| = O(1)$, $\|H_{r,s} - H_{0,s}\| = O(r)$,

b) $\|\partial_r^k \partial_s^\beta H_{r,s}\| = O(1)$,

where the norms are in $C^{j,\nu}(\theta)$, $j \geqslant 0$, $0 < \nu < 1$, *and O depends only on* (j, ν, β, k).

PROOF. The proof of *a*) uses (2.2), corollary 2.3*a*, lemma 2.4, (2.16) and (2.17) and is identical to the corresponding argument in [3]. By (2.2), (2.16) and corollary 2.3*b*, part *b* is reduced to showing that

$$\|\partial_r^k \partial_s^\beta \xi_{r,s}\|_{j,\nu} \equiv \|\partial_r^k \partial_s^\beta \mu_{r,s}\|_{j-1,\nu} = O(1) \|\varphi\|_{j,\nu} .$$

To this purpose we (formally) differentiate (2.6) with respect to r and s to obtain

$$S_{r,s}[\partial_r \mu_{r,s}] = -(\partial_r S_{r,s})[\mu_{r,s}] ,$$

$$S_{r,s}[\partial_s \mu_{r,s}] = -(\partial_s S_{r,s})[\mu_{r,s}] .$$

The right hand sides involve $\partial_r C_{r,s}$ and $\partial_s C_{r,s}$, which are bounded by corollary 2.3*b* and the r and s derivatives of

$$\frac{t_\theta(\theta\ r\ s)}{t_\sigma(\sigma, r, s)} = \frac{t_\theta^1(\theta, s) + w_\theta(\theta, r, s)/r}{t_\sigma^1(\sigma, s) + w_\sigma(\sigma, r, s)/r} ,$$

which are bounded by lemma 2.2*b*. Lemma 2.4 gives

$$\|\delta\mu_{r,s}\|_{j-1,\nu} = O(1)\|\mu_{r,s}\|_{j-1,\nu} = O(1)\|\varphi\|_{j,\nu},$$

where δ is either ∂_r or ∂_s. The continuity of μ in r and s follows from that of $S_{r,s}$. The formal differentiation is then justified in the usual manner by considering difference quotients. An inductive procedure on the total order of derivatives gives *b*). □

The study of $H_{r,s}$ for $r \geqslant r_0$, as in [3], is an easier version of the foregoing argument. It yields the following proposition, provided that δ is so small that θ can be used to parameterize $\gamma_{r,s}$.

PROPOSITION 2.6. *Let* $\gamma_{r,s}$, $H_{r,s}$ *be as above. Then* $H(\varphi, r, s) \equiv H_{r,s}[\varphi]$ *is a* C^∞ *mapping from* $C^{j,\nu}(\theta) \times [0 < r < \delta] \times [|s| < \varepsilon_0]$ *to* $C^{j,\nu}(\theta)$ *for* $j \geqslant 0$, $0 < \nu < 1$.

3. – The implicit function theorem.

In this section we shall start with the mapping $T: z = z(t, u, s)$ given by (1.9) and deform it so as to make the boundaries of the analytic discs lie on M. Thus, we seek $\tilde{A}(t, u, s) \in \mathbf{C}^n$, holomorphic in t and satisfying $R(z + \tilde{A}) = 0$ when $r^2 \equiv u = (q + f)(t, s)$.

To achieve this goal it turns out to be convenient to make both a change of frame, $\partial/\partial z^j \mapsto X_j$, and a change of defining function, $R \mapsto \hat{R}$. We introduce the vector fields

$$X_1 = \frac{\partial z}{\partial t}, \quad X_\alpha = \frac{\partial z}{\partial s^\alpha}, \quad X_n = (0, 0, 1)^t, \tag{3.1}$$

where the derivatives are evaluated at (t, u, s). These vectors are positioned along the image of T at $z(t, u, s)$ and vary holomorphically with t when r and s are fixed. They are linearly independent over $\mathbf{C}$ for $|s| < \varepsilon_0$ and r sufficiently small. As differential operators

$$X_j = b^i_j(t, u, s)\frac{\partial}{\partial z^i}, \quad \partial_{\bar{i}} b^i_j = 0\,. \tag{3.1a}$$

With this change of frame we have

$$R_j \tilde{A}^j = tX_1 RA^1 + iX_\alpha RA^\alpha + iX_n RA^n, \tag{3.2}$$

where

$$tA^1 = \tilde{A}^j b_j^{-1i}, \quad iA^\alpha = \tilde{A}^j b_j^{-1\alpha}, \quad iA^n = \tilde{A}^j b_j^{-1n}, \tag{3.2a}$$

are also holomorphic in t. We make the normalization

$$\operatorname{Im} A^j(0, u, s) = 0, \quad 1 \leqslant j \leqslant n,$$

so that

$$A^j = A_{r,s}[\varphi^j] \equiv \varphi^j + iH_{r,s}[\varphi^j], \tag{3.3}$$

$H_{r,s}$ being the Hilbert transform on $\gamma_{r,s}$ (2.1).

With R as in section 1 we set

$$\begin{cases} \hat{R}^1 = R^1 + c_\alpha^1(x) R^\alpha, \\ \hat{R}^n = R^n + c_\alpha^n(x) R^\alpha + c_1^n(x) R^1, \\ \hat{R}^\alpha = R^\alpha, \end{cases} \tag{3.4}$$

where the real coefficients $c_i^j(x)$ are to be chosen later. Using the second order Taylor expansion (1.14) and (3.2), we arrive at the functional equation

$$F(\varphi, r, s) \equiv \hat{R}(z + \tilde{A}[\varphi]) = \hat{R}(z) + L_{r,s}[\varphi] + \hat{E}(z, A) = 0, \tag{3.5}$$

where

$$L_{r,s}[\varphi] = 2 \operatorname{Re} \{tX_1 \hat{R} A[\varphi^1] + iX_\alpha \hat{R} A[\varphi^\alpha] + iX_n \hat{R} A[\varphi^n]\}, \tag{3.6}$$

and $\hat{E}$ is the remainder rearranged according to (3.2a). By the results of section 2, F is a C^∞ mapping from $C^{j,\nu}(\theta)^n \times [0 < r < \delta_0] \times [|s| < \varepsilon_0]$ into $C^{j,\nu}(\theta)^n$, for every $j \geqslant 0$, $0 < \nu < 1$.

The main problem at this point is to invert the operator $L_{r,s}$. We write out the equation $L_{r,s}[\varphi] = \psi$ more explicitly, using the notation

$$K_{r,s}^j[\varphi^1] = 2 \operatorname{Re} \left(X_1 \hat{R}^j t A[\varphi^1]\right), \tag{3.6a}$$

and the relations

$$2 \operatorname{Re} \left(iX_\beta \hat{R} A[\varphi^\beta]\right) = 2 \operatorname{Re} (iX_\beta \hat{R}) \varphi^\beta - 2 \operatorname{Re} (X_\beta \hat{R}) H[\varphi^\beta],$$

$$X_n \hat{R}^\alpha = X_n R^\alpha = 0, \quad X_n \hat{R}^1 = X_n R^1 = \tfrac{1}{2},$$

$$X_n \hat{R}^n = X_n R^n + c_1^n(x) X_n R^1 = \tfrac{1}{2}(-i + c_1^n(x)).$$

Setting $a(x) = c_1^n(x)$, we have

$$K^1[\varphi^1] + 2\,\mathrm{Re}\,(iX_\beta \hat{R}^1)\varphi^\beta - 2\,\mathrm{Re}\,(X_\beta \hat{R}^1)H[\varphi^\beta] - H[\varphi^n] = \psi^1\,, \tag{3.7}$$

$$K^n[\varphi^1] + 2\,\mathrm{Re}\,(iX_\beta \hat{R}^n)\varphi^\beta - 2\,\mathrm{Re}\,(X_\beta \hat{R})^n H[\varphi^\beta] + \varphi^n - aH[\varphi^n] = \psi^n\,, \tag{3.8}$$

$$K^\alpha[\varphi^1] + 2\,\mathrm{Re}\,(iX_\beta \hat{R}^\alpha)\varphi^\beta - 2\,\mathrm{Re}\,(X_\beta \hat{R}^\alpha)H[\varphi^\beta] = \psi^\alpha\,. \tag{3.9}$$

Recall that $|t(\theta, r, s)| \sim r$.

LEMMA (3.1). *Independently of l the coefficients $c_j^i(x)$ in (3.4) can be chosen so that $c_i^j(0) = 0$, and for $|s| < \varepsilon_0$, r sufficiently small, and $r^2 = (q + f)(t, s)$, the following hold*:

i) $2\,\mathrm{Re}\,(iX_\beta \hat{R}^\alpha)$ *is an invertible matrix*,

ii) $X_1 \hat{R} = O(r)\,,$

iii) $2\,\mathrm{Re}\,(X_\beta \hat{R}) = O(r^2)\,,$

iv) $2\,\mathrm{Re}\,(iX_\beta \hat{R}^1) = O(r)\,,$

v) $2\,\mathrm{Re}\,(iX_\beta \hat{R}^n) = O(r)\,,$

vi) $X_1 \hat{R}^n = O(r^2)\,.$

ii)-vi) *also hold if the $C^{j,\nu}(\theta, s)$ norm is taken on the left hand side.*

PROOF. By (3.4) the condition (1.11) holds with R replaced by $\hat{R}$. We differentiate this with respect to t and s^α, obtaining

$$\begin{cases} O(r^l) = X_1 \hat{R} + \partial_u \hat{R}(z)(q + f)_t\,, \\ O(r^{l+1}) = 2\,\mathrm{Re}\,(X_\alpha \hat{R}) + \partial_u \hat{R}(z)(q + f)_{s^\alpha}\,. \end{cases} \tag{3.10}$$

Since derivatives with respect to θ and s do not affect the order of vanishing in r (lemma 2.2), (ii) and (iii) follow. We next compute (iv), (v), and $\partial_u \hat{R}^n(z)$ along $r = 0$, where $x = x_* = s$. Setting these quantities equal to zero yields the equations

$$0 = 2\,\mathrm{Re}\,(iX_\beta R^1) + 2\,\mathrm{Re}\,(iX_\beta R^\alpha)c_\alpha^1(s)\,, \tag{3.11}$$

$$0 = 2\,\mathrm{Re}\,(iX_\beta R^n) + 2\,\mathrm{Re}\,(iX_\beta R^\alpha)c_\alpha^n(s) + 2\,\mathrm{Re}\,(iX_\beta R^1)c_1^n(s)\,, \tag{3.12}$$

$$0 = \partial_u R^n(z) + \partial_u R^\beta(z)c_\beta^n(s) + \partial_u R^1(z)c_1^n(s)\,. \tag{3.13}$$

At the origin $X_\beta = \partial/\partial z^\beta$, so by (1.3), (1.4),

$$2\,\mathrm{Re}\,(iX_\beta R^\alpha)(0) = \delta^\alpha_\beta\,, \qquad X_\beta R^1(0) = X_\beta R^n(0) = 0\,.$$

It follows that for s sufficiently small, (3.11) can be solved with $c^1_\alpha(0) = 0$. At $u = 0$, $t = 0$,

$$\begin{aligned}\partial_u R^i(z) &= 2\,\mathrm{Re}\,(R^i_j z^j_u) = 2\,\mathrm{Re}\,(R^i_j B^j_{21})\\ &= 2\,\mathrm{Re}\,(R^i_\alpha B^\alpha_{21} + R^i_n B^n_{21})\,,\end{aligned}$$

by (1.10). So at the origin, $R^i_\alpha = 0$, for $i = 1, n$, and

$$\partial_u R^1(z)(0) = 2\,\mathrm{Re}\,\frac{1}{2}B^n_{21}(0) = 1\,,$$

$$\partial_u R^n(z)(0) = 2\,\mathrm{Re}\,\frac{1}{2i}B^n_{21}(0) = 0\,,$$

by (1.19). Since also $\mathrm{Re}\,(iX_\beta R^n)(0) = 0$, (3.12) and (3.13) can be solved for small s with $c^n_\alpha(0) = c^n_1(0) = 0$. Shrinking ε_0 (if necessary) and replacing s by x in $c^i_j(s)$ gives the lemma. □

Next we begin to solve $L[\varphi] = \psi$ for φ. By (i) and (iii) of lemma 3.1 we may invert the operator

$$W: \varphi \mapsto 2\,\mathrm{Re}\,(iX_\beta \hat{R}^\alpha)\varphi^\beta - 2\,\mathrm{Re}\,(X_\beta \hat{R}^\alpha)H[\varphi^\beta]\,,$$

appearing in (3.9), for r sufficiently small. Putting $* = (2, \ldots, n-1)$, we have, by (ii) and (3.6a),

$$\varphi^* = W^{-1}[\psi^* - K^*[\varphi^1]] = O(1)[\psi^*] + O(r^2)[\varphi^1]\,. \tag{3.14}$$

We use this to eliminate φ^* in (3.7) and (3.8). By (vi), (v), and (iii) equation (3.8) gives

$$\varphi^n - aH[\varphi^n] = O(1)[\psi] + O(r^3)[\varphi^1]\,.$$

We must next invert the operator

$$I - a\big(x(t, u, s)\big)H = I - a(s)H + O(r)H\,,$$

$\big(x(t, u, s) - s = O(r)\big)$. With $\underline{m}$ denoting the value of the imaginary part at the origin, we have

$$H^2 = -I + \underline{m}\,, \quad H^3 = -H\,, \quad H^4 = I - \underline{m}\,, \quad H^5 = H\,.$$

Hence, formally

$$(I - a(s)H)^{-1} = \sum a^j H^j = \left(\sum a^{5j}\right)\{H + aH^2 + a^2H^3 + a^3H^4 + a^4H\}.$$

This is valid and represents a bounded operator on $C^{j,\nu}(\theta)$, for any $j \geqslant 0$, $0 < \nu < 1$, provided $\sum |a(s)|^j < \infty$. But we may assume $|a(s)| < \frac{1}{2}$ for $|s| < \varepsilon_0$, since $a(0) = 0$. Thus we have

$$\begin{aligned}(3.15)\qquad \varphi^n &= (I - aH)^{-1}[O(r^3)H[\varphi^1] + O(1)[\psi]] \\ &= O(1)[\psi] + O(r^3)[\varphi^1].\end{aligned}$$

We substitute (3.15) into (3.7) to get

$$(3.16)\qquad K^1[\varphi^1] + O(r^3)[\varphi^1] = O(1)[\psi].$$

To solve (3.16) for φ^1 we shall show that K^1 is r^2 times an operator invertible uniformly in (r, s). To this purpose let R^0, $\hat{R}^0$ be the corresponding defining functions for the surface $z_0 = z_*(s) + B_1(t, u, s) + B_2(t, u, s)$. Let X_j^0, $t^0(\theta, r, s)$, A^0, H^0, and $K_{r,s}^{10}$ be the corresponding objects.

LEMMA (3.2). $K_{r,s}^1[\varphi^1] = r^2 K_{1,s}^{01}[\varphi^1] + O(r^3)[\varphi^1]$. *$K_{1,s}^{01}$ is invertible with bounds independent of s, $|s| < \varepsilon_0$, on every space $C^{j,\nu}(\theta)$, $j \geqslant 0$, $0 < \nu < 1$.*

PROOF. We clearly have

$$z = z_0 + O(r^3), \qquad X_j = X_j^0 + O(r^2)\partial/\partial z,$$

and from section 2

$$t = t^0 + O(r^2), \qquad H_{r,s} = H_{r,s}^0 + O(r).$$

Also, $\hat{R} - \hat{R}^0 = O(|z^1 - P(x)|^3)$, since it holds for $R - R^0$, and the coefficients $c_j^i(x)$ are the same for both R and R_0. If these are substituted into (3.6a) ($j = 1$), and if lemma (3.1) is applied to R^0, one gets

$$K_{r,s}^1[\varphi^1] = K_{r,s}^{01}[\varphi^1] + O(r^3)[\varphi^1].$$

The first statement of the lemma follows from $t^0(\theta, r, s) = rt^0(\theta, 1, s) \equiv rt^1$.

We claim that

$$\hat{R}^{01}(z_0(t, u, s)) = \hat{p}(t, u, s)(u - q(t, s)),$$

where $\hat{p}$ is a smooth factor with $\hat{p}(0,0,0)=1$. Given this, we have

$$X_1^0 \hat{R}^{01}(z_0) = \frac{\partial}{\partial t}\hat{R}^{01}(z_0(t,u,s)) = -\hat{p}(t,u,s)\,q_t(t,s)\,,$$

along $u=q(t,s)$. Since $t^0=rt^1$, $H^0_{r,s}=H^0_{1,s}$, we have

$$r^{-2}K^{01}_{r,s}[\varphi^1] = -\,2\hat{p}(t,u,s)\,\mathrm{Re}\,\{q_t(t^1,s)\,t^1 A^0_{1,s}[\varphi^1]\}\,.$$

We assume $\hat{p}(0,0,s)\geqslant\frac{1}{2}$ for $|s|<\varepsilon_0$. Then the operator on the right hand side will be invertible on every $C^{j,\nu}(\theta)$ with bounds independent of s. This follows essentially from lemma (1.2) of [3]. We only need to observe that the Riemann map onto the ellipse $q(t,s)=1$ varies continuously (in $C^{j,\nu}$-norm) with s. A completely elementary argument for this can be based on the explicit formula of H. A. Schwarz.

It remains to verify the claim. We have

$$R^{01}(z_0(t,u,s)) = p(t,u,s)(u-q(t,s))\,,$$
$$R^{0\alpha}(z_0(t,u,s)) = p^{\alpha}(t,u,s)(u-q(t,s))\,,$$

for smooth p, p^{α}, so that

$$\hat{p}(t,u,s) = p(t,u,s) + c^1_{\alpha}(x)p^{\alpha}(t,u,s)\,.$$

Since $c^1_{\alpha}(0)=0$, we want $p(0)=1$. But

$$\begin{aligned} p(0) &= \partial_u R^{01}(z_0(t,u,s))(0) = \partial_u(x^{0n}-h_0(z^{01},x_0))(0)\\ &= \partial_u x^{0n}(0) = \partial_u\{x^n_*(s)+\mathrm{Re}\,(B^n_1+B^n_2)\}(0)\\ &= \mathrm{Re}\,B^n_{21}(0) = \beta(0) = 1\,, \qquad \square \end{aligned}$$

by (1.21).

From now on we let * run from 2 to n. By lemma (3.2), (3.16), (3.15), and (3.14), the equation $\varphi = L^{-1}[\psi]$ has the form

$$\text{(3.17)}\qquad \begin{cases} \varphi^1 = r^{-2}O^{\mathrm{I}}_{r,s}(1)[\psi]\,, \\ \varphi^* = O^*_{r,s}(1)[\psi]\,, \qquad \varphi^* = (\varphi^2,\dots,\varphi^n)\,, \end{cases}$$

where $O_{r,s}(1)$ denotes the operators on $C^{j,\nu}(\theta)^n$ bounded uniformly in (r,s).

PROPOSITION (3.3). *Let $l\geqslant 4$, $j\geqslant 0$, $0<\nu<1$, and let F be constructed from the approximating family of discs in prop.* (1.1). *Then there exists $\delta>0$*

such that for $0 < r < \delta$, $0 < |s| < \varepsilon_0$ *the equation* (3.5) *has a unique soluion* $\varphi_{r,s}$ *satisfying*

$$\|\varphi^1_{s,r}\|_{j,\nu} \leq r^{l-1}, \quad \|\varphi^i_{r,s}\|_{j,\nu} \leq r^{l+1}, \quad 2 \leq i \leq n. \tag{3.18}$$

$\varphi_{r,s}$ *is a smooth function of* (r, s) *and satisfies*

$$\|\partial^a_s \partial^b_r \varphi_{r,s}\|_j = O(r^{l-1-2|a|-2b}) \tag{3.19}$$

for $|a| + b \leq l + 1$, *as* $r \to 0$.

PROOF. Solving (3.5) is equivalent to finding a fixed point φ of the operator

$$T_{r,s}[\varphi] \equiv - L^{-1}_{r,s}[\psi(\varphi, r, s)], \qquad \psi(\varphi, r, s) \equiv \hat{R}(z) + \hat{E}(z, \tilde{A}). \tag{3.20}$$

We let $\|\cdot\| = \|\cdot\|_{j,\nu}$ and define

$$B_r = \{\varphi \in C^{j,\nu}(\theta) \colon \|\varphi^1\| \leq r^2, \ \|\varphi^i\| \leq r^3, \ 2 \leq i \leq n\}.$$

By (3.2) $\tilde{A}^i = tb^i_1 A^1 + i \sum_{\alpha=2}^{n} b^i_\alpha A^\alpha$, $1 \leq i \leq n$, so that

$$\begin{cases} \|\tilde{A}^i\| = O(r\|\varphi^1\| + \|\varphi^*\|), \\ \|\psi(\varphi, r, s)\| = O(r^{l+1}) + O(r^2\|\varphi^1\|^2 + \|\varphi^*\|^2). \end{cases} \tag{3.21}$$

For $\varphi \in B_r$, $\|\psi\| = O(r^{l+1}) + O(r^6)$; so by (3.17)

$$\|T^1_{r,s}[\varphi]\| = O(r^{l-1}) + O(r^4),$$
$$\|T^*_{r,s}[\varphi]\| = O(r^{l+1}) + O(r^6).$$

Hence, for r sufficiently small, $T_{r,s}$ maps B_r into itself. For $\varphi_i \in B_r$, $i = 1, 2$, set $\tilde{A}_i = \tilde{A}[\varphi_i]$, $\psi_i = \psi(\varphi_i, r, s)$. In the Taylor remainder $\hat{E}$ we write

$$\begin{aligned} &\hat{R}_{ab}(z + \sigma\tilde{A}_2)\tilde{A}^a_2\tilde{A}^b_2 - \hat{R}_{ab}(z + \sigma\tilde{A}_1)\tilde{A}^a_1\tilde{A}^b_1 \\ &\qquad = [\hat{R}_{ab}(z + \sigma\tilde{A}_2) - \hat{R}_{ab}(z + \sigma\tilde{A}_1)]\tilde{A}^a_2\tilde{A}^b_2 \\ &\qquad + \hat{R}_{ab}(z + \sigma\tilde{A}_1)[(\tilde{A}^a_2 - \tilde{A}^a_1)\tilde{A}^b_2 + \tilde{A}^a_1(\tilde{A}^b_2 - \tilde{A}^b_1)]. \end{aligned}$$

From this and the fact that $\tilde{A} = O(r^3)$ on B_r, we get

$$\begin{aligned} \|\psi_2 - \psi_1\| &= O(\|\tilde{A}_1\| + \|\tilde{A}_2\|)\|\tilde{A}_2 - \tilde{A}_1\| \\ &= O(r^3)(r\|\varphi^1_2 - \varphi^1_1\| + \|\varphi^*_2 - \varphi^*_1\|). \end{aligned}$$

Thus,

$$\|T_{r,s}[\varphi_2]-T_{r,s}[\varphi_1]\| \leqslant \|L_{r,s}^{-1}\|\cdot\|\psi_2-\psi_1\| = O(r)\big(r\|\varphi_2^1-\varphi_1^1\|+\|\,\|\varphi_2^*-\varphi_1^*\|\big),$$

so that $T_{r,s}$ is contracting on B_r, if r is small enough. A Picard iteration argument $\varphi_{a+1}=T_{r,s}[\varphi_a]$, $\varphi_0\equiv 0$, yields the existence (and uniqueness) of $\varphi_{r,s}\in B_r$. The convergence is uniform in (r, s), so that $\varphi_{r,s}$ depends continuously on (r, s). By (3.17), (3.20), and (3.21), we have

$$\|\varphi_{r,s}^1\| = \|T_{r,s}^1[\varphi_{rs}]\| = O(r^{l-1}) + O\big(\|\varphi_{r,s}^1\|^2 + r^{-2}\|\varphi_{r,s}^*\|^2\big),$$

$$\|\varphi_{r,s}^*\| = \|T_{r,s}^*[\varphi_{r,s}]\| = O(r^{l+1}) + O\big(r^2\|\varphi_{r,s}^1\|^2 + \|\varphi_{r,s}^*\|^2\big).$$

These two equations imply (3.18).

It remains to prove the smoothness of $\varphi_{r,s}$ in (r, s) and the bounds (3.19). For this we linearize the functional equation (3.5) with respect to φ

$$\partial_\varphi F(\varphi, r, s)[\dot\varphi] = L_{r,s}\dot\varphi] + \partial_\varphi \hat E(\varphi, r, s)[\dot\varphi]. \tag{3.22}$$

The last term is the integral with respect to σ of

$$2\hat R_{ab}\tilde A^a[\varphi]\tilde A^b[\dot\varphi] + \sigma\hat R_{abc}\tilde A^a[\varphi]\tilde A^b[\varphi]\tilde A^c[\dot\varphi],$$

so that

$$\partial_\varphi \hat E(\varphi_{r,s}, r, s)[\dot\varphi] = O(r^{l-1})[\dot\varphi].$$

Since $l\geqslant 4$, this implies that (3.22) is invertible for small r and

$$\partial_\varphi F(\varphi_{r,s}, r, s)^{-1}[\dot\varphi] = O(r^{-2})[\dot\varphi]. \tag{3.23}$$

The smoothness of the map $(r, s)\mapsto\varphi_{r,s}$ from $[0<r<\delta]\times\big[|s|<\varepsilon_0\big]$ into $C^{j,\nu}(\theta)^n$ follows from the implicit function theorem.

To simplify the argument for (3.19) we let $S=(r, s)$ and ∂_S denote differentiation with respect to r or s. $\alpha, \alpha_0, \alpha_1, \ldots, \alpha_k$ will serve as multi-indices. For $|\alpha|\geqslant 1$, the chain rule gives

$$\begin{aligned}(3.24)\quad & \partial_\varphi F(\varphi_S, S)[\partial_S^\alpha\varphi_S] + \\ &+\sum\sum c(\alpha_0, \alpha_1, \ldots, \alpha_k, k, \alpha)\partial_S^{\alpha_0}\partial_\varphi^k F(\varphi_S, S)[\partial_S^{\alpha_1}\varphi_S, \ldots, \partial_S^{\alpha_k}\varphi_S] + \partial_S^\alpha F(\varphi_S, S) = 0.\end{aligned}$$

Here the first summation is over (α_0, k), $1 < |\alpha_0| + k \leqslant |\alpha|$, $k > 0$, the second over $(\alpha_1, \dots, \alpha_k)$, $|\alpha_0| + |\alpha_1| + \dots + |\alpha_k| = |\alpha|$, and the $c(\alpha_0, \dots, \alpha)$ are certain non-negative integers. We claim that

$$\partial_S^\alpha F(\varphi_S, S) = O(r^{l+1-|\alpha|}), \quad \text{for } |\alpha| \leqslant l+1, \tag{3.25}$$

and

$$\partial_S^\alpha \partial_\varphi^k F(\varphi_S, S)[\varphi_1, \dots, \varphi_k] = O(\|\varphi_1\| \cdot \|\varphi_2\| \dots \|\varphi_k\|), \quad \text{for } k \geqslant 1. \tag{3.26}$$

Since,

$$F(\varphi_S, S) = \hat{R}(z(S)) + \int_0^1 \hat{R}_a(z + \sigma A[\varphi_S]) \tilde{A}^a[\varphi_S] d\sigma,$$

the results of section 2 and (3.18) give

$$\partial_S^\alpha F(\varphi_S, S) = O(r^{l+1-|\alpha|}) + O(r^{l-1}).$$

Hence, (3.25) holds. (3.26) follows from

$$\begin{aligned} &\partial_\varphi^k F(\varphi, S)[\varphi_1, \dots, \varphi_k] \\ &= c_k' \int \hat{R}_{aa_1 \dots a_k} \tilde{A}^a[\varphi] \tilde{A}^{a_1}[\varphi_1] \dots \tilde{A}^{a_k}[\varphi^k] \sigma^k d\sigma + c_k'' \int \hat{R}_{a_1 \dots a_k} \tilde{A}^{a_1}[\varphi_1] \dots \tilde{A}^{a_k}[\varphi_k] \sigma^{k-1} d\sigma. \end{aligned}$$

In our notation (3.19) is written

$$\|\partial_S^\alpha \varphi_S\| = O(r^{l-1-2|\alpha|}),$$

which we assume to be true for $|\alpha| \leqslant l$. In (3.24) with $|\alpha| = l+1$, we solve for $\partial_S^\alpha \varphi_S$ using (3.23). Using (3.25) and (3.26) we get

$$\partial_S^\alpha \varphi_S = O(r^{l-1-|\alpha|}) + O(r^{-2}) \sum \sum O(r^{l-1-|\alpha_1|} \dots r^{l-1-2|\alpha_k|}),$$

where the summations are as in (3.24). This gives

$$\partial \varphi_S = O(r^{l-1-|\alpha|}) + \sum O(r^{k(l-1)-2|\alpha|+2|\alpha_0|-2}),$$

where $1 < |\alpha_0| + k \leqslant |\alpha|$. Now $k \geqslant 1$, and $k \geqslant 2$ if $|\alpha_0| = 0$, so $k(l-1) - 2|\alpha| + 2|\alpha_0| - 2 \geqslant l - 1 - 2|\alpha|$. This proves (3.19) and the proposition. □

4. – The family of analytic discs.

The remaining arguments needed to prove the theorem stated in the introduction are essentially contained in section 4 of [3]. We shall give only a brief sketch. Given l, j, r, and $\varphi_{r,s}$ as in proposition (3.3) we define

$$A(\xi, r, s) = \frac{1}{2\pi i}\int \frac{A_{r,s}[\varphi_{r,s}](\theta)t_\theta(\theta, r, s)\,d\theta}{t(\theta, r, s) - \xi},$$

for ξ inside $\gamma_{r,s}$. $\tilde{A}$ is then defined by (3.2a) and T by $T(t, u, s) = z(t, u, s) + A(t, r, s)$. As in lemma 4.1 of [3] we have

$$\partial_r^b \partial_s^a \partial_t^k \tilde{A}(t, r, s) = O(r^{l-1-2b-2|a|-k}),$$

for $k \leqslant j$, $b + |a| \leqslant l + 1$. Since $\partial_r = 2r\partial_u$, this gives

$$\partial_s^a \partial_u^b \partial_t^k \tilde{A} = O(r^{l-1-4b-2|a|-k})$$

for $k \leqslant j$, $b + |a| \leqslant l + 1$. These derivatives remain bounded if $4b + 2|a| + k \leqslant l - 1$, so T is of class $C^{m-1,1}$ if we choose $j = l - 1$, $m = (l-1)/7$. For $l \geqslant 6$ the Jacobian matrix of T has maximal rank, so $\hat{M}_l = T(D_0)$ is a regularly embedded complex-foliated manifold with boundary of class $C^{m-1,1}$.

Away from $r = 0$, the map T is of class C^∞. This follows from the reflection-principle argument given in the proof of theorem 4.4 in [3]. In fact, we need only to replace the variable $z' \in \mathbf{C}^2$ there by $z' \in \mathbf{C}^n$, the parameter r by (r, s), and let B and α range from 1 to n in (4.11) of [3].

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