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On the isoperimetric inequality for minimal surfaces


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For any compact minimal submanifold of dimension \( k \) in \( \mathbb{R}^n \), it is known that there exists a constant \( \overline{C}_k \) depending only on \( k \), such that

\[
V(\partial M)^{k-1} > \overline{C}_k V(M),
\]

where \( V(\partial M) \) and \( V(M) \) are the \((k-1)\)-dimensional and \( k\)-dimensional volumes of \( \partial M \) and \( M \) respectively. We refer to [6] for a more detailed reference on the inequality. An open question [6] is to determine the best possible value of \( \overline{C}_k \). When \( M \) is a bounded domain in \( \mathbb{R}^2 \subset \mathbb{R}^3 \), the sharp constant is given by

\[
C_k = \frac{V(\partial D)^{k-1}}{V(D)},
\]

where \( D \) is the unit disk in \( \mathbb{R}^2 \). One speculates that \( C_k \) is indeed the sharp constant for general minimal submanifolds in \( \mathbb{R}^n \).

In the case \( k = 2 \), \( C_4 = 4\pi \), it was proved [1] (see [7]) that if \( \Sigma \) is a simply-connected minimal surface in \( \mathbb{R}^3 \), then

\[
l(\partial \Sigma)^2 > 4\pi A(\Sigma),
\]

where \( l(\partial \Sigma) \) and \( A(\Sigma) \) denote the length of \( \partial \Sigma \) and the area of \( \Sigma \) respectively.

In 1975, Osserman-Schiffer [5] showed that (2) is valid with a strict inequality for doubly-connected minimal surfaces in \( \mathbb{R}^3 \). Feinberg [2] later generalized this to doubly-connected minimal surfaces in \( \mathbb{R}^n \) for all \( n \). So far, the sharp constant, (1), has been established for minimal surfaces with topological restrictions.

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The purpose of this article is to prove the isoperimetric inequality (2) for those minimal surfaces in \( \mathbb{R}^n \) whose boundaries satisfy some connectedness assumption (see Theorem 1). This has the advantage that the topology of the minimal surface itself can be arbitrary. An immediate consequence of Theorem 1 is a generalization of the theorem of Osserman-Schiffer. In fact, Theorem 2 states that any minimal surface (not necessarily doubly-connected) in \( \mathbb{R}^3 \) whose boundary has at most two connected components must satisfy inequality (2).

Finally, in Theorem 3, we also generalize the non-existence theorem of Hildebrandt [3], Osserman [4], and Osserman-Schiffer [5] to higher codimension.

1. Isoperimetric inequality.

**Definition.** The boundary \( \partial \Sigma \) of a surface \( \Sigma \) in \( \mathbb{R}^n \) is weakly connected if there exists a rectangular coordinate system \( \{x^\alpha\}_{\alpha=1}^n \) of \( \mathbb{R}^n \), such that, for every affine hypersurface \( H = \{x^\alpha = \text{const.}\} \) in \( \mathbb{R}^n \), \( H \) does not separate \( \partial \Sigma \). This means, if \( H \cap \partial \Sigma = \emptyset \), then \( \partial \Sigma \) must lie on one side of \( H \).

In particular, if \( \partial \Sigma \) is a connected set, then \( \partial \Sigma \) is weakly connected.

**Theorem 1.** Let \( \Sigma \) be a compact minimal surface in \( \mathbb{R}^n \). If \( \partial \Sigma \) is weakly connected, then

\[
\text{area}(\partial \Sigma) \geq 4\pi \lambda(\Sigma) .
\]

Moreover, equality holds iff \( \Sigma \) is a flat disk in some affine 2-plane of \( \mathbb{R}^n \).

**Proof.** Let us first prove the case when \( \partial \Sigma \) is connected. By translation, we may assume that the center of mass of \( \partial \Sigma \) is at the origin, i.e.,

\[
\int_{\partial \Sigma} x^\alpha = 0, \quad \text{for all } 1 \leq \alpha \leq n .
\]

By the assumption on the connectedness of \( \partial \Sigma \), any coordinate system \( \{x^\alpha\}_{\alpha=1}^n \) satisfies the definition of weakly connectedness.

Let \( X = (x^1, ..., x^n) \) be the position vector, then \( |X|^2 = \sum_{\alpha=1}^n (x^\alpha)^2 \) must satisfy

\[
\lambda(|X|^2) = 4 ,
\]

due to the minimality assumption on \( \Sigma \). Here \( \lambda \) denotes the Laplacian on \( \Sigma \) with respect to the induced metric from \( \mathbb{R}^n \). Integrating (4) over \( \Sigma \), and
applying the divergence theorem, we have

\[ 4A(\Sigma) = 2 \int_{\partial \Sigma} X \cdot \frac{\partial X}{\partial \nu}, \]

where \( \partial / \partial \nu \) is the outward unit normal vector to \( \partial \Sigma \) on \( \Sigma \). Since \( \partial |X| / \partial \nu < 1 \), we have

\[ 2A(\Sigma) < \int |X| (\partial \Sigma)^{1/4} \int (|X|^2)^{1/4}. \]

In order to estimate the right hand side of (6), we will estimate \( (x^\alpha)^{\alpha} \) for each \( 1 < \alpha < n \). By (3), the Poincaré inequality implies that \( \partial \Sigma \)

\[ \int_{\partial \Sigma} (x^\alpha)^2 < \frac{l(\partial \Sigma)^2}{4\pi^2} \int_{\partial \Sigma} \left( \frac{dx^\alpha}{ds} \right)^2, \]

where \( d/\partial s \) is differentiation with respect to arc-length. Combining with (6) yields

\[ 4\pi A(\Sigma) = l(\partial \Sigma)^{1/4} \int_{\partial \Sigma} \left( \frac{dx^\alpha}{ds} \right)^{1/4} = l(\partial \Sigma)^{1/4}, \]  

because \( dX/\partial s \) is just the unit tangent vector to \( \partial \Sigma \).

Equality holds at (8), implies

\[ \frac{\partial |X|}{\partial \nu} = 1 \]

\[ |X| = \text{constant} = R \]

and equality at (7). The latter implies that

\[ x^\alpha = a^\alpha \sin \frac{2\pi s}{l(\partial \Sigma)} + b^\alpha \cos \frac{2\pi s}{l(\partial \Sigma)} \]

where \( a^\alpha \) and \( b^\alpha \)'s are constants for all \( 1 < \alpha < n \). By rotation, we may assume that

\[
\begin{align*}
    X(0) &= (R, 0, 0, \ldots, 0) \\
    \frac{dX}{ds}(0) &= (0, 1, 0, \ldots, 0),
\end{align*}
\]
because (10) implies that $\partial \Sigma$ lies on the sphere of radius $R$. Evaluating (11) at $s = 0$, we deduce that

$$b_1 = R, \quad b_\alpha = 0 \quad \text{for } 2 < \alpha < n$$

(13) and

$$a_1 = \frac{l(\partial \Sigma)}{2\pi}, \quad a_\alpha = 0 \quad \text{for } \alpha \neq 2.$$

On the other hand, summing over $1 < \alpha < n$ on (7), we derive

$$R^2 l(\partial \Sigma) = \int_{\partial \Sigma} |X|^2 = \left(\frac{l(\partial \Sigma)}{2\pi}\right)^2 l(\partial \Sigma),$$

(14) Hence

$$R = \frac{l(\partial \Sigma)}{2\pi}.$$ 

Combining with (13), (11) becomes

$$\begin{align*}
x^1 &= R \cos \left(\frac{s}{R}\right) \\
x^2 &= R \sin \left(\frac{s}{R}\right)
\end{align*}$$

(15) and

$$x^2 = 0 \quad \text{for } 3 < \alpha < n.$$

This implies $\partial \Sigma$ is a circle on the $x^1 x^2$-plane centered at the origin of radius $R$. Equation (9) shows that $\Sigma$ is tangent to the $x^1 x^2$-plane along $\partial \Sigma$. By the Hopf boundary lemma, this proves that $\Sigma$ must be the disk spanning $\partial \Sigma$.

For the general case when $\partial \Sigma$ is not connected. Let $\partial \Sigma = \bigcup_{i=1}^p \sigma_i$, where $\sigma_i$'s are connected closed curves. By the assumption on weakly connectedness, we may choose $\{x^1\}_{i=1}^p$ to be the appropriate coordinate system. For any fixed $1 < \alpha < n$, we claim that there exist translations $A_{\alpha}^i$, $2 < i < p$, generated by vectors $v^i_\alpha$ perpendicular to $\partial x^\alpha$, such that the union of the set of translated curves $\{A^i_\alpha \sigma_i\}_{i=2}^p$ together with $\sigma_1$ form a connected set. We prove the claim by induction on the number of curves, $p$. When $p = 2$, we observe that since no planes of the form $x^2 = \text{constant}$ separates $\sigma_1$ and $\sigma_2$, this is equivalent to the fact that there exists a number $x$, such that the plane $H = \{x^2 = x\}$ must intersect both $\sigma_1$ and $\sigma_2$. Let $q_1$ and $q_2$ be the points of intersection between $H$ with $\sigma_1$ and $\sigma_2$ respectively.
Clearly one can translate $q_2$ along $H$ to $q_i$. Denote this by $A^\sigma_{q_2}$, and $\sigma_1 \cup A^\sigma_{q_2} \sigma_2$ is connected now. For general $p$, we consider the set of numbers defined by

$$y_i = \max \{x^\sigma |_{a_i} \}.$$

Without loss of generality, we may assume $y_1 < y_2 < \ldots < y_p$. Now we claim that the set $\bigcup_{i=2}^{p} \sigma_i$ cannot be separated by hyperspaces of the form $H = \{x^\sigma = \text{constant}\}$. If so, say $H = \{x^\sigma = x\}$ separates $\bigcup_{i=2}^{p} \sigma_i$, then $x$ must be in the range of $x^\sigma |_{a_1}$. This is because $\bigcup_{i=2}^{p} \sigma_i$ cannot be separated hence $H \cap \sigma_1 \neq \emptyset$. On the other hand, since $H$ separates $\bigcup_{i=2}^{p} \sigma_i$, this means there exists some $\sigma_i$, $2 < i < p$, lying on the left of $H$, hence $y_i < x < y_1$, for some $2 < i < p$, which is a contradiction. By induction, there exist translations, $A^\sigma_i$, $3 < i < p$, perpendicular to $\partial/\partial x$ such that $\sigma = \sigma_1 \cup \bigcup_{i=3}^{p} A^\sigma_i \sigma_i$ is connected. However, $\bigcup_{i=2}^{p} \sigma_i$ is non-separable by $H = \{x^\sigma = \text{constant}\}$ implies $\sigma_1 \cup \sigma$ is non-separable also. Hence, there exists a translation $A^\sigma$ perpendicular to $\partial/\partial x$, such that $\sigma \cup A^\sigma$ is connected. The set $A = A_2, A A_3, A A_4, \ldots, A A_p$ gives the desired translations. Notice that since all translations are perpendicular to $\partial/\partial x^\sigma$, then

$$x^\sigma |_{a_1} \equiv x^\sigma |_{A^\sigma a_1}, \quad \text{for all } i.$$

By the connectedness of $\sigma^\sigma = \sigma_1 \cup A^\sigma_2 \sigma_2 \cup \ldots \cup A^\sigma_p \sigma_p$ we can view $\sigma^\sigma$ as a Lipschitz curve in $R^n$. Clearly

$$\int_{\sigma^\sigma} x^\sigma = \sum_{i=1}^{p} \int_{\sigma_i} x^\sigma = 0,$$

hence the Poincaré inequality can be applied to yield

$$\sum_{i=1}^{p} \int_{\sigma_i} (x^\sigma)^2 = \int_{\sigma^\sigma} (x^\sigma)^2 \leq \frac{l(\partial \Sigma)^2}{4\pi} \int_{\sigma^\sigma} \left( \frac{dx^\sigma}{ds} \right)^2 = \frac{l(\partial \Sigma)^2}{4\pi} \sum_{i=1}^{p} \int_{\sigma_i} \left( \frac{dx^\sigma}{ds} \right)^2.$$

Summing over all $1 < \alpha < n$ and proceeding as the connected case we derived the inequality (8).

When equality occurs, we will show that $\partial \Sigma$ is actually connected, and hence by the previous argument it must be a circle and $\Sigma$ must be a disk. To see this, we observe that (10) still holds on $\partial \Sigma$. In particular, we may
assume that $X(0)$ is a point on $\sigma_1$, and (12) is valid. However, Poincaré inequality is now applied on $\sigma^2$ instead of $\partial \Sigma$, therefore equation (11) only applies to the curve $\sigma^2$. On the other hand, since $X(0) \in \sigma_1$, and $\sigma^2 = \sigma_1$ \cup \bigcup_{i=2}^{p} A_i^\sigma \sigma_i$, the argument concerning the coefficients $a_\sigma$ and $b_\sigma$'s is still valid. Equations (15) can still be concluded on each $\sigma^i$, hence on $\partial \Sigma$, by (17). This implies $\partial \Sigma$ is a circle, and the Theorem is proved.

**Theorem 2.** Let $\Sigma$ be a compact minimal surface in $\mathbb{R}^3$. If $\partial \Sigma$ consists of at most two components, then

$$l(\partial \Sigma)^2 > 4\pi A(\Sigma).$$

Moreover, equality holds iff $\Sigma$ is a flat disk in some affine 2-plane of $\mathbb{R}^3$.

**Proof.** In view of Theorem 1, it suffices to prove that when $\partial \Sigma = \sigma_1 \cup \sigma_2$ has exactly two connected components and is not weakly connected, $\Sigma$ must be disconnected into two components $\Sigma_1$ and $\Sigma_2$ with $\partial \Sigma_1 = \sigma_1$ and $\partial \Sigma_2 = \sigma_2$. Indeed, if this is the case, we simply apply Theorem 1 to $\Sigma_1$ and $\Sigma_2$ separately and derive

$$l(\partial \Sigma)^2 = (l(\sigma_1) + l(\sigma_2))^2$$

$$> l(\sigma_1)^2 + l(\sigma_2)^2$$

$$> 4\pi (A(\Sigma_1) + A(\Sigma_2))$$

$$= 4\pi A(\Sigma).$$

In this case, equality will never be achieved for (2).

To prove the above assertion, we assume that $\partial \Sigma = \sigma_1 \cup \sigma_2$ is not weakly connected. This implies, there exists an affine plane $P'$ in $\mathbb{R}^3$ separating $\sigma_1$ and $\sigma_2$. For any oriented affine 2-plane in $\mathbb{R}^3$ must be divided into two open half-spaces. Defining the sign of these half-spaces in the manner corresponding to the orientation of the 2-plane, we consider the sets $S_i^+$ (or $S_i^-$) as follows: a 2-plane $P$ is said to be in $S_i^+$ (or $S_i^-$) for $i = 1$ or 2, if $\sigma_i$ is contained in the positive (or negative) open half-space defined by $P$. Obviously, $P' \in S_1^+ \cap S_2^-$ for a fixed orientation of $P'$. However, by the compactness of $\partial \Sigma = \sigma_1 \cup \sigma_2$, $S_1^+ \cap S_2^- \neq \emptyset$ and $S_2^- \cap S_1^+ \neq \emptyset$. Hence $\partial S_1^+ \cap \partial S_2^- \neq \emptyset$, by virtue of the fact that both $S_1^+$ and $S_2^-$ are connected sets. This gives a 2-plane in $\mathbb{R}^3$, $P_1$, which has the property that $\sigma_1$ (and $\sigma_2$) lies in the closed positive (respectively negative) half-space defined by $P_1$. Moreover, both the sets $\sigma_1 \cap P_1$ and $\sigma_2 \cap P_1$ are nonempty.

By the assumption that $\partial \Sigma$ is not weakly connected and since $P_1$ does not separate $\sigma_1$ and $\sigma_2$, there exists an affine 2-plane in $\mathbb{R}^3$, $P'$, which is perpendicular to $P_1$ and separating $\sigma_1$ and $\sigma_2$. Let us define $\bar{S}$ to be the set of
oriented affine 2-planes in \( \mathbb{R}^3 \) which are perpendicular to \( P_i \). Setting \( \overline{S}_i^+ \) (or \( \overline{S}_i^- \)) to be \( S_i^+ \cap \overline{S} \) (or \( S_i^- \cap \overline{S} \)), and as before, we conclude that \( \partial \overline{S}_1^+ \cap \partial \overline{S}_2^- \neq \emptyset \). Hence, there exists an affine 2-plane, \( P_z \), perpendicular to \( P_1 \), and having the property that \( \sigma_1 \) (and \( \sigma_2 \)) lie in the closed positive (respectively negative) half-space defined by \( P_z \) and both sets \( \sigma_1 \cap P_z \) and \( \sigma_2 \cap P_z \) are nonempty.

Arguing once more that \( P_1 \) and \( P_z \) do not separate the \( \sigma_i \)'s, there must be an affine 2-plane \( P_3 \) perpendicular to both \( P_1 \) and \( P_z \). Moreover, \( P_3 \) must separate \( \sigma_1 \) and \( \sigma_2 \) by the assumption the \( \partial \Sigma \) is not weakly connected. We defined a rectangular coordinate system \( xyz \) such that \( P_1, P_z \) and \( P_3 \) are the \( xy, yz, \) and \( xz \) planes respectively. Clearly by the properties of the 2-planes \( P_i \)'s, \( \sigma_1 \) and \( \sigma_2 \) are contained in the closed octant \( \{ x > 0, y > 0, z > 0 \} \) and the closed octant \( \{ x < 0, y < 0, z < 0 \} \) respectively. In particular, \( \sigma_1 \) is contained in the cone defined by \( C_1 = \{ X \in \mathbb{R}^3 | X \cdot V > |X|/\sqrt{3}, \text{ where } V = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}) \} \) and \( \sigma_2 \) is contained in the cone \( C_2 = \{ X \in \mathbb{R}^3 | X \cdot V < -|X|/\sqrt{3}, \text{ where } V = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}) \} \). However, one verifies that the two cones \( C_i, i = 1, 2 \), are contained in the positive and negative cones defined by the catenoid obtained from rotating the catenary along the line given by \( V \). In view of Theorem 6 in [4], the minimal surface \( \Sigma \) must be disconnected. This concludes our proof.

2. – Nonexistence.

Let \( (x^1, ... , x^n) \) be a rectangular coordinate system in \( \mathbb{R}^n \). We consider the \( (n-1) \)-dimensional surface of revolution \( S_n \) obtained by rotating the catenary \( x^{n-1} = \cosh (x^n/a) \) around the \( x^n \)-axis. One readily computes that its principal curvatures are

\[
(\cosh^{-1}(z/a), -\cosh^{-1}(z/a), -\cosh^{-1}(z/a), ..., -\cosh^{-1}(z/a))
\]

with respect to the inward normal vector (i.e. the normal vector pointing towards the \( x^n \)-axis). The set of hypersurfaces \( \{ S_n \}_{a>0} \) defines a cone in \( \mathbb{R}^n \) as in the case when \( n = 3 \) (see [4]). This cone (positive and negative halves) is given by

\[
C = \{ (x^1, ..., x^n) \in \mathbb{R}^n | (x^1)^2 + ... + (x^{n-1})^2 < (x^n)^2 \sinh^2 \tau \}
\]

where \( \tau \) is the unique positive number satisfying \( \cosh \tau - \tau \sinh \tau = 0 \).

If \( \Sigma \) is a compact connected minimal surface in \( \mathbb{R}^n \) with boundary decomposed into \( \partial \Sigma = \sigma_1 \cup \sigma_2 \), where \( \sigma_1 \) and \( \sigma_2 \) (each could have more than one connected component) lie inside the positive and negative part of \( C \) respect-
ively, then arguing as in [5], $\Sigma$ must intersect one of the surfaces $S_\alpha$ tangentially. Moreover, $\Sigma$ must lie in the interior (the part containing the $x^\alpha$-axis) of $S_\alpha$, except at those points of intersection. This violates the maximum principle since $\Sigma$ is minimal and any 2-dimensional subspace of the tangent space of $S_\alpha$ must have nonpositive mean curvature. Hence $\Sigma$ must be disconnected. This gives the following:

**Theorem 3.** Let $C^+$ and $C^-$ be the positive and negative halves of the cone in $R^n$ defined by (18). Suppose $\Sigma$ is a minimal surface spanning its boundary $\partial \Sigma = \sigma_1 \cup \sigma_2$. If $\sigma_1 \subset C^+$ and $\sigma_2 \subset C^-$, then $\Sigma$ must be disconnected.

We remark that using similar arguments, one can use surfaces of revolution having principal curvatures of the form $(k\lambda, -\lambda, -\lambda, ..., -\lambda)$ as barrier to yield nonexistence type theorems for $(k + 1)$-dimensional minimal submanifolds in $R^n$.

**References**


