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Interpolation manifolds

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INTRODUCTION AND DEFINITIONS

In this paper we study several interpolation and peaking questions for function algebras on interpolation manifolds in the boundary of strictly pseudoconvex domains and in the distinguished boundary of the unit polydisc.

We first recall the definitions of some basic concepts used throughout the text.

We denote by $A^k(D)$ with $0 \leqq k \leqq \infty$, the algebra of all functions that are analytic on the bounded domain D and for which all the derivatives of order less than or equal to k extend continuously to the closure of D. The algebra $A^0(D)$ of functions analytic on D and continuous on $\bar{D}$, will be denoted by $A(D)$.

A compact subset E of the boundary bD of D is called a *peak set for* $A^k(D)$ if there is a function G in $A^k(D)$ which is identically one on E and such that $|G(z)| < 1$ for all z in $\bar{D} \setminus E$.

The set E is called an *interpolation set for* $A^k(D)$ if for each f in $C^k(E)$, there exists an $F \in A^k(D)$ which equals f on E.

If we can choose the above function F to have the additional property that $|F(z)| < \max_E |f|$ whenever z belongs to $\bar{D} \setminus E$ and $f \not\equiv 0$, then E is called a *peak-interpolation set for* $A^k(D)$.

The set E is called a *local peak set* (or a *local interpolation set* or a *local peak-interpolation set*) *for* $A^k(D)$ if for each point p in E, there is some neighborhood U_p such that $E \cap \bar{U}_p$ is a peak set (or an interpolation set or a peak-interpolation set) for $A^k(D)$.

In what follows we will mostly be concerned with these properties for compact subsets of interpolation sets. More general peak sets and interpolation sets are studied in [12], [13], [16] and [39].

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For a strictly pseudoconvex domain in $\mathbf{C}^N$, the concept of interpolation manifold was introduced (although not called this) by Davie and Øksendal [8]. A C^1-submanifold Σ of the boundary bD of a strictly pseudoconvex domain D, is called an *interpolation manifold for D* if at each point in p in Σ, its tangent space $T_p(\Sigma)$ is contained in $T_p^{\mathbf{C}}(bD)$, the maximal complex tangent space to bD at p. The manifold Σ is also said to *point in the complex direction.*

Every compact subset of such a manifold is a peak-interpolation set for $A(D)$, (see [6], [8], [16], [22], [26]).

The algebra $A(D)$ is the only one for which peak-interpolation results are known. Interpolation and peaking results are known for other algebras. Under the assumptions that both Σ and bD are of class C^∞, Hakim and Sibony [14] proved that every compact set of Σ is an interpolation set and a local peak set for $A^\infty(D)$, while Chaumat and Chollet [4] showed that they are also peak sets. Both papers contain partial local converses. Fornaess and Henriksen [11] obtained global converses.

Another interpolation result of the same kind is due to Burns and Stout [3] who proved that if Σ is closed and real-analytic, then every real-analytic function on Σ extends to a function holomorphic in a neighborhood of $\bar{D}$. They, also, have a converse result.

Geometric questions concerning interpolation manifolds in the boundary of a strongly pseudoconvex domain were studied by Stout in [36].

For the unit polydisc U^N in $\mathbf{C}^N$, the concept of interpolation manifold has been studied much less. Burns and Stout [3] proved an interpolation result analogous to the one on strictly pseudoconvex domains, for real-analytic functions on a closed real-analytic submanifold Σ' of the distinguished boundary $\mathbf{T}^N$ satisfying the following *cone condition*: At each point $p \in \Sigma'$, the intersection of $T_p(\Sigma')$ with $\bar{C}_p$ is zero, where $\bar{C}_p$ is the closure of the positive cone in $T_p(\mathbf{T}^N)$ generated by the vector fields $\partial/\partial\theta_1, \ldots, \partial/\partial\theta_N$ at p, $\left(i.e. \sum_{i=1}^N a_i(\partial/\partial\theta_i) \text{ with } a_i > 0 \text{ for all } i\right)$. We will call these manifolds *interpolation manifolds for U^N* and show that interpolation theorems similar to those for strictly pseudoconvex domains hold for these manifolds.

In the appendix, we prove that the interpolation theories for C^k-functions on the unit polydisc and on strictly pseudoconvex domains differ significantly.

There are also two major geometrical differences between the strictly pseudoconvex case and the polydisc case. The condition of pointing in the complex direction is a closed condition (*i.e.* there are arbitrarily small perturbations of Σ which do not satisfy the condition) while the cone condition is an open condition. This first difference yields the fact that most theorems for the polydisc case only have partial converses (see also [3]).

In both cases, the conditions on the tangent spaces imply that the real dimension of Σ is at most $N-1$. (See [26] for the strictly pseudoconvex case and [37] for more general peak sets.) For the unit ball B^N in $\mathbf{C}^N$, the standard examples of closed, maximal dimensional interpolation manifolds are the sphere $\Sigma_1 = bB^N \cap \mathbf{R}^N$ and the torus

$$\Sigma_2 = \left\{(z_1, \ldots, z_N) \in bB^N \colon \prod_{j=1}^{N} z_j = N^{-N/2}\right\}.$$

(See [9] for more examples.) For the unit polydisc, we have however:

THEOREM. *A closed, $(N-1)$-dimensional interpolation manifold for U^N is a torus.*

PROOF. Let π, the projection of such a manifold Σ to $\mathbf{T}^{N-1}$, be given by

$$\pi(z_1, \ldots, z_N) = (z_1, \ldots, z_{N-1}).$$

We show that π is a submersion. Choose $p \in \Sigma$ and consider a coordinate neighborhood U of p in Σ so small that $\exp(y_1, \ldots, y_N) = (\exp(iy_1), \ldots, \exp(iy_N))$ is a diffeomorphism between $\tilde{\Sigma}$, an $(N$—$1)$-dimensional submanifold of $\mathbf{R}^N$, and U. We denote the map $\pi \circ \exp$ by φ and $\exp^{-1}(p)$ by q. The map $d\varphi(q)$ maps $T_q(\tilde{\Sigma})$ onto $T_{\varphi(q)}(\mathbf{T}^{N-1})$. Indeed, the cone condition on Σ is equivalent to the condition that for each y in $\tilde{\Sigma}$,

$$T_y(\tilde{\Sigma}) \cap \bar{\mathbf{R}}_+^N = \{0\}.$$

Hence there exists a vector $v = (v_1, \ldots, v_N)$ in $\mathbf{R}_+^N$ such that $T_q(\tilde{\Sigma}) = \{w \in \mathbf{R}^N \colon (w, v) = \sum w_j v_j = 0\}$. If $(w_1, \ldots, w_{N-1})$ belongs to $T_{\varphi(q)}(\mathbf{T}^{N-1})$ but not to the image of $T_{q^0}(\tilde{\Sigma})$ under $d\varphi(q^0)$, then for all y in $\mathbf{R}$, $\sum_{j=1}^{N-1} w_j v_j + v_N y \neq 0$ but this is impossible since $v_N \neq 0$. Hence $d\varphi(q^0)$ is surjective. But this and

$$d\varphi(q) = d\pi(p) \circ d\exp(q)$$

implies that π is a submersion.

Since Σ is compact and $\mathbf{T}^{N-1}$ is connected, π maps Σ onto $\mathbf{T}^{N-1}$ and is a covering projection. All the compact covering spaces of a torus are tori.

For the unit disc D in $\mathbf{C}$, the Fatou-Rudin-Carleson theorem (see [17]) ensures that peak sets, interpolation sets and peak-interpolation sets for $A(D)$ coincide and that they are precisely the sets with Lebesgue measure zero. The peak sets for $A^k(D)$ $(k \geqq 1)$ are all finite sets (see [39]). More details about known results in $\mathbf{C}$ can be found in the survey article [7].

PART I

PEAK-INTERPOLATION SETS FOR BOUNDED, MEASURABLE FUNCTIONS

1. – Statement of the results.

In [31], Stein proved that every function in $H^\infty(D)$, the space of bounded holomorphic functions on a smoothly bounded domain D in $\mathbf{C}^N$, has nontangential limits almost everywhere on the boundary with respect to the surface measure. Nagel and Rudin [23] proved that if bD is of class C^1 and if γ is a closed curve of class $\Lambda_{1+\alpha}$ (*i.e.*, γ is of class C^1 and its derivatives satisfy $|\gamma_j'(t) - \gamma_j'(t')| \leqq C|t-t'|^\alpha$), and if the tangent never points in the complex direction, then every function in $H^\infty(D)$ has nontangential boundary values almost everywhere on γ with respect to the measure induced by any choice of local coordinates for the curve. On the other hand, if D is assumed to be strictly pseudoconvex and γ is an interpolation curve in its boundary, then there is a function in $H^\infty(D)$ which has no limit along any curve in D that ends on γ, [23].

We show that given a bounded measurable function f on γ which points in the complex direction, there exists a function in $H^\infty(D)$ whose nontangential limits exist and equal f almost everywhere on γ.

We denote by $\mathcal{M}(E)$ the space of all nonnegative, σ-finite, regular Borel measures on the Borel subset E of the boundary of the domain D. The trivial extension of μ in $\mathcal{M}(E)$ to all of $\mathbf{C}^N$ is denoted by $\tilde{\mu}$.

A Borel subset E of bD is called a *peak-interpolation set for* μ in $\mathcal{M}(E)$ if for every function f in $L^\infty(E,\mu)\setminus\{0\}$, there is an $\tilde{f}$ in $H^\infty(D) \cap$ $\cap\, C(\bar{D}\setminus \operatorname{supp}\tilde{\mu})$ such that

(1) $\tilde{f}^* = f\,[\mu]$—a.e. on E,

(2) $|\tilde{f}(z)| < \|f\|_{L^\infty(E,\mu)}$ for all $z \in D$,

(3) $|\tilde{f}^*(x)| < \|f\|_{L^\infty(E,\mu)}$ for all $x \in bD\setminus E$ where $\tilde{f}^*$ exists.

(Here and throughout the rest of the paper we denote by $\tilde{f}^*$ the nontangential boundary values of $\tilde{f}$.)

It is clear that if E is a peak-interpolation set for every measure μ in $\mathcal{M}(E)$, then every Borel subset F of E has the same property for every Borel measure in $\mathcal{M}(F)$. The Borel set E need not be closed in order to have the above property. We will see that the set $E_1 = \{(\exp(i\theta), \exp(-i\alpha\theta)) : \theta \in \mathbf{R}\}$ has this property as a subset of $\mathbf{T}^2$, while

$$E_2 = \left\{\sqrt{\frac{\alpha}{\alpha+1}}\exp(i\theta),\quad \frac{1}{\sqrt{\alpha+1}}\exp(-i\alpha\theta) : \theta \in \mathbf{R}\right\}$$

has it as a subset of bB^2, where in both examples α is a positive, irrational number, (see Part II).

The following result indicates a sufficient condition for E to have this property when D has a $C^{1,1}$-boundary, *i.e.* when the normals to the boundary bD satisfy a Lipschitz condition of order 1.

THEOREM I.1. *Let D be a bounded domain in $\mathbf{C}^N$ ($N \geqq 1$), with $C^{1,1}$-boundary. If E is a Borel subset of bD, every compact subset of which is a peak-interpolation set for $A(D)$, then E is a peak-interpolation set for every measure μ in $\mathcal{M}(E)$.*

Note that in the case of the unit polydisc in $\mathbf{C}^N$ ($N \geqq 2$) the condition on E forces it to be a subset of $\mathbf{T}^N$. By Rudin's main result in [26] (on see Part II for the statement) and by Theorem II.1, interpolation manifolds satisfy the hypothesis both in the strictly pseudoconvex case and in the polydisc case. It should also be pointed out that the theorem is apparently new even in the case $N = 1$.

In the case of a strictly pseudoconvex domain or of the unit polydisc in $\mathbf{C}^N$ ($N \geqq 2$), the condition on E in Theorem I.1 is also necessary.

THEOREM I.2. *Let D be the unit polydisc or a strictly pseudoconvex domain in $\mathbf{C}^N$ ($N \geqq 2$). If a Borel subset E of bD is a peak-interpolation set for every measure μ in $\mathcal{M}(E)$, then each compact subset of E is a peak-interpolation set for $A(D)$.*

The following theorem gives other examples of such a set E. For the unit ball it was proved in [28], p. 190.

THEOREM I.3. *Let D be a bounded strictly pseudoconvex domain in $\mathbf{C}^N$ with boundary of class C^3 and let f be a zero-free function in $H^\infty(D)$. The set $E = \{z \in bD : f^*(z) = 0\}$ is a Borel set with the property that every compact subset of it is a peak-interpolation set for $A(D)$.*

Note. Although f is zero-free on D, E need not be empty.

2. – Proof of Theorem I.1.

We first consider the case where D has $\mathcal{C}^{1,1}$-boundary. This regularity hypothesis implies that we can find a $\delta_0 > 0$ such that if $L_p = \{p - tN_{bD}(p): 0 < t < \delta_0\}$, then $L_p \cap L_q = \emptyset$ whenever $p \neq q$ and $\bigcup_{p \in bD} L_p$ is contained in D, where $N_{bD}(p)$ denotes the outward unit normal to bD at p. (See [20].) We fix such a δ_0.

Since for every σ-finite, regular Borel measure μ, there exists a finite, regular Borel measure ν such that μ is absolutely continuous with respect to ν and ν with respect to μ, we may assume that μ itself is finite. We abbreviate $L^\infty(E, \mu)$ to $L^\infty(\mu)$. We denote by $\mathcal{A}_\mu$ the algebra

$$\{f \in H^\infty(D): f^* \text{ exists } [\mu]\text{-a.e. on } E\}$$

and by π the map from $\mathcal{A}_\mu$ to $L^\infty(\mu)$ defined by $\pi(f) = f^*|E$. The algebra $\mathcal{A}_\mu$ is uniformly closed. We break the proof of the theorem into several lemmas.

Lemma I.4. *Let K be a compact subset of E and $0 < \alpha < \frac{1}{2}$. There exists a function g in $\mathcal{A}_\mu \cap \mathcal{C}(\bar{D} \setminus \operatorname{supp} \tilde{\mu})$ such that*

(i) *$g^*(x) = 1$ whenever $x \in K$,*

(ii) *$g^* = 0$ $[\mu]$-a.e. on $E \setminus K$,*

(iii) *at every point x of $bD \setminus E$ where g^* exists, $|g^*(x)| < 1$,*

(iv) *the image of D under g is contained in the set $\{z \in \mathbf{C}: |z| < 1$ and $|\arg z| < \alpha\pi\}$.*

Proof. By the regularity of μ, we can find compact subsets K_j of $E \setminus K$ such that

$$\mu\Big(E \setminus \Big(K \cup \bigcup_{j=1}^{\infty} K_j\Big)\Big) = 0\,.$$

We choose open neighborhoods W_j of $\operatorname{supp} \tilde{\mu}$ in $\bar{D}$ such that $W_j \supset W_{j+1}$ and $\bigcap_{j=1}^{\infty} W_j = \operatorname{supp} \tilde{\mu}$. For each $j \geq 1$, the function h_j that equals 1 on K and -1 on K_j, is continuous on $K \cup K_j$. Hence by the hypothesis on E, there exists, for each j, a function H_j in $A(D)$ that peak-interpolates h_j on $K \cup K_j$, *i.e.*,

(1) $H_j | K \cup K_j = h_j$,

(2) $|H_j(z)| < 1$ for all z in $\bar{D} \setminus K \cup K_j$.

For δ_0 as given before, we denote by L the compact set $\{p - tN_{bD}(p)\colon 0 \leqq t \leqq \delta_0,\ p \in K\}$. For each j, we can find a positive δ_j so small that for all z in $L \cup (\bar{D} \setminus W_j)$, we have

$$(*) \qquad |2^{-\delta_j}(H_j(z)+1)^{\delta_j} - 1| < 2^{-j}$$

and $\sum \delta_j < 2\alpha$. Denote the element $2^{-\delta_j}(H_j+1)^{\delta_j}$ of $A(D)$ by G_j. By the choice of the δ_j, the product $\prod_{j=1}^{\infty} G_j$ converges uniformly on compact subsets of D to g, a bounded holomorphic function on D. Since for $[\mu]$-almost every point x in $E \setminus K$, $G_n(x) = 0$ for some n, g satisfies (ii). The condition $(*)$ implies that the product $\prod_{j=1}^{\infty} G_j$ converges uniformly on L.

This, together with the statement due to Čirka, that if $h \in H^\infty(D)$ has a limit along the normal at $p \in bD$ then it has also nontangential limit at p and they are the same, (p. 629, [5]), implies that g belongs to $\mathcal{A}_\mu$ and satisfies (i). Clearly g belongs to $\mathcal{C}(\bar{D} \setminus \operatorname{supp} \tilde{\mu})$. The property (iv) holds by the choices of the $\{\delta_n\}_{n=1}^{\infty}$ and $\{H_n\}_{n=1}^{\infty}$. We prove now (iii). Denote by A_0 the set of points in bD where g^* exists. For each m, we can deal with the product $p_m = \prod_{n \neq m} G_n$ as we dealt with $\prod_{n=1}^{\infty} G_n$ and see that it converges and belongs to $\mathcal{A}_\mu$. Denote by A_m the subset of bD where p_m^* exists. We have that for all k, $g = p_k\, G_k$ and $A_0 \setminus E = A_k \setminus E$. Hence for all x in $A_0 \setminus E$ we have $|g^*(x)| = |p_k^*(x) G_k(x)| \leqq |G_k(x)| < 1$ which completes the proof of Lemma I.4.

Lemma I.5. *If F is any Borel subset of E and $0 < \alpha < \frac{1}{2}$, then there exists a function g in $\mathcal{A}_\mu \cap \mathcal{C}(\bar{D} \setminus \operatorname{supp} \tilde{\mu})$ such that*

(i) $g^* = \chi_F$ $[\mu]$-a.e. *on E, where χ_F is the characteristic function of F,*

(ii) *the image of D under g is contained in the set $\{z \in \mathbf{C}\colon |z| < 1$ and $|\arg z| < \alpha\pi\}$.*

Proof. By the regularity of μ we can find compact subsets F_j of F such that $F_j \subset F_{j+1}$ and $\mu\Big(F \setminus \bigcup_{j=1}^{\infty} F_j\Big) = 0$. Similarly there exist compact subsets K_j of $E \setminus F$ such that $K_j \subset K_{j+1}$ and $\mu\Big(E \setminus \Big(F \cup \bigcup_{j=1}^{\infty} K_j\Big)\Big) = 0$. For each j we choose a function g_j in $\mathcal{A}_\mu \cap \mathcal{C}(\bar{D} \setminus \operatorname{supp} \tilde{\mu})$ which satisfies the properties of Lemma I.4 for K_j and some $\alpha_j < \frac{1}{8}$. Since for each j, $K_j \cap F_j = \emptyset$ the functions $h_j = 1$ on F_j and $= -1$ on K_j, are continuous on $K_j \cup F_j$. By the hypothesis on E we can find H_j in $A(D)$ which peak-interpolates h_j

on $K_j \cup F_j$. By choosing the $\delta'_j < \frac{1}{4}$ sufficiently small we can assume that

$$|2^{-\delta'_j}(H_j(z)+1)^{\delta'_j} - 1| < 16^{-j}$$

for all z in $L'_j \cup (\bar{D} \setminus W_j)$ where

$$L'_j = \{p - tN_{bD}(p) : 0 \leqq t \leqq \delta_0,\ p \in F_j\}$$

and where W_j are open neighborhoods of supp $\tilde{\mu}$ in $\bar{D}$ such that $W_j \supset W_{j+1}$ and $\bigcap_{j=1}^{\infty} W_j = \operatorname{supp} \tilde{\mu}$. We denote the functions

$$[1 - 2^{-\delta'_j}(H_j + 1)^{\delta'_j}]^{\frac{1}{2}}$$

in $A(D)$ by F_j. Choose a sequence of positive numbers $\beta_j < 1$ such that $\sum_{j=1}^{\infty} \beta_j < 2\alpha$ and $\sin(\beta_j \pi/4) < 2^{-(j+1)}$. By the choices of $\{g_j\}_{j=1}^{\infty}$ in $\mathcal{A}_\mu \cap \cap\, C(\bar{D} \setminus \operatorname{supp} \tilde{\mu})$, $\{F_j\}_{j=1}^{\infty}$ in $A(D)$, $\{\delta'_j\}_{j=1}^{\infty}$ and $\{\beta_j\}_{j=1}^{\infty}$, the product $\prod_{j=1}^{\infty}(1 - g_j F_j)^{\beta_j}$ converges uniformly on compact subsets of D to a bounded holomorphic function g which is continuous on $\bar{D} \setminus \operatorname{supp} \tilde{\mu}$ and clearly satisfies (ii).

We denote by A'_j the subset of E where g_j^* does not exist. If A'_0 is the set

$$\left(\bigcup_{j=1}^{\infty} A'_j\right) \cup \left(F \setminus \bigcup_{j=1}^{\infty} F_j\right) \cup \left(E \setminus F \cup \left(\bigcup_{j=1}^{\infty} K_j\right)\right)$$

then $\mu(A'_0) = 0$. We show that for all x in $E \setminus A'_0$, $g^*(x)$ exists and that $g^* = 0$ on $E \setminus F \cup A'_0$, $g^* = 1$ on $F \setminus A'_0$. Indeed, if x belongs to $E \setminus F \cup A'_0$, there exists some j such that $x \in K_j \setminus A'_0$, but then

$$|g(z)| < |1 - g_j(z) F_j(z)|^{\beta_j},$$

and the latter term goes to zero when z approaches x along the normal by the choice of g_j and F_j. If $x \in F \setminus A'_0$ and $\varepsilon > 0$, we choose a j_0 such that $x \in F_j \setminus A'_0$ for all $j \geqq j_0$ and $2^{-j_0} < \varepsilon/16$. We have that for some j_1 depending on z

$$|g(z) - 1| \leqq \varepsilon/2 + \left|1 - \prod_{j=1}^{j_1}(1 - g_j(z)F_j(z))^{\beta_j}\right|.$$

Since for all z in D

$$|1 - g_j(z)F_j(z)| < 1,$$

we can use the inequality

$$|\alpha\beta - 1| \leqq |\alpha||\beta - 1| + |\alpha - 1|$$

repeatedly to obtain

$$\left|1 - \prod_{j=1}^{j_1}(1 - g_j(z)F_j(z))^{\beta_j}\right| \leqq \sum_{j=1}^{j_0-1}\left|(1 - g_j(z)F_j(z))^{\beta_j} - 1\right| + \sum_{j=j_0}^{\infty}\left|(1 - g_j(z)F_j(z))^{\beta_j} - 1\right|.$$

By using the fact that when $r < 1$ and $-\pi/2 < \theta < \pi/2$,

$$|r \exp(i\theta) - 1| \leqq |r - 1| + |r||\exp(i\theta) - 1| \leqq 1 - r + 2r \sin\frac{\theta}{2}$$

we obtain

$$\sum_{j=j_0}^{\infty}|(1 - g_j(z)F_j(z))^{\beta_j} - 1| \leqq \sum_{j=j_0}^{\infty}(1 - |1 - g_j(z)F_j(z)|^{\beta_j}) + 2\sum_{j=j_0}^{\infty}\sin\frac{\beta_j\pi}{4}.$$

But if $|w| < 1$ and $0 < \beta < 1$, then

$$1 - |w|^{\beta} \leqq 1 - |w| \leqslant |w - 1|.$$

Thus we can majorize

$$\sum_{j=j_0}^{\infty}|(1 - g_j(z)F_j(z))^{\beta_j} - 1|$$

by

$$\sum_{j=j_0}^{\infty}|g_j(z)||F_j(z)| + 2\sum_{j=j_0}^{\infty}\sin\frac{\beta_j\pi}{4}$$

which by choice of $\{\beta_j\}_{j=1}^{\infty}$, $\{g_j\}_{j=1}^{\infty}$, $\{F_j\}_{j=1}^{\infty}$ and j_0, is majorized by $\varepsilon/4$ for all z in L'_{j_0}. Hence

$$|g(z) - 1| \leqq 3\varepsilon/4 + \sum_{j=1}^{j_0-1}\left|(1 - g_j(z)F_j(z))^{\beta_j} - 1\right|$$

where j_0 depends only on x in $F \diagdown A_0'$, and therefore $g(z)$ approaches 1 whenever z approaches x in $F \diagdown A_0'$ along the normal. This completes the proof of Lemma I.5.

Since the space $\pi(\mathcal{A}_\mu)$ is a linear space and the simple functions are dense in $L^\infty(\mu)$, we can conclude from Lemma I.5 that $\pi(\mathcal{A}_\mu)$ is dense in $L^\infty(\mu)$. Thus the proof of the existence of an interpolating $\tilde{f}$ in $\mathcal{A}_\mu$ for any f in $L^\infty(\mu)$, is reduced to showing that $\pi(\mathcal{A}_\mu)$ is closed in $L^\infty(\mu)$. This can be done by using the Gelfand transform in order to represent $L^\infty(\mu)$ as the algebra of continuous functions on its maximal ideal space and then applying a result of Bade and Curtis [1]: *If Λ is a compact F-space and $\mathcal{A}$ is a Banach subalgebra of $\mathcal{C}(\Lambda)$ that is dense in $\mathcal{C}(\Lambda)$, then $A = \mathcal{C}(\Lambda)$.* (Recall that a compact space Λ is an F-space if disjoint open F_σ-sets in Λ have disjoint closures.)

We give a more elementary argument which allows us to get at the same time the other properties stated in the theorem.

PROOF OF THEOREM I.1. It is clearly sufficient to prove the theorem for functions in $L^\infty(\mu)$ with norm 1. We first prove the theorem for a non-negative function f.

We define a sequence of Borel sets E_j in E and a sequence of f_j in $L^\infty(\mu)$ as follows: Let $E_0 = \{x \in E\colon \frac{1}{2} < f(x) \leqq 1\}$ and $f_1 = f - \frac{1}{2}\chi_{E_0}$. Inductively, let $E_j = \{x \in E\colon 2^{-(j+1)} < f_j(x) \leqq 2^{-j}\}$ and $f_j = f - \sum_{k=0}^{j-1} 2^{-(k+1)}\chi_{E_k}$. The $\sum_{k=0}^{j-1} 2^{-(k+1)}\chi_{E_k}$ converges to f in $L^\infty(\mu)$. Pick for each j, a function g_j in $\mathcal{A}_\mu \cap \mathcal{C}(\bar{D} \setminus \operatorname{supp}\tilde{\mu})$ which satisfies Lemma I.5 for E_j and some $\alpha_j < \frac{1}{4}$. Consider a conformal map h from the set $\Lambda = \{z \in \mathbf{C}\colon |z| < 1 \text{ and } |\arg z| < \pi/4\}$ to the set $\Lambda' = \{z \in \mathbf{C}\colon |z| < 1,\ |z - \frac{1}{2}| < \frac{1}{2} \text{ and } 0 < \arg z < \pi/4\}$ such that $h(0) = 0$, $h(1) = 1$. Since the function h is continuous up to the boundary, the functions $\tilde{f}_j = h \circ g_j$ belong to $\mathcal{A}_\mu \cap \mathcal{C}(\bar{D} \setminus \operatorname{supp}\tilde{\mu})$ and $\tilde{f}_j^* = \chi_{E_j}$ $[\mu]$-a.e. on E. The sum $\sum_{j=0}^{\infty} 2^{-(j+1)}\tilde{f}_j$ converges in $H^\infty(D)$ to a function $\tilde{f}$ which belongs to $\mathcal{A}_\mu \cap \mathcal{C}(\bar{D} \setminus \operatorname{supp}\tilde{\mu})$ since $\mathcal{A}_\mu \cap \mathcal{C}(\bar{D} \setminus \operatorname{supp}\tilde{\mu})$ is a closed subalgebra of $H^\infty(D)$. By the choice of the $\tilde{f}_j$, we have that $\tilde{f}^* = f$ $[\mu]$-a.e on E, $|\tilde{f}^*(x)| < 1$ for all x in $bD \setminus E$ where $\tilde{f}^*$ exists, $|\tilde{f}(z)| < 1$ and $0 < \operatorname{Im}\tilde{f}(z)$ whenever $z \in D$.

We use this special case to prove the theorem for any function f in $L^\infty(\mu)$ with norm 1. Let $S = \{x \in E\colon f(x) \neq 0\}$. Choose a function $\tilde{f}_1$ in $\mathcal{A}_\mu \cap \mathcal{C}(\bar{D} \setminus \operatorname{supp}\tilde{\mu})$ which satisfies Lemma I.5 for S. On S, we may write $f/|f| = \exp(2\pi i\varphi)$ for some φ in $L^\infty(\mu)$ and $0 \leqq \varphi < 1$. The functions $|f|$ and φ can be interpolated as in the first part of the proof of the theorem by functions $\tilde{f}_2$ and $\tilde{f}_3$ respectively in $\mathcal{A}_\mu \cap \mathcal{C}(\bar{D} \setminus \operatorname{supp}\tilde{\mu})$ such that $|\tilde{f}_j^*(x)| < 1$ for all x in $bD \setminus E$ where $\tilde{f}_j^*$ exists and $|\tilde{f}_j(z)| < 1$ and $\operatorname{Im}\tilde{f}_j(z) > 0$ when $z \in D$, $(j = 2, 3)$. The function $\tilde{f} = \tilde{f}_1\tilde{f}_2\exp(2\pi i\tilde{f}_3)$ belongs to $\mathcal{A}_\mu \cap \mathcal{C}(\bar{D} \setminus \operatorname{supp}\tilde{\mu})$, interpolates f on E, $|\tilde{f}(z)| < 1$ whenever $z \in D$ and $|\tilde{f}^*(x)| < 1$ for all x in $bD \setminus E$ where $\tilde{f}^*$ exists.

This concludes the proof of Theorem I.1 for domains D with $\mathcal{C}^{1,1}$-boundary. This regularity condition was, however, used in only two places: first to ensure the existence of a δ_0-strip under E inside D and secondly to use a theorem due to Čirka (p. 629, [5]) about nontangential limits.

These properties hold on a much larger class of domains in $\mathbb{C}^N$, for example for the polydisc. For instance Theorem I.1 on polydisc can be restated as follows.

THEOREM I.6. *Let E be a Borel subset of $\mathbb{T}^N$ of which each compact subset is a peak set for $A(U^N)$ and let μ be an element of $\mathcal{M}(E)$. For each $f \in L^\infty(E, \mu)$ there exists a function $\tilde{f}$ in $H^\infty(U^N) \cap C(\overline{U}^N \setminus \operatorname{supp} \mu)$ such that*

(i) $\lim_{r\to 1} \tilde{f}(rx) = f(x)$ *for $[\mu]$-almost all x in E.*

(ii) $|\tilde{f}(z)| < \|f\|_{L^\infty(E,\mu)}$ *whenever $z \in U^N$.*

(iii) $|\lim_{r\to 1} \tilde{f}(rx)| < \|f\|_{L^\infty(E,\mu)}$ *whenever x is in $bU^N \setminus E$ and the limit exists.*

3. – Proof of Theorem I.2.

We want to show that in the case of strictly pseudoconvex domains or the unit polydisc the condition on E in Theorem I.1 is also necessary. We first prove the following general result.

LEMMA I.7. *Let D be a bounded, starshaped domain in $\mathbb{C}^N$ and let E be a Borel subset of bD which is a peak-interpolation set for every measure in $\mathcal{M}(E)$. If K is a compact subset of E and μ annihilates $A(D)$, then $|\mu|(K) = 0$.*

PROOF. We may assume that D is starshaped with respect to the origin. We fix a compact K in E and a measure μ which annihilates $A(D)$ and prove that $\mu(K_0) = 0$ for every compact subset K_0 of K.

Denote by ν the measure $|\mu|$ restricted to K. Since K is compact, $\operatorname{supp}\tilde{\nu} = \operatorname{supp}\nu$ where $\tilde{\nu}$ denotes as before the trivial extension of ν to $\mathbb{C}^N$. Hence by the hypothesis on E, we can find a function f in $H^\infty(D) \cap C(\overline{D} \setminus \operatorname{supp}\tilde{\nu})$ with $f^* = \chi_{K_0}$ $[\tilde{\nu}]$ a.e. on bD and $|f^*| < 1$ on $\overline{D} \setminus \operatorname{supp}\tilde{\nu}$. We define the functions $f_m(z) = f((1 - 1/m)z)$. These functions belong to $A(D)$ since D is starshaped. Therefore for all m and n,

$$\int_{\overline{D}} f_m^n(z)\, d\mu(z) = 0 .$$

Hence $0 = \lim_{n\to\infty} \lim_{m\to\infty} \int_{\overline{D}} f_m^n(z)\, d\mu(z)$.

We define $h_n \in L^\infty(\bar{D}, \mu)$ by

$$h_n(z) = f^n(z) \qquad \text{if } z \in \bar{D} \setminus \operatorname{supp} \tilde{\nu}$$
$$= (f^*(z))^n \qquad \text{if } z \in \operatorname{supp} \tilde{\nu} \cap A_0$$

where $A_0 = \{x \in bD : f^*(x) \text{ exists}\}$. Then $0 = \lim\limits_{n\to\infty} \int\limits_{\bar{D}} h_n(z)\, d\mu$.

But $h_n \to \chi_{K_0}$ $[\mu]$-a.e. This implies that $\mu(K_0) = 0$, which completes the proof of the lemma.

This lemma combined with Theorem 6.1.2 of [25] yields immediately Theorem I.2 in the case of the unit polydisc.

PROOF OF THEOREM I.2 *for strictly pseudoconvex domains.* By Weinstock [40] it is sufficient to show that for every x in E there exists a ball centered at x such that $\mu(K) = 0$ whenever K is compact in $E \cap \bar{B}$ and whenever μ annihilates $A(D \cap B)$.

But every point x in bD has a neighborhood U in $\bar{D}$ such that U is strictly convex for a suitable choice of holomorphic local coordinates. Therefore the above statement holds by Lemma I.7.

4. – Proof of Theorem I.3.

We first prove Theorem I.3 for a strictly convex domain D. We may assume D contains the origin.

By Bishop's lemma [2] it is sufficient to show that for every compact subset K of E and every measure μ which annihilates $A(D)$, $\mu(K) = 0$.

After composing f with a conformal map on its range if necessary, we may assume that $|f(z)| < 1$ and $\operatorname{Re} f(z) > 0$ on D. As $|\operatorname{Im} \log f| < \pi/2$, $\log |f|^p$ $(p \in (0, \infty))$ has a pluriharmonic majorant (p. 151, [28]) and so belongs to Lumer's Hardy space, (see [21]), and $\log f_\tau(z) = \log f(\tau z)$ converges to $\log f^*$ as $\tau \to 1$ in $L^p(bD, d\sigma)$ where $d\sigma$ denotes the surface measure on bD.

We denote by A_0 the Borel subset of bD on which f^* exists. The set E is clearly a Borel set. Let K be a fixed compact subset of E. Choose neighborhoods V_n and W_n of K such that $W_n \supset \bar{V}_n$ and $\bigcap\limits_{n=1}^{\infty} W_n = K$. Let k_n be $\mathcal{C}^\infty$-functions on $\mathbf{C}^N$ with support contained in W_n and which are identically one on $\bar{V}_n$, $0 \leqq k_n < 1$ anywhere else and which satisfy the inequality $|k_n(z) - k_n(z')| \leqq A_n |z - z'|$ for some positive constant A_n for all z, z' in $\bar{D}$.

We define

$$g_n(z) = \int k_n(w) \log f^*(w) \frac{K(z, w)}{[\Phi(z, w)]^N} d\sigma(w)$$

where $K(z, w)/[\Phi(z, w)]^N$ is the Cauchy-Fantappiè kernel as described in [35] or in [19] (p. 190 ff.). By the properties of the Cauchy-Fantappiè kernel the functions g_n are holomorphic in D. They have moreover the following properties:

(1) $\operatorname{Re} g_n \leqq C_n$ for some $C_n > 1$.

(2) If $x \in E_0 = (bD \setminus E) \cap A_0$ or $x \in \{z \in bD : k_n(z) = 0\}$ then $\lim\limits_{\substack{\varrho \to 1 \\ 0<\varrho<1}} g_n(\varrho x)$ exists (as a finite number).

(3) $\lim\limits_{\substack{\varrho \to 1 \\ 0<\varrho<1}} g_n(\varrho x) = -\infty$ whenever x is in $E \setminus \{z \in bD : k_n(z) = 0\}$.

Let us assume first that these properties hold. We define then

$$\varphi_n(z) = \frac{g_n(z)}{g_n(z) - 2C_n}$$

These functions form a uniformly bounded family in $H^\infty(D)$. Let $\varphi_{n,m}(z) = \varphi_n\big((1 - 1/m)z\big)$, then the $\varphi_{n,m}$ are in $A(D)$ and for all z in bD which are not in $E_n = E_0 \cap \{w \in bD : k_n(w) \neq 0\}$ we have that $\lim\limits_{m\to\infty} \varphi_{n,m}(z) = \varphi_n^*(z)$. Hence whenever μ annihilates $A(D)$, we have

$$\int_{bD} \varphi_{n,m} d\mu = 0 \qquad \text{or} \qquad \int_{E_n} \varphi_{n,m} d\mu = -\int_{bD\setminus E_n} \varphi_{n,m} d\mu$$

which gives us that

$$\lim_{m\to\infty} \int_{E_n} \varphi_{n,m} d\mu = -\int_{bD\setminus E_n} \varphi_n^* d\mu$$

When $n \to \infty$, we have that the characteristic functions of $\{w \in bD : k_n(w) \neq 0\}$ approaches χ_K, the characteristic function of K. Hence since for all n, $\varphi_n^* \equiv 1$ on K, we obtain that

$$\lim_{n\to\infty} \lim_{m\to\infty} \int_{E_n} \varphi_{n,m} d\mu = -\mu(K)$$

or

$$\begin{aligned} |\mu(K)| &= \lim_{n\to\infty} \lim_{m\to\infty} \left| \int_{bD} \varphi_{n,m} \chi_{E_n} d\mu \right| \\ &\leqq \lim_{n\to\infty} \lim_{m\to\infty} \int \chi_{E_n} d|\mu| \\ &\leqq \lim_{n\to\infty} |\mu|(E_n) . \end{aligned}$$

But

$$\bigcap_{n=1}^{\infty} E_n = E_0 \cap \bigcap_{n=1}^{\infty} \{w \in bD: k_n(w) \neq 0\}$$
$$= E_0 \cap K \subset E_0 \cap E = \emptyset ,$$

lets us conclude that $\mu(K) = 0$.

The proof of Theorem I.3 in the case of a strictly convex domain D will be completed if we show that all g_n satisfy the properties (1)-(3). In order to do so we will make use of the estimates about the Cauchy-Fantappiè kernel obtained in [35].

Let $w_0 \in bD$. We consider (for $\varrho < 1$)

$$\begin{aligned}
&|g_n(\varrho w_0) - k_n(w_0) \log f(\varrho w_0)| \\
&\quad \leqq \left| \int_{bD} [k_n(w) - k_n(w_0)] \log f^*(w) \frac{K(\varrho w_0, w)}{[\Phi(\varrho w_0, w)]^N} d\sigma(w) \right| \\
&\quad + \left| \int_{bD} k_n(w_0)[\log f^*(w) - \log f_\tau(w)] \frac{K(\varrho w_0, w)}{[\Phi(\varrho w_0, w)]^N} d\sigma(w) \right| \\
&\quad + |k_n(w_0)[\log f_\tau(\varrho w_0) - \log f(\varrho w_0)]| \,.
\end{aligned}$$

This inequality holds for all τ in $(\varrho, 1)$ because of the reproducing properties of the Cauchy-Fantappiè kernel for functions in $A(D)$.

The first term is bounded from above by a constant independent of ϱ and w_0 by estimates on the Cauchy-Fantappiè kernel given on p. 828 of [35]. (See also [34]). The second term is bounded from above by $C_{\varrho,n} \|\log f^* - \log f_\tau\|_p$ where $C_{\varrho,n}$ depends on ϱ and n. By choosing τ close enough to 1 this can be bounded from above by 1.

The last term can similarly be bounded by 1 by choosing τ close to 1 since $\log f_{\varrho\tau}$ goes uniformly on bD to $\log f_\varrho$ for τ approaching 1 and $\varrho < 1$. Thus we have that

$$|g_n(\varrho w_0) - k_n(w_0) \log f(\varrho w_0)| \leqq C_n$$

for some constant independent of ϱ and w_0. This implies properties (1) and (3) for the g_n's.

In order to prove (2), we look at the differences

$$\begin{aligned}
|g_n(\varrho w_0) - g_n(\varrho' w_0)| &\leqq \left| \int_{bD} [k_n(w) - k_n(w_0)] \log f^*(w) L(\varrho, \varrho', w_0, w) \, d\sigma(w) \right| \\
&\quad + \left| \int_{bD} k_n(w_0)[\log f^*(w) - \log f_\tau(w)] L(\varrho, \varrho', w_0, w) \, d\sigma(w) \right| \\
&\quad + k_n(w_0) [\log f_\tau(\varrho w_0) - \log f_\tau(\varrho' w_0)]
\end{aligned}$$

where

$$L(\varrho, \varrho', w_0, w) = \frac{K(\varrho w_0, w)}{[\Phi(\varrho w_0, w)]^N} - \frac{K(\varrho' w_0, w)}{[\Phi(\varrho' w_0, w)]^N}.$$

The third term is small when w_0 is in $\{k_n = 0\}$ or in $(bD\setminus E) \cap A_0$ and $\varrho, \varrho', \tau \to 1$. The second term is bounded from above by

$$C_{\varrho,\varrho'}|k_n(w_0)|\,\|\log f^* - \log f_\tau\|_p$$

For any $\varepsilon > 0$ and ϱ, ϱ' we can find τ sufficiently close to 1, such that this term is less than ε. We estimate the first term from above by

$$K_n\|\log f^*\|_p\left(\int\limits_{bD} \frac{|w - w_0|^q}{[\Phi(\varrho w_0, w)]^{Nq}}\left(|\tilde{H}_{w_0}(\varrho, \varrho', w)|^q d\sigma(w)\right)\right)^{1/q}$$

where

$$\tilde{H}_{w_0}(\varrho, \varrho', w) = \frac{K(\varrho w_0, w)\,\Phi^N(\varrho' w_0, w) - K(\varrho' w_0, w)\,\Phi^N(\varrho w_0, w)}{[\Phi(\varrho' w_0, w)]^N}$$

and $1/p + 1/q = 1$. The function $\left(|w - w_0|^q\right)/[\Phi(\varrho w_0, w)]^{Nq}$ is in $L^1(bD, d\sigma)$ if q is chosen sufficiently small (see p. 830, [35]) while $H_{w_0}(\varrho, \varrho', w)$ is uniformly bounded and approaches zero when ϱ and ϱ' approach 1.

This completes the proof of property (2) for the functions g_n and hence the proof of Theorem I.3 in the case of a strictly convex domain D.

The proof for a strictly pseudoconvex domain follows from this in the same way as we proved Theorem I.2 by using Weinstock's result [40].

It should be pointed out that an analogous result does not hold on the unit polydisc without additional restrictions on the function f. Indeed, for the unit polydisc, it is not true in general that such a set E is contained in the distinguished boundary as can be seen by looking at $f(z_1, z_2) = 1 - z_1$.

PART II

PEAK-INTERPOLATION SETS FOR THE POLYDISC ALGEBRA

1. – Results.

For strictly pseudoconvex domains in $\mathbf{C}^N$, Rudin [26] proved the following theorems.

THEOREM 1. *Let D be a bounded strictly pseudoconvex domain in $\mathbf{C}^N$*

with C^2-boundary. Let Ω be an open set in $\mathbf{R}^m$ and $\Phi\colon \Omega \to bD$ a nonsingular C^1-mapping that satisfies the orthogonality condition

$$\langle \Phi'(x)v, N_{bD}(\Phi(x))\rangle = 0 \qquad \textit{for all } x\in\Omega \textit{ and } v\in\mathbf{R}^m .$$

For every compact subset K of Ω, $\Phi(K)$ is a peak-interpolation set for $A(D)$.

THEOREM 2. *With D and Ω as in the preceding result, if $\Phi\colon \Omega\to bD$ is of class C^1, if Φ' satisfies a Hölder condition of positive order (i.e. $|\Phi'(z)-\Phi'(z')|< C|z-z'|^\alpha$ for some $\alpha>0$), and if $\Phi(K)$ is a peak-interpolation set for $A(D)$, for every compact subset K in Ω, then $\langle \Phi'(x)v, N_{bD}(\Phi(x))\rangle = 0$ for all $x\in\Omega$ and $v\in\mathbf{R}^m$.*

In case the set $\Phi(\Omega)$ is a manifold, the orthogonality condition of Theorem 1 is equivalent to saying that $\Phi(\Omega)$ is an interpolation manifold for bD. However, the set $\Phi(\Omega)$ need not be a manifold as the example mentioned in Part I illustrates. Let $\Phi\colon \mathbf{R}\to bB^2(0,1)$ be given by

$$\Phi(t) = \left(\sqrt{\frac{\alpha}{\alpha^2+1}}\exp(it),\ \frac{1}{\sqrt{\alpha^2+1}}\exp(-i\alpha t)\right)$$

for some positive irrational number α. The set $\Phi(\mathbf{R})$ is not a manifold but the map Φ satisfies the required orthogonality condition.

For the unit polydisc, we have obtained the following results, which show that interpolation manifolds are those submanifolds of $\mathbf{T}^N$ that satisfy the cone condition of Burns and Stout [3].

THEOREM II.1. *Let Ω be an open set in $\mathbf{R}^m$ and let $\Psi\colon \Omega\to\mathbf{R}^N$ be a nonsingular map of class C^2 such that for all x in Ω*

$$\{\Psi'(x)v\colon v\in\mathbf{R}^m\}\cap\bar{\mathbf{R}}^N=\{0\}.$$

If K is a compact subset of Ω, then $\exp\circ\Psi(K)$ is a peak-interpolation set for $A(U^N)$ where

$$\exp(x_1,\ldots,x_N) = (\exp(ix_1),\ldots,\exp(ix_N)).$$

THEOREM II.2. *If $\Psi\colon\Omega\to\mathbf{R}^N$ is a nonsingular map of class C^1 such that all ψ_j' satisfy a Hölder condition of positive order α and such that, for every compact set K in Ω, $\exp\circ\Psi(K)$ is a peak-interpolation set for $A(U^N)$, then*

$$\{\Psi'(x)v\colon v\in\mathbf{R}^m\}\cap\mathbf{R}^N_+=\{0\}\qquad\text{for all } x\in\Omega .$$

Just as in the strictly pseudoconvex case, $\exp\circ\Psi(\Omega)$ need not be a manifold as can be seen from the example mentioned in Part I, namely

$$E_1 = \{(\exp(i\theta), \exp(-i\alpha\theta)) : \theta \in \mathbf{R}\}$$

where α is a positive irrational number. If $\exp\circ\Psi(\Omega)$ is a manifold the condition of Ψ is equivalent to saying that it is an interpolation manifold for U^N.

The methods used in proving Theorem II.1 consist of mapping, locally around any point p in $\Psi(\Omega)$, the set $\exp\circ\Psi(\Omega)$ into an interpolation manifold of class $\mathcal{C}^1$ in the boundary of the unit ball in $\mathbf{C}^N$. The rest of the proof follows then from Theorem 1 of Rudin [26].

The proof of Theorem II.2 uses the same methods as used by Nagel and Rudin [23] for studying the boundary behavior of holomorphic functions on strictly pseudoconvex domains.

Both Theorem II.1 and Theorem II.2 should be compared with analogous results due to Burns and Stout [3] for real analytic functions.

We also point out that in certain cases Theorem II.1 is a consequence of an interpolation result of Forelli [10] involving the concept of null S-width. Examples of this relation to Forelli's result are given after the proof of the theorems.

2. – Proof of Theorem II.1.

The key ingredient of the proof of Theorem II.1 is Bishop's lemma [2]. In order to prove Theorem II.1 it is sufficient to prove that for every point p in Ω there is an open neighborhood Ω_p of p in Ω such that $\nu(\exp\circ\Psi(K)) = 0$ whenever K is compact in Ω_p and ν annihilates $A(U^N)$.

Fix p in Ω. The condition

$$\{\Psi'(x)v : v \in \mathbf{R}^m\} \cap \bar{\mathbf{R}}^N_+ = \{0\}$$

implies that $m \leq N-1$. Hence there exist an open neighborhood W_p of p, relatively compact in Ω, and a map $\beta = (\beta_1, \ldots, \beta_N) : W_p \to \mathbf{R}^N_+$ of class $\mathcal{C}^1$ such that on W_p

$$(1) \qquad \sum_{j=1}^{N} \beta_j \frac{\partial \psi_j}{\partial x_k} = 0 \qquad \text{for all } k = 1, \ldots, m\,.$$

$$(2) \qquad \sum_{j=1}^{N} \beta_j = 1\,.$$

We denote $\sqrt{\beta_j}$ by α_j. The following lemma is the key remark which enables us to link Theorem II.1 to Rudin's Theorem 1 for the unit ball B^N. (See [26] and [28].)

Lemma II.3. *The map $\Phi\colon W_p \to \mathbb{C}^N$ with $\varphi_j = \alpha_j(\exp\circ\Psi)_j$, sends W_p into bB^N. It is a nonsingular map of class C^1 such that $\langle \Phi'(x)v, \Phi(x)\rangle = 0$ whenever $x \in W_p$ and $v \in \mathbb{R}^m$.*

Proof. The smoothness of Φ and the fact that $\Phi(W_p) \subset bB^N$ are direct consequences of the choice of the α_j.

In order to see that Φ is nonsingular, we notice that

$$\varrho(z_1, \ldots, z_N) = \big(|z_1|, \ldots, |z_N|, \arg z_1, \ldots, \arg z_N\big)$$

is a local diffeomorphism away from the coordinate planes. Hence $\varrho\circ\Phi(x) = \big(\alpha_1(x), \ldots, \alpha_N(x), \psi_1(x), \ldots, \psi_N(x)\big)$ has real rank m since Ψ is nonsingular, which proves that Φ is nonsingular.

For any $x \in W_p$ and $v \in \mathbb{R}^m$,

$$\langle \Phi'(x)v, \Phi(x)\rangle = \sum_{k=1}^{m} v_k \left(\sum_{j=1}^{N} \alpha_j(x)\frac{\partial\alpha_j}{\partial x_k}(x) + i\sum_{j=1}^{N}\alpha_i^2(x)\frac{\partial\psi_j}{\partial x_k}(x)\right) = 0$$

by the choice of α_j.

The rest of the proof of Theorem II.1 follows the general lines of Rudin's proof of Theorem 1 in [26], (see also [28], p. 215 ff.), by using Lemma II.3.

We choose Ω_p to be an open neighborhood of p, relatively compact in W_p, on which the following holds

(1) $|\Phi'(x)| \geqq c|v|$ for all $x \in \bar{\Omega}_p$, $v \in \mathbb{R}^m$,

(2) $|\Phi(x) - \Phi(y)| \geqq c|x - y|$ whenever $x, y \in \bar{\Omega}_p$

for some suitable constant c.

Let γ be defined on W_p by

$$\gamma_j(x) = \alpha_j^2(x)(\exp\circ\Psi)_j(x).$$

Lemma II.4. *There exists a constant $c' > 0$ such that for all z in $\bar{U}^N$*

$$\operatorname{Re}\langle \exp\Psi(x) - z, \gamma(x)\rangle \geqq c'\|\exp\circ\Psi(x) - z\|^2$$

whenever $x \in \bar{\Omega}_p$.

Proof. By the definition of γ we have that

$$\langle \exp\circ\Psi(x) - z, \gamma(x)\rangle = \langle \Phi(x) - w, \Phi(x)\rangle$$

where $w=(w_1,\dots,w_N)$ is given by $w_j=\alpha_j(x)z_j$. Since $w\in\bar{B}^N$ by the choice of the α_j, we have by Lemma 2.1 of [26] that

$$\begin{aligned}\operatorname{Re}\langle \exp\circ\Psi(x)-z,\gamma(x)\rangle &\geqq \tfrac{1}{2}\|\Phi(x)-w\|^2\\ &\geqq c'\|\exp\circ\Psi(x)-z\|^2\end{aligned}$$

where $c'=\frac{1}{2}\inf\{\alpha_j^2(x)\colon x\in\bar{\Omega}_p;\ j=1,\dots,N\}$.

LEMMA II.5. *For all x in Ω_p and v in $\mathbb{R}^m$ the inner product*

$$\left\langle\frac{\exp\circ\Psi(x+\delta v)-\exp\circ\Psi(x)}{\delta^2},\ \gamma(x+\delta v)\right\rangle$$

converges to

$$F_x(v)=\frac{1}{2}\sum_{k,l=1}^{m}v_kv_l\left(\sum_{j=1}^{N}\alpha_j^2(x)\frac{\partial\psi_j}{\partial x_k}(x)\frac{\partial\psi_j}{\partial x_l}(x)+2i\alpha_j(x)\frac{\partial\alpha_j}{\partial x_k}\frac{\partial\psi_j}{\partial x_l}\ x)\right)$$

as $\delta\to 0$. Moreover $\operatorname{Re}F_x(v)>0$ *unless* $v=0$.

PROOF. We have that, for fixed $x\in\Omega_p$, $v\in\mathbb{R}^m$:

$$\begin{aligned}&\frac{1}{\delta^2}\langle\exp\circ\Psi(x+\delta v)-\exp\circ\Psi(x),\gamma(x+\delta v)\rangle\\ &\qquad=\frac{1}{\delta^2}\sum_{j=1}^{N}\alpha_j^2(x+\delta v)\Big(1-\exp\big[i(\psi_j(x)-\psi_j(x+\delta v))\big]\Big)\,.\end{aligned}$$

We expand $\alpha_j^2(x+\delta v)$ and $\exp\big[i(\psi_j(x)-\psi_j(x+\delta v))\big]$ about x to obtain that

$$\begin{aligned}&\frac{1}{\delta^2}\langle\exp\circ\Psi(x+\delta v)-\exp\circ\Psi(x),\gamma(x+\delta v)\rangle=\frac{1}{\delta}\sum_{j=1}^{N}\alpha_j^2(x)\left(\sum_{k=1}^{m}i\frac{\partial\psi_j}{\partial x_k}(x)v_k\right)\\ &\quad+2i\sum_{j=1}^{N}\alpha_j(x+\delta_j'v)\left(\sum_{k=1}^{m}\frac{\partial\alpha_j}{\partial x_k}(x+\delta_j'v)v_k\right)\left(\sum_{l=1}^{m}\frac{\partial\psi_j}{\partial x_l}(x)v_l\right)\\ &\quad+\frac{1}{2}\sum_{j=1}^{N}\alpha_j^2(x)\left(\sum_{k=1}^{m}\frac{\partial\psi_j}{\partial x_k}(x+\delta_j''v)v_k\right)^2\exp\big[i(\psi_j(x)-\psi_j(x+\delta_j''v))\big]\\ &\quad+\frac{i}{2}\sum_{j=1}^{N}\alpha_j^2(x)\left(\sum_{k,l=1}^{m}\frac{\partial^2\psi_j}{\partial x_k\partial x_l}(x+\delta_j''v)v_kv_l\right)\exp\big[i(\psi_j(x)-\psi_j(x+\delta_j''v))\big]+o(\delta)\,.\end{aligned}$$

Where δ_j' and δ_j'' are between 0 and δ.

The first term is zero because, by choice of the α_j,

$$\sum_{j=1}^{N}\alpha_j^2(x)\frac{\partial\psi_j}{\partial x_k}(x)\equiv 0\qquad\text{for all }k\,.$$

By letting $\delta \to 0$ and by using the fact that

$$\sum_{j=1}^{N} \alpha_j^2(x) \frac{\partial^2 \psi_j}{\partial x_k \partial x_l}(x) = -2 \sum_{j=1}^{N} \alpha_j(x) \frac{\partial \alpha_j}{\partial x_k}(x) \frac{\partial \psi_j}{\partial x_l}(x)$$

we obtain that

$$\begin{aligned}\lim_{\delta\to 0} \frac{1}{\delta^2} &\langle \exp\circ\Psi(x+\delta v) - \exp\circ\Psi(x), \gamma(x+\delta v)\rangle \\ &= \frac{1}{2}\sum_{k,l=1}^{m} v_k v_l \left(\sum_{j=1}^{N}\left(\alpha_j^2 \frac{\partial \psi_j}{\partial x_k}(x) \frac{\partial \psi_j}{\partial x_l}(x) + 2i\alpha_j(x) \frac{\partial \alpha_j}{\partial x_k}(x) \frac{\partial \psi_j}{\partial x_l}(x)\right)\right) = F_x(v)\,.\end{aligned}$$

We prove now that $\operatorname{Re} F_x(v) > 0$ unless $v = 0$. Suppose that for some x in Ω_p and v in $\mathbb{R}^m$,

$$\operatorname{Re} F_x(v) = \sum_{j=1}^{N}\left(\sum_{k=1}^{m} \alpha_j(x) \frac{\partial \psi_j}{\partial x_k}(x) v_k\right)\left(\sum_{l=1}^{m} \alpha_j(x) \frac{\partial \psi_j}{\partial x_l}(x) v_l\right) = 0$$

then for all $j = 1, \ldots, N$ we have

$$\sum_{k=1}^{N} \alpha_j(x) \frac{\partial \psi_j}{\partial x_k}(x) v_k = 0$$

which can only happen if $v = 0$ since all the $\alpha_j > 0$ on Ω_p and Ψ is non-singular. Since

$$\operatorname{Re} F_x(v) = \sum_{j=1}^{N} \alpha_j^2(x) \left(\frac{\partial \psi_j}{\partial x_l}(x)\right)^2$$

when $v = (1, 0, \ldots, 0)$ we must have $\operatorname{Re} F_x(v) > 0$ unless $v = 0$.

We are now ready to prove Theorem II.1.

Proof of Theorem II.1. By Lemma II.5 and by Lemma 2.4 of [26] we have that the function

$$g(y) = \int_{\mathbb{R}^m} \frac{dv}{[1 + F_y(v)]^m} \quad (y \in \bar{\Omega}_p)$$

is zero-free on $\bar{\Omega}_p$; it is plainly continuous there.

Let $f\colon \mathbb{R}^m \to \mathbb{C}$ be continuous with support in Ω_p. For $\delta > 0$ we define

$$h_\delta(z) = \int_{\Omega_p} \frac{\delta^m (f/g)(x)\, dx}{[\delta^2 + \langle \exp\circ\Psi(x) - z, \gamma(x)\rangle]^m}\,.$$

By Lemma II.4, the real part of the inner product in the integrand is non: negative when $z \in \overline{U}^N$, so h_δ belongs to $A(U^N)$ for δ sufficiently small. Moreover the family $\{h_\delta\}$ has the following properties (with δ sufficiently small)-

(1) $\sup\{|h_\delta(z)|: z \in \overline{U}^N, \delta > 0\} < \infty$,

(2) $\lim_{\delta \to 0} h_\delta(z) = 0$ for all z in $\overline{U}^N \setminus \exp\circ\Psi(\overline{\Omega}_p)$,

(3) $\lim_{\delta \to 0} h_\delta(\exp\circ\Psi(y)) = f(y)$ whenever $y \in \overline{\Omega}_p$.

These properties are proved by using the above lemmas and arguments similar to those used in [26] and will not be given here. The rest of Theorem II.1 follows the same lines as the proof of the main result in [26]: Let K be a compact subset of Ω_p. Choose compact subsets K_i such that $K_i \supset K_{i+1}$ and $\bigcap_{i=1}^{\infty} K_i = K$, on which there exist continuous functions f_i with $\operatorname{supp} f_i \subset K_i$ and $f_i \equiv 1$ on K. Since $\exp\circ\Psi$ is one-to-one on $\overline{\Omega}_p$, we have that $f_i \circ (\exp\circ\Psi)^{-1}$ is defined and continuous on some compact subset of $\mathbf{T}^N$. Let $h_{i,\delta}$ be the functions constructed as above for $f_i \circ (\exp\circ\Psi)^{-1}$. Then $\int h_{i,\delta}\, d\nu = 0$ whenever ν annihilates $A(U^N)$. By letting $\delta \to 0$, we have

$$0 = \int_{\exp\circ\Psi(\Omega_p)} f_i \circ (\exp\circ\Psi)^{-1} d\nu$$

which for $i \to \infty$ gives that $\nu(\exp\circ\Psi(K)) = 0$.

By Bishop's lemma and the remark made in the beginning this completes the proof of Theorem II.1.

3. – Proof of Theorem II.2.

We obtain Theorem II.2 as a direct consequence of Lemma 4.2 in [23] and the following theorem.

THEOREM II.6. *Let $\varphi: (0,1) \to \mathbb{R}^N$ be a map of class C^1, such that φ' satisfies a Hölder condition of order $\alpha (\alpha > 0)$ and such that $\varphi'(x)$ belongs to $\mathbb{R}^N_+$ for all x in $(0,1)$. Then for all points x in $(0,1)$, there exists a nontangential curve γ_x in U^N such that $\lim_{t \to 0} \gamma_x(t) = \exp\circ\varphi(x)$ and with the property that for every F in $H^\infty(U^N)$, $\lim_{t \to 0} F(\gamma_x(t))$ exists for almost all x in $(0,1)$.*

With a nontangential curve γ_x we mean a curve of which every coordinate projection is a nontangential curve in the unit disc $\mathbf{C}$.

The proof of this theorem is similar to that of Theorem 1 in [23].

PROOF OF THEOREM II.6. It is sufficient to prove that the conclusion of the theorem holds for any compact $K = [a, b]$ contained in $(0, 1)$. Fix such a K and pick nonnegative functions α_j with support in $(0, 1)$ such that $\alpha_j = c_j'$ on K and which satisfy a Hölder condition of order α on $\mathbb{R}$. Let $u_j(x, t)$ be the Poisson integral of α_j for $t > 0$. Let Q be the set

$$\{(x, t) \in \mathbb{R}^2 : x \in K,\ t \in [0, h]\}$$

for some $h > 0$. Let $\Gamma(x + it) = \varphi(x) + itu(x, t)$ where $u(x, t) = \big(u_1(x, t), \ldots, u_N(x, t)\big)$ and let $\gamma_x(t) = \exp\circ\Gamma(x + it)$. Since

$$(\exp\circ\Gamma)_j(x + it) = \exp\big(-tu_j(x, t)\big)\exp\big(i\varphi_j(x)\big)\,,$$

each coordinate projection of γ_x is clearly a nontangential curve in the unit disc and $\lim_{t\to 0} \gamma_x(t) = \exp\circ\varphi(x)$ when $x \in K$.

Let F be any bounded holomorphic function on U^N and denote $F\big(\exp\circ\Gamma(z)\big)$ by $f(z)$ where $z = x + it$. Fix for the moment z in the interior of Q. The point $w = \exp\circ\Gamma(z)$ is the center of a polydisc with polyradius

$$\Big(1 - \exp\big(-tu_1(x, t)\big), \ldots, 1 - \exp\big(-tu_N(x, t)\big)\Big)$$

in U^N. By Schwarz's lemma we have that

$$\left|\frac{\partial F}{\partial w_j}(w)\right| \leqq \|F\|_\infty\big[1 - \exp\big(-tu_j(x, t)\big)\big]^{-1}$$

and hence

$$\left|\frac{\partial f}{\partial \bar z}(w)\right| \leqq \frac{1}{2}\,\|F\|_\infty \sum_{j=1}^{N} h_j(x, t)$$

where

$$h_j(x, t) = \frac{\big|i\big(\varphi_j'(x) - u_j(x, t) - t(\partial u_j/\partial t)(x, t)\big) - t\big((\partial u_j/\partial x)(x, t)\big)\big|}{\big|\exp\big(tu_j(x, t)\big) - 1\big|}\,.$$

By using the estimate for the numerator given on p. 582 of [23] and the inequality

$$\exp\big(tu_j(x, t)\big) - 1 > tu_j(x, t) \geqq mt$$

on Q, we have that $\partial f/\partial \bar z$ is in $L^p(Q)$ for some $p > 1$.

Hence by Theorem 4 of [23] we can conclude that for almost all x in K,

$$\lim_{t\to 0} f(x + it) = \lim_{t\to 0} F\big(\gamma_x(t)\big)$$

exists.

4. – Examples.

In this section we consider the relations between the cone condition used in Theorem II.1 and the concept of null S-width due to Forelli [10].

Let S be a set of unit vectors in $\mathbb{R}^N$. A set E in $\mathbb{R}^N$ is said to have *null S-width* if for all $\varepsilon > 0$, there exist a family $\{u^j\}_{j=1}^{\infty}$ in S and a family $\{I_j\}_{j=1}^{\infty}$ of open intervals in $\mathbb{R}$ such that $\sum_{j=1}^{\infty} l(I_j) < \varepsilon$ and

$$E \subset \bigcup_{j=1}^{\infty} \{x \in \mathbb{R}^N \colon \langle x, u^j \rangle \in I_j\} ,$$

where $l(I_j)$ denotes the length of the interval I_j.

Forelli [10] obtained the following interpolation result: *Let G be a countable union of sets of null S-width where S varies over compact subsets of unit vectors in $\mathbb{R}_+^N$. Then, for every compact K of G,* $\exp(K)$ *is a peak-interpolation set for $A(U^N)$.*

The following theorem, analogous to Theorem 6.3.5 of [25], shows that for $m = 1$, Theorem II.1 is a special case of Forelli's result.

THEOREM II.7: *Let Ω be an open set in $\mathbb{R}$ and let $\psi\colon \Omega \to \mathbb{R}^N$ be a map of class C^2 such that for all $x \in \Omega$, $\psi'(x) \notin \overline{\mathbb{R}}_+^N \cup \overline{\mathbb{R}}_-^N$. Then for every compact set K in Ω, $\Psi(K)$ has null S-width where S is a compact set of unit vectors in $\mathbb{R}_+^N$.*

PROOF. We may assume that for x in Ω all $\psi_j'(x)$ with $1 \leqq j \leqq k$ belong to $\mathbb{R}_-$ and $\psi_l'(x)$ with $k+1 \leqq l \leqq N$ to $\mathbb{R}_+$. Consider the map $\varphi\colon \Omega \to \mathbb{R}^2$ defined by

$$\varphi(x) = (\varphi_1(x), \varphi_2(x)) = \Big(\sum_{j=1}^{k} \psi_j(x), \sum_{j=k+1}^{N} \psi_j(x)\Big).$$

By Theorem 5 in [10], there exists a compact set S' of unit vectors in $\mathbb{R}_+^2$ such that $\varphi(K)$ has null S'-width. (See also Theorem 6.3.5 of [25].) We consider now the map $\alpha\colon \mathbb{R}^2 \to \mathbb{R}^N$ given by

$$\alpha_j(u_1, u_2) = \begin{cases} \dfrac{u_1}{\sqrt{ku_1^2 + (N-k)u_2^2}} & \text{when } 1 \leqq j \leqq k \\[2ex] \dfrac{u_2}{\sqrt{ku_1^2 + (N-k)u_2^2}} & \text{when } k+1 \leqq j \leqq N . \end{cases}$$

Then $\alpha(S') = S$ is a compact set of unit vectors in $\mathbf{R}^N_+$. We prove that $\Psi(K)$ has null S-width. Indeed, for a given $\varepsilon > 0$, we can find a family $\{u^j\}_{j=1}^\infty$ in S' and a family of open intervals $\{I_j\}_{j=1}^\infty$ in $\mathbf{R}$ with midpoints a_j and $\sum_{j=1}^\infty l(I_j) < \varepsilon$, such that $\varphi(K) \subset \bigcup_{j=1}^\infty \{y \in \mathbf{R}^2 : \langle y, u^j \rangle \in I_j\}$. Denote $\alpha(u^j)$ by v^j and by I'_j the open interval with midpoint $a'_j = a_j[k(u^j_1)^2 + (N-k)(u^j_2)^2]^{-\frac{1}{2}}$ and length equal to $l(I_j)$. Let $x \in K$ be given and j be such that $\langle \varphi(x), u^j \rangle \in I_j$. Then

$$|\langle \psi(x), v^j \rangle - a'_j| = [k(u^j_1)^2 + (N-k)(u^j_2)^2]^{-\frac{1}{2}} |\langle \varphi(x), u^j \rangle - a_j|$$

which is less than $\frac{1}{2} l(I_j)$. Hence $\Psi(K)$ is contained in

$$\bigcup_{j=1}^\infty \{x \in \mathbf{R}^N : \langle x, v^j \rangle \in I'_j\} \quad \text{and} \quad \sum_{j=1}^\infty l(I'_j) < \varepsilon ,$$

which completes the proof of the theorem.

For certain higher dimensional manifolds, the cone condition implies null S-width. This is true, for example, for cylindrical hypersurfaces in $\mathbf{R}^N$ $(N \geqq 3)$.

THEOREM II.8. *Let Ω be an open set in $\mathbf{R}^{N-1}$ and let $\Phi: \Omega \to \mathbf{R}^N$ be a nonsingular map of class C^1 given by*

$$\Phi(t) = T \circ R(x_1(t_1), x_2(t_1), t_2, \dots, t_{N-1})$$

where T is a translation and R a rotation of $\mathbf{R}^N$ such that for all t in Ω

$$\{\Phi'(t)v : v \in \mathbf{R}^{N-1}\} \cap \bar{\mathbf{R}}^N_+ = \{0\} .$$

If K is a compact in Ω, then $\Phi(K)$ has null S-width for some compact set S of unit vectors in $\mathbf{R}^N_+$.

PROOF. We first consider the curve γ given by $\gamma(t) = (x_1(t), x_2(t))$ *in* $\mathbf{R}^2$. By Theorem 6.3.5 in [25], we know that $\gamma(\pi_1(K))$ has null S'-width where π_1 denotes the projection on the t_1-axis in $\mathbf{R}^{N-1}$ and where

$$S' = \left\{ \left(\frac{x'_2(t)}{\|x'(t)\|}, \frac{-x'_1(t)}{\|x'(t)\|} \right), \ t \in \pi_1(K) \right\}.$$

Hence for a given $\varepsilon > 0$, we can find $\{u^j\}_{j=1}^{\infty}$ in S' and $\{I_j\}_{j=1}^{\infty}$ open intervals in $\mathbf{R}$ with midpoints a_j such that $\sum_{j=1}^{\infty} l(I_j) < \varepsilon$ and

$$\gamma(\pi_1(K)) \subset \bigcup_{j=1}^{\infty} \{y \in \mathbf{R}^2 \colon \langle y, u^j \rangle \in I_j\} .$$

By the choice of the set S' and by the cone-condition on Φ, the set $R(S' \times \{0\})$ is a compact set of unit vectors in $\mathbf{R}^N_+$ or $\mathbf{R}^N_-$. Let us assume it is in $\mathbf{R}^N_+$ and let us denote $R(u_1^j, u_2^j, 0, \ldots, 0)$ by v^j and by I'_j the open interval with center $a'_j = a_j + \langle T(0), v^j \rangle$ and length $l(I_j)$. Clearly

$$\Phi(K) \subset \bigcup_{j=1}^{\infty} \{u \in \mathbf{R}^N \colon \langle y, v^j \rangle \in I'_j\} .$$

If S is in $\mathbf{R}^N_-$, we replace it by $-S$. The set $\Phi(K)$ has then clearly null $(-S)$-width.

The previous result is false for general surfaces.

THEOREM II.9. *Let M be a compact, strictly convex hypersurface of class $\mathbf{C}^2$ in $\mathbf{R}^N$ ($N \geq 3$). Any compact subset K of M with null S-width has surface measure zero.*

By strictly convex, we mean here that M has a defining function whose Hessian is strictly positive definite.

This theorem implies for example that the peak-interpolation properties for $A(U^3)$ of compact subsets of $\exp(\Omega)$ where $\Omega = \{(x, y, z) \in \mathbf{R}^3_+ : x^2 + y^2 + z^2 = 1\}$ is only a consequence of Theorem II.1 and not of Forelli's result.

PROOF. Suppose I_j is an interval of length ε_j in $\mathbf{R}$. Then $M_j = \{x \in M\colon \langle x, u^j \rangle \in I_j\}$ is part of M contained between two hyperplanes with distance ε_j apart from each other.

The area of M_j can be bounded from above by $C\varepsilon_j$ where C is a constant which depends only on M and is independent of ε_j, the hyperplanes bordering M_j and any choice of coordinates. This estimate on the area of M_j is given in [29].

If K is a compact subset of M with null S-width then for any $\varepsilon > 0$, there exist $\{u^j\}_{j=1}^{k}$ in S and open intervals $\{I_j\}_{j=1}^{k}$ with $l(I_j) = \varepsilon_j$ such that $\sum_{j=1}^{k} \varepsilon_j < \varepsilon$ and such that $K \subset \bigcup_{j=1}^{k} \{x \in \mathbf{R}^N \colon \langle x, u^j \rangle \in I_j\}$. Thus $K \subset \bigcup_{j=1}^{k} M_j$ where

$M_j = \{x \in M : \langle x, u^j \rangle \in I_j\}$. But then by the above estimate we have that area $(K) \leqq \sum_{j=1}^{k}$ area $(M_j) \leqq C \sum_{j=1}^{k} \varepsilon_j < C\varepsilon$. Since this holds for all ε, we conclude that area $(K) = 0$.

PART III

INTERPOLATION OF $\mathcal{C}^\infty$-FUNCTIONS

1. – Results.

Hakim and Sibony [14] proved the following interpolation and peaking results for strictly pseudoconvex domains.

THEOREM 1. *Let D be a bounded, strictly pseudoconvex domain in $\mathbb{C}^N$ with $\mathcal{C}^2$-boundary. If a subset E of bD is a local peak set for $A^2(D)$ then it is locally contained in a $\mathcal{C}^1$-interpolation manifold Σ of real dimension $(N-1)$.*

THEOREM 2. *Let D be a bounded, strictly pseudoconvex domain in $\mathbb{C}^N$, with $\mathcal{C}^\infty$-boundary and let Σ be an interpolation manifold of bD of class $\mathcal{C}^\infty$. If K is a compact subset of Σ, then K is an interpolation set and a local peak set for $A^\infty(D)$.*

The first theorem implies that in general the union of two peak sets for $A^k(D)$ $(k \geqq 2)$ need not be a peak set for $A^2(D)$ and hence a simple argument by partition of unity cannot be given to improve the local peaking property of Theorem 2 to a global one. Chaumat and Chollet [4] used methods involving $\bar\partial$-problems to prove.

THEOREM 3. *Let D be a bounded, strictly pseudoconvex domain in $\mathbb{C}^N$ with $\mathcal{C}^\infty$-boundary and let Σ be an interpolation manifold of class $\mathcal{C}^\infty$. Every compact subset K of Σ is a peak set for $A^\infty(D)$.*

For the unit polydisc, we have obtained the following similar result.

THEOREM III.1. *Every compact subset K of an interpolation manifold Σ of $\mathbb{T}^N$ of class $\mathcal{C}^\infty$, is an interpolation set and a local peak set for $A^\infty(U^N)$.*

The technique involved in proving this consists essentially in constructing strictly pseudoconvex domains in $\mathbb{C}^N$ that contain the open polydisc and for which, at least locally, Σ is an interpolation manifold.

A global peaking result and a result similar to Theorem 1 of Hakim and Sibony are not known yet for the polydisc.

2. – Technical lemmas.

If Σ is a k-dimensional interpolation submanifold of $\mathbb{T}^N$ then as we noted before, k is less than or equal to $N-1$. We begin by proving that any such manifold is locally contained in an $(N-1)$-dimensional interpolation manifold of $\mathbb{T}^N$.

LEMMA III.2. *If Σ is a k-dimensional interpolation manifold of $\mathbb{T}^N$ of class $\mathcal{C}^\infty$, then for every $p \in \Sigma$, there exists a neighborhood V in $\mathbb{T}^N$ and an $(N-1)$-dimensional interpolation manifold $\tilde{\Sigma}$ of $\mathbb{T}^N$ of class $\mathcal{C}^\infty$ such that $\Sigma \cap V$ is contained in $\tilde{\Sigma}$.*

PROOF. Let W be an open set in $\mathbb{R}^k$ and $\varphi\colon W \to \mathbb{R}^N$ be a $\mathcal{C}^\infty$-map which composed with exp is a parameterization of Σ is some neighborhood U of p. The set $\varphi(W)$ is a k-dimensional submanifold of $\mathbb{R}^N$ such that the cone condition $T_q(\varphi(W)) \cap \bar{\mathbb{R}}^N_+ = \{0\}$ holds for all q in $\varphi(W)$. The matrix

$$A = \begin{pmatrix} \dfrac{\partial\varphi_1}{\partial x_1} \cdots \dfrac{\partial\varphi_1}{\partial x_k} \\ \vdots \qquad \vdots \\ \dfrac{\partial\varphi_N}{\partial x_1} \cdots \dfrac{\partial\varphi_N}{\partial x_k} \end{pmatrix}$$

has maximal rank and we may as well assume that on W

$$\det \begin{pmatrix} \dfrac{\partial\varphi_1}{\partial x_1} \cdots \dfrac{\partial\varphi_1}{\partial x_k} \\ \vdots \qquad \vdots \\ \dfrac{\partial\varphi_k}{\partial x_1} \cdots \dfrac{\partial\varphi_k}{\partial x_k} \end{pmatrix} \neq 0 .$$

We construct a map $\gamma\colon W' \to \mathbb{R}^N$ rank $N-1$, where $W' = W \times \mathbb{R}^{N-1-k}$ and such that on $W \times \{0\}$, $\varphi = \gamma$ while for all $q \in \gamma(W')$ the cone condition holds. We denote the coordinates in W' by (x, x') where $x \in W$ and $x' = (x_{k+1}, \dots, x_{N-1})$ in $\mathbb{R}^{N-1-k}$, and we choose

$$\gamma_j(x, x') = \varphi_j(x) \quad \text{when } j \leqq k$$

$$\gamma_j(x, x') = \varphi_j(x) + \sum_{l=1}^{N-1-k} \alpha_{l,j} x_{k+l} \qquad \text{when } k+1 \leqq j \leqq N,$$

where the $(N-1-k)$ vectors $\alpha_l = (\alpha_{l,k+1}, \ldots, \alpha_{l,N-1})$ are linearly independent in $\mathbf{R}^{N-1-k}$ and none is contained in the positive or negative cone of $\mathbf{R}^{N-1-k}$. The map γ has the desired property and the manifold $\tilde{\Sigma} = \exp\circ\gamma(W')$ satisfies the conclusion of Lemma III.2.

Since at first we are interested only in local results, we consider a maximal dimensional interpolation manifold Σ, and we assume we are working in a neighborhood U of the point $p = (1, \ldots, 1)$ in Σ. Whenever necessary throughout this section we shrink this neighborhood without mention.

We have $\mathbf{T}^N = \{(\exp(i\theta_1), \ldots, \exp(i\theta_N)) : \theta_1, \ldots, \theta_N \in \mathbf{R}\}$, so the tangent space of $\mathbf{T}^N$ is spanned by $\partial/\partial\theta_1, \ldots, \partial/\partial\theta_N$. We denote the coordinates in the ambient $\mathbf{C}^N$ by $(z_1, \ldots, z_N)$. If $|z_j| = r_j$, we have that away from the plane $\{z_j = 0\}$,

$$\frac{\partial}{\partial z_j} = \frac{1}{2z_j}\left(r_j \frac{\partial}{\partial r_j} - i\frac{\partial}{\partial \theta_j}\right).$$

On some neighborhood U' of p in $\mathbf{T}^N$ containing U, there exists a real-valued function $\beta \in C^\infty(\mathbf{T}^N)$ such that

(1) $\Sigma \cap U' = \{(\exp(i\theta_1), \ldots, \exp(i\theta_N)) : \beta(\exp(i\theta_1), \ldots, \exp(i\theta_N)) = 0\}$.

(2) $\left(\frac{\partial\beta}{\partial\theta_1}(q), \ldots, \frac{\partial\beta}{\partial\theta_N}(q)\right) \in C_q$ for all q in $U' \cap \Sigma$.

We denote by $\tilde{\beta}$ the extension to some neighborhood V of U' in $\mathbf{C}^N$, given by

$$\tilde{\beta}(z_1, \ldots, z_N) = \beta\left(\frac{z_1}{|z_1|}, \ldots, \frac{z_N}{|z_N|}\right).$$

For simplicity we assume $\Sigma \cap V = U$. Using the function $\tilde{\beta}$, we construct a strictly pseudoconvex hypersurface with defining function ϱ, which contains Σ and for which Σ points in the complex direction.

LEMMA III.3. *If* $D' = \{z \in V : \varrho(z) < 0\}$ *where*

$$\varrho(z) = i\sum_{k=1}^{N}\left(z_k \frac{\partial\tilde{\beta}}{\partial z_k}(z) - \bar{z}_k \frac{\partial\tilde{\beta}}{\partial \bar{z}_k}(z)\right)(|z_k|^2 - 1)$$

then the following conditions are satisfied.

(1) $d\varrho(p) \neq 0$,

(2) $V \cap U^N$ *is contained in* D',

(3) *The function* ϱ *is strictly pluriharmonic near the point* p,

(4) $\mathbf{T}^N \cap V$ *is contained in* bD' *and for any point of* $\Sigma \cap V$, $T_q(\Sigma \cap V) \subset T_q^{\mathbf{C}}(bD')$.

Since

$$\frac{\partial}{\partial z_j} = \frac{1}{2z_j}\left(r_j \frac{\partial}{\partial r_j} - i \frac{\partial}{\partial \theta_j}\right),$$

we remark that

$$i\left(z_k \frac{\partial \tilde{\beta}}{\partial z_k}(z) - \bar{z}_k \frac{\partial \tilde{\beta}}{\partial \bar{z}_k}(z)\right) = \frac{\partial \beta}{\partial \theta_k}(z)$$

when restricted to Σ, which is the k-th component of the normal to Σ in $\mathbf{T}^N$.

PROOF OF LEMMA III.3. The conclusions (1) and (2), as well as the first part of (4), are easy to verify using the fact that for all k, $\partial\beta/\partial\theta_k > 0$ on V (after shrinking V if necessary).

A straightforward calculation using the fact that $p = (1, \ldots, 1)$, gives us that when $j \neq l$

$$\frac{\partial^2 \varrho}{\partial z_j \partial \bar{z}_l}(p) = i\left[\frac{\partial^2 \beta}{\partial z_j \partial z_l}(p) - \frac{\partial^2 \beta}{\partial \bar{z}_j \partial \bar{z}_l}(p)\right]$$

But since $(\partial\tilde{\beta}/\partial r_j)(z) = 0$, we have that

$$\frac{\partial^2 \tilde{\beta}}{\partial z_j \partial z_l}(p) = -\frac{1}{2}\,\frac{\partial^2 \beta}{\partial \theta_j \partial \theta_l}(p).$$

Hence for all $j \neq l$,

$$\frac{\partial^2 \varrho}{\partial z_j \partial \bar{z}_l}(p) = 0.$$

The conclusion of (3) follows then immediately from the fact that

$$\begin{aligned}\frac{\partial^2 \varrho}{\partial z_j \partial \bar{z}_j}(p) &= i\left[\frac{\partial^2 \tilde{\beta}}{\partial z_j \partial z_j}(p) - \frac{\partial^2 \tilde{\beta}}{\partial \bar{z}_j \partial \bar{z}_j}(p) + 2\frac{\partial \tilde{\beta}}{\partial z_j}(p) - 2\frac{\partial \tilde{\beta}}{\partial \bar{z}_j}(p)\right] \\ &= 2i\left[\frac{\partial \tilde{\beta}}{\partial z_j} - \frac{\partial \tilde{\beta}}{\partial \bar{z}_j}\right] \\ &= 2\frac{\partial \beta}{\partial \theta_j}(p) > 0.\end{aligned}$$

In order to prove the second part of (4), we remark that, (see [4]),

$$T_q^{\mathbf{C}}(bD') = \left\{\operatorname{Re} X \colon X = \sum_{j=1}^{N} a_j \frac{\partial}{\partial z_j},\ a_j \in \mathbf{C} \text{ and } \sum_{j=1}^{N} a_j \frac{\partial \varrho}{\partial z_j}(q) = 0\right\}.$$

For every q in the neighborhood of p, the space $T_q(\Sigma \cap V)$ is spanned by $Y^j(q)$ where $Y^j = \sum_{k=1}^{N} \alpha_{j,k}(\partial/\partial\theta_k)$ with the property $\alpha_{j,k}$ are real-valued functions such that $\sum_{k=1}^{N} \alpha_{j,k}(\partial\beta/\partial\theta_k) \equiv 0$ on U whenever $1 \leqq j \leqq N-1$. If we choose $X^j = i\sum_{k=1}^{N} \alpha_{j,k} z_k(\partial/\partial z_k)$ and use $\partial/\partial z_k = [r_k(\partial/\partial r_k) - i(\partial/\partial\theta_k)]/2z_k$, it follows that $\operatorname{Re} X^j = Y^j$ for all $j \leqq N-1$, on U. For all $q \in U$ we also have that $\operatorname{Re} X^j(q)$ belongs to $T_q^{\mathbf{C}}(bD')$ since

$$\begin{aligned} X^j \varrho &= i \sum_{k=1}^{N} 2\alpha_{j,k} z_k \left[\left(z_k \frac{\partial\tilde{\beta}}{\partial z_k} - \bar{z}_k \frac{\partial\tilde{\beta}}{\partial \bar{z}_k}\right)\bar{z}_k\right] \\ &= 2 \sum_{k=1}^{N} \alpha_{j,k} \frac{\partial\beta}{\partial\theta_k} = 0 . \end{aligned}$$

This concludes the proof of Lemma III.3.

LEMMA III.4. *For each point p in Σ, there exists a strictly pseudoconvex domain D in $\mathbf{C}^N$ with boundary of class $\mathbf{C}^\infty$ which contains the unit polydisc and for which, near p, Σ is an interpolation manifold.*

REMARK. More explicitly, there is a neighborhood W of p in $\mathbf{C}^N$ such that $\Sigma \cap W$ is contained in bD and $\Sigma \cap W$ is an interpolation manifold for D.

PROOF OF LEMMA III.4. Let D' be as constructed in Lemma III.3. We extend the function ϱ to a $\mathbf{C}^\infty$-function on $\mathbf{C}^N$, which we denote again by ϱ. Let χ be a nonnegative $\mathbf{C}^\infty$-function on $\mathbf{C}^N$ such that $\chi \equiv 0$ on U', a neighborhood of p in Σ such that $\Sigma \cap \overline{U}' \subset U$. We also choose χ such that $U^N \cap V \subset \{z \in V : \varrho' < 0\}$ where $\varrho' = \varrho - \varepsilon\chi$ and such that $\operatorname{dist}(\overline{U}^N, b\{z \in V : \varrho' = 0\}) > 0$ where ε is so small that $d\varrho' \neq 0$ and ϱ' is strictly pluriharmonic in V. Since $\overline{U}^N$ has a neighborhood basis of strictly pseudoconvex domains of class $\mathbf{C}^\infty$, we can find such a domain C with a $\mathbf{C}^\infty$ defining function such that $C \cap V \neq \emptyset$ but $b\{z \in V : \varrho' = 0\} \subset \mathbf{C}^N \setminus \overline{C}$. The domain D is obtained by modifying the functions ϱ' and τ on a small neighborhood of $bC \cap \{z \in V : \varrho' = 0\}$ as described in [36], pp. 384-386.

3. – Interpolation results.

We use Lemma III.4 to prove a local peak result and a local interpolation result.

THEOREM III.5. *If Σ is a k-dimensional interpolation manifold of class $\mathcal{C}^\infty$ for $\mathbf{T}^N$, then each compact subset K of Σ is a local peak set and a local interpolation set for $A^\infty(U^N)$.*

PROOF. Fix a point q in K and let $\tilde{\Sigma}$ and D be as constructed in Lemma III.2 and Lemma III.4. Let U be the open neighborhood of q in $\tilde{\Sigma}$ contained in bD. We choose a neighborhood W of q in $\tilde{\Sigma}$, compactly contained in U and denote the compact set $K \cap \overline{W}$ by L. By [14] (theorem 2) we can conclude that there exists a neighborhood W_q of q in $\bar{D}$ and a function $\tilde{G}$ in $A^\infty(D)$ such that $\tilde{G} = 0$ on $L \cap \overline{W}_q$ and $\operatorname{Re} \tilde{G} < 0$ on $\bar{D} \setminus L \cap \overline{W}_q$. We may assume that $\overline{W}_q \cap \Sigma$ is contained in W. Denote the neighborhood $W_q \cap \bar{U}^N$ of q in $\bar{U}^N$ by V_q. By the construction of D then, $G = \tilde{G}|\bar{U}^N$ is a function in $A^\infty(U^N)$ such that $G = 0$ on $K \cap \bar{V}_q$ and $\operatorname{Re} G < 0$ on $\bar{U}^N \setminus K \cap \bar{V}_q$.

The same theorem of [14] gives us that for any function f in $\mathcal{C}^\infty(L)$ there exists a neighborhood W_q in $\bar{D}$ and a function $\tilde{F}$ in $A^\infty(D)$ such that $\tilde{F} = f$ on $L \cap \overline{W}_q$. But then again by letting $W_q \cap \bar{U}^N = V_q$ and $F = \tilde{F}|\bar{U}^N$ we have that $F \in A^\infty(U^N)$ and $F = f$ on $K \cap \bar{V}_q$.

Note. The interpolation process gives interpolating functions which are not merely in $A^\infty(U^N)$ but which continue analytically across $bU^N \setminus \mathbf{T}^N \cap V$.

The proof of the global interpolation result follows from Theorem III.5 by constructing a suitable partition of unity exactly as done in [14] for the strictly pseudoconvex case.

APPENDIX

NON-EXISTENCE OF DIFFERENTIABLE PEAK-INTERPOLATION RESULTS

Interpolation and peak results are known for various function algebras on compact subsets of interpolation manifolds for a strictly pseudoconvex domain D. The following theorem shows that $\mathcal{C}^m$-functions on such sets cannot be interpolated in $A^m(D)$.

THEOREM A.1. *Let D be a strictly pseudoconvex domain in $\mathbf{C}^N$ with boundary of class $\mathcal{C}^k$ and let M be a compact $\mathcal{C}^k$-interpolation manifold for D. If $A^l(D)$ interpolates $\mathcal{C}^m(M)$, then $l \leq [m/2]$.*

In the proof of this theorem we use the function spaces Λ_α which are defined on p. 441 of [32] as generalisation to manifolds of the spaces

$$\Lambda_\alpha(\mathbf{R}^n) = \left\{ f \in L^\infty(\mathbf{R}^n) : \sup_x \left| \frac{\partial^k}{\partial y^k} u(x,y) \right| \leqq A y^{-k+\alpha} \right\}$$

where k is the smallest integer greater than α and $u(x, y)$ denotes the convolution of f with the Poisson kernel on the upper halfspace of $\mathbf{R}^{n+1}$.

PROOF. Choose a function f in $C^m(M)$ such that some derivative of order $m-1$ does not belong to $\bigcup_{\alpha<1} \Lambda_\alpha$: We assume that some function F in $A^l(D)$ interpolates f on M with $l \geqq [m/2]+1$. Since all the derivatives of order $[m/2]+1$ of F exist and are continuous on $\bar{D}$, we know that all of its derivatives of order $[m/2]$ belong to Λ_1 (see [31], p. 146). By Proposition 9, p. 147 of [31], this means that F belongs to $\Lambda_{1+[m/2]}$ and by Corollary 2 on p. 443 in [32], that f, the restriction of F to M, belongs to $\Lambda_{2(1+[m/2])}$ But clearly $p = 2(1+[m/2]) > m$ which implies by Proposition 9 on p. 147 of [31] that all derivatives of order $m-1$ of f belong to Λ_{p-m+1} contradicting the choice of f.

The loss of differentiability of the order of $[m/2]$, does not happen in the polydisc case.

THEOREM A.2. *Let $\Sigma = \{(\exp(i\theta), \exp(-i\theta)) : \theta \in \mathbf{R}\}$. Any function in $C^m(\Sigma)$ can be interpolated in $A^{m-1}(U^2)$.*

PROOF. Any function f in $C^m(\Sigma)$ can be written as $\sum_{n=-\infty}^{\infty} c_n \exp(in\theta)$ where for n sufficiently large

$$|c_n| \leqslant A \frac{1}{|n|^m} \beta_n$$

where $\{\beta_n\}$ is a square-summable sequence. (By using repeatedly partial integration, $|c_n| = |n|^{-m} |\widehat{f^{(m)}(n)}|$.) We interpolate $\exp(i\theta)$ by z_1 and $\exp(-i\theta)$ by z_2. The functions $F(z_1, z_2) = \sum_{n=0}^{\infty} c_n z_1^n$ and $G(z_1, z_2) = \sum_{n=1}^{\infty} c_{-n} z_2^n$ clearly belong to $A^{m-1}(U^2)$ and their sum interpolates f on Σ.

Although up to now we have not succeeded in obtaining this result for more general interpolation manifolds than Σ, it allows us to show that there cannot exist differentiable peak-interpolation results for compact interpolation manifolds for strictly pseudoconvex domains.

THEOREM A.3. *Let* $M = \{(z_1, z_2) \in bB^2 : z_1 z_2 = \frac{1}{2}\}$. *For no* $l \geqq 2$ *does there exist* Φ *in* $A^l(B^2)$ *such that* $\Phi = \varphi$ *on* M *and* $|\Phi(z)| < 1$ *for all* z *in* $\bar{B}^2 \setminus M$ *where* $\varphi\colon M \to \mathbf{T}$ *is given by* $\varphi(z_1, z_2) = \sqrt{2} z_1$.

PROOF. We denote by ψ the map from M to $\mathbf{T}$ given by $\psi(z_1, z_2) = \sqrt{2} z_2$. The map (φ, ψ) maps M bijectively onto $\Sigma = \{(\exp(i\theta), \exp(-i\theta)) : \theta \in \mathbf{R}\}$. If there exists a function Φ in $A^2(B^2)$ which peakinterpolates φ, then the map $\alpha = (\Phi, \Psi)$ where $\Psi(z_1, z_2) = \Phi(z_2, z_1)$, has the property that $\alpha(\bar{B}^2 \setminus M) \subset U^2$ and $\alpha(M) = \Sigma$.

For any function f in $\mathcal{C}^3(M)$, the function $\tilde{f} = f \circ \alpha^{-1}$ belongs to $\mathcal{C}^3(\Sigma)$. Hence by the above theorem this function can be interpolated by a function $\tilde{F}$ in $A^2(U^2)$. The function $F = \tilde{F} \circ \alpha$ belongs to $A^2(B^2)$ and interpolates f. This shows that $A^2(B^2)$ interpolates $\mathcal{C}^3(M)$ contradicting Theorem A.1.

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