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Star Shaped Coincidence Sets in the Obstacle Problem.

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1. – Introduction.

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Give an obstacle $\psi \in C^1(\bar{\Omega})$ which is negative on $\partial\Omega$. We consider the following obstacle problem:

(1.1) Find a function u in the closed convex set

$$\mathbb{K}(\psi) = \{u \in H_0^1(\Omega);\ u \geqq \psi \text{ in } \Omega\}$$

which minimizes the integral

$$\int_\Omega |\nabla u|^2 \, dx \, .$$

It was shown in [10] and [3] that this problem has a unique solution u which belongs to $C^1(\bar{\Omega})$. Furthermore, if $\psi \in C^2(\bar{\Omega})$, it follows from the results of [1] that u belongs to $C^{1,1}(\bar{\Omega})$. Let $I(\psi)$ be the coincidence set

$$I(\psi) = \{x \in \Omega;\ u(x) = \psi(x)\} \, . \tag{1.2}$$

Note that u satisfies

$$\Delta u = 0 \quad \text{in } \Omega - I(\psi) \, . \tag{1.3}$$

The principal open questions here concern the nature of the coincidence set $I(\psi)$. For $n = 2$, under the hypotheses of convexity of Ω and analyticity

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and strong concavity of ψ, it was shown in [10] and [6] that $\partial I(\psi)$ is a regular analytic Jordan curve (see also D. Kinderlehrer - G. Stampacchia [8]). For $n>2$, it is not known whether or not the same hypotheses imply the same conclusion. However H. Lewy [9] considered the inverse problem and gave examples in which the obstacle problem is solved by the inverse method.

Motivated by this, we consider the obstacle problem (1.1) under the hypotheses of convexity of Ω and concavity of ψ. By using only the maximum principle, we obtain certain properties of the coincidence sets for certain obstacles, that is,

THEOREM 1. *Let Ω be a bounded convex domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Suppose that the origin is contained in Ω. Let $f \in C^1(\bar{\Omega})$ be a nonnegative convex function which is positive on $\partial\Omega$ and homogeneous of degree $s>1$. Consider the obstacle $\psi \in C^1(\bar{\Omega})$, which is negative on $\partial\Omega$, defined by*

$$\psi(x) = -f(x) + c \tag{1.4}$$

with positive constant c. Then the coincidence set $I(\psi)$ is star shaped with respect to the origin. In particular, if $f(x)$ is a positive definite quadratic form, that is,

$$f(x) = \sum a_{ij} x_i x_j$$

with positive definite constant $n\times n$-matrix $[a_{ij}]$, then the coincidence set $I(\psi)$ is star shaped with respect to any point in $B_\varepsilon(0)$ for some $\varepsilon>0$. Here $B_\varepsilon(0)$ denotes an open ball in $\mathbb{R}^n$ centered at the origin with radius ε.

REMARK. (i) It is not known whether or not the hypotheses of convexity of Ω and concavity of ψ always imply star-shaped-ness of the coincidence set. (ii) It was shown in [5] that star-shaped-ness of the free boundary with respect to any point in some ball implies its Lipschitz character (see [5, Lemma 4.1, p. 1023]).

It follows from Theorem 1 that the coincidence set $I(\psi)$ satisfies an interior cone condition (that is, each $x \in \partial I(\psi)$ is the vertex of a cone $V(x) \subset I(\psi)$), when $f(x)$ is a positive definite quadratic form. Therefore, applying the results of L. A. Caffarelli [2] and D. Kinderlehrer - L. Nirenberg [7] to this, we obtain

COROLLARY 2. *Under the hypotheses of* Theorem 1, *if $f(x)$ is a positive definite quadratic form, then $\partial I(\psi)$ is a regular analytic hypersurface in $\mathbb{R}^n$.*

In section 2, applying an idea of L. A. Caffarelli - J. Spruck [4] (see [4, Lemma 2.2, p. 1341]) to the obstacle problem (1.1), we prove Theorem 1.

2. – Proof of Theorem 1.

First of all, we introduce the function $v \in C^0(\bar{\Omega})$

$$v(x) = x \cdot \nabla(u - \psi)(x) - s(u - \psi)(x) , \tag{2.1}$$

where s is the homogeneous degree of $f(x)$. Since $f(x)$ is homogeneous of degree $s > 1$, we have

$$x \cdot \nabla f(x) = sf(x) \quad \text{in } \Omega . \tag{2.2}$$

Then we obtain from (1.4)

$$v(x) = x \cdot \nabla u(x) - su(x) + sc . \tag{2.3}$$

Hence it follows from (1.3) that

$$\Delta v = 0 \quad \text{in } \Omega - I(\psi) . \tag{2.4}$$

On the other hand, since the C^1 function $u - \psi$ attains its minimum at any point in $I(\psi)$, we have

$$u - \psi = 0 \quad \text{and } \nabla(u - \psi) = 0 \quad \text{in } I(\psi) .$$

Then we obtain from (2.1)

$$v = 0 \quad \text{in } I(\psi) . \tag{2.5}$$

LEMMA 2.1. *v is positive on $\partial\Omega$.*

PROOF. Since $u = 0$ on $\partial\Omega$, we have from (2.3)

$$v(x) = x \cdot \nabla u(x) + sc \quad \text{on } \partial\Omega . \tag{2.6}$$

Here, we recall an idea of H. Lewy - G. Stampacchia [10] used in order to obtain some topological properties of the coincidence set in the two-dimensional obstacle problem (see [10, p. 179] or [8, p. 176]). Let I_1 be the set of points $y \in \Omega$ for which the tangent plane of the graph $(\cdot, \psi(\cdot))$ at $(y, \psi(y))$

$$\Pi_y;\ x_{n+1} = W_y(x) = \nabla\psi(y)\cdot(x-y) + \psi(y) \tag{2.7}$$

does not meet $\Omega \times \{0\}$. Since $\max_{\Omega} \psi(x) = \psi(0) = c > 0$, we see that $0 \in I_1$. Furthermore, since $\psi \in C^1(\bar{\Omega})$, so I_1 contains a neighborhood of 0. By the same argument as in [8] (see [8, p. 176]), we have

$$I_1 \subset I(\psi)\,. \tag{2.8}$$

Now, fix any point $x^0 \in \partial\Omega$. Since Ω is convex, we can find a plane through the tangent to $\partial\Omega$ at x^0 which is tangent to the graph $(\cdot, \psi(\cdot))$ at some point $(z, \psi(z)) \in \Omega \times \mathbb{R}$. Then $z \in I_1$. Therefore we have

$$\psi(z) > 0\,, \tag{2.9}$$

and

$$W_z(x^0) = \nabla\psi(z)\cdot(x^0 - z) + \psi(z) = 0\,. \tag{2.10}$$

On the other hand we see that

$$W_z \geqq 0 = u \quad \text{on } \partial\Omega\,, \qquad W_z \geqq \psi = u \quad \text{in } I(\psi)\,,$$

and

$$\Delta W_z = \Delta u = 0 \quad \text{in } \Omega - I(\psi)\,.$$

Hence by the maximum principle we obtain

$$W_z \geqq u \quad \text{in } \Omega - I(\psi)\,.$$

Also, since $W_z(x^0) = u(x^0) = 0$ and x^0 is regarded as an outward directed vector from Ω at $x^0 \in \partial\Omega$, we have

$$x^0\cdot\nabla u(x^0) \geqq x^0\cdot\nabla W_z(x^0)\,. \tag{2.11}$$

Here, it follows from (2.7) that $\nabla W_z(x^0) = \nabla\psi(z)$. Then we have from (2.10)

$$x^0\cdot\nabla u(x^0) \geqq x^0\cdot\nabla\psi(z) = -\psi(z) + z\cdot\nabla\psi(z)\,.$$

Hence from (2.2) and (1.4) we obtain

$$x^0\cdot\nabla u(x^0) + sc \geqq (s-1)\psi(z)\,.$$

Therefore, since $s > 1$, it follows from (2.6) and (2.9) that

$$v(x^0) > 0 .$$

This completes the proof of Lemma 2.1.

In view of (2.4), (2.5), and Lemma 2.1, we obtain by the maximum principle

$$v(x) \geqq 0 \quad \text{in } \Omega - I(\psi) . \tag{2.12}$$

Therefore, since $u - \psi > 0$ in $\Omega - I(\psi)$, we have

$$x \cdot \nabla(u - \psi)(x) > 0 \quad \text{in } \Omega - I(\psi) . \tag{2.13}$$

This shows that $I(\psi)$ is star shaped with respect to the origin. Indeed, if this is false, there exist a unit vector $\xi \in \mathbf{R}^n$ and positive numbers t_i $(i = 1, 2)$ with $t_1 < t_2$ such that $t_i \xi \in I(\psi)$ $(i = 1, 2)$ and $t\xi \in \Omega - I(\psi)$ for all $t \in (t_1, t_2)$. Then, since $(u - \psi)(t_i \xi) = 0$ $(i = 1, 2)$, it follows from the mean value theorem that $\xi \cdot \nabla(u - \psi)(\tau\xi) = 0$ for some $\tau \in (t_1, t_2)$. This contradicts (2.13) and $\tau\xi \in \Omega - I(\psi)$.

Now, we consider the case in which $f(x)$ is a positive definite quadratic form. Since $0 \in \operatorname{Int} I(\psi)$, there exists a number $\varepsilon > 0$ with $B_\varepsilon(0) \subset I(\psi)$. Fix any point $x^\sim \in B_\varepsilon(0)$ and put

$$v^\sim(x) = (x - x^\sim) \cdot \nabla(u - \psi)(x) - 2(u - \psi)(x) . \tag{2.14}$$

Since $\Delta\psi(x)$ is a constant, we see that

$$\Delta v^\sim(x) = 0 \quad \text{in } \Omega - I(\psi) . \tag{2.15}$$

Of course, $v^\sim(x) = 0$ in $I(\psi)$. Since $u - \psi \in C^1(\bar{\Omega})$, if we choose $\varepsilon > 0$ sufficiently small, we obtain from Lemma 2.1

$$v^\sim(x) > 0 \quad \text{on } \partial\Omega$$

for any $x^\sim \in B_\varepsilon(0)$. Therefore, by the maximum principle we have

$$v^\sim(x) \geqq 0 \quad \text{in } \Omega - I(\psi) .$$

Proceeding as in the case of $x^\sim = 0$, we see that $I(\psi)$ is star shaped with respect to $x^\sim \in B_\varepsilon(0)$. This completes the proof of Theorem 1.

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