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On functions, whose lines of steepest descent bend proportionally to level lines

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On Functions, Whose Lines of Steepest Descent Bend Proportionally to Level Lines.

GIORGIO TALENTI

1. – Introduction.

Suppose u is a smooth real-valued function of two real variables x and y; consider:

$$k = -|Du|^{-3}[u_y^2 u_{xx} - 2u_y u_x u_{xy} + u_x^2 u_{yy}], \tag{1.1}$$

$$h = |Du|^{-3}[(u_{xx} - u_{yy})u_x u_y - u_{xy}(u_x^2 - u_y^2)], \tag{1.2}$$

$$\varphi = k + ih. \tag{1.3}$$

Here subscripts stand for differentiation, D stands for gradient, $|Du| = (u_x^2 + u_y^2)^{\frac{1}{2}}$, $i = \sqrt{-1}$. In other words,

$$k = \text{curvature of the level lines of } u.$$

More precisely, the value of $|k|$ at any point (x, y), where the gradient of u does not vanish, is the curvature at (x, y) of $u^{-1}(u(x, y))$, the level line of u passing through (x, y); the sign of k is that which makes kDu, a normal vector field to the level lines of u, orientate towards the center of curvature. Analogously,

$$h = \text{curvature of the lines of steepest descent of } u,$$

where lines of steepest descent = trajectories of the gradient = orthogonal trajectories of the level lines.

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In section 2 we prove

THEOREM 1.

$$2\left|\frac{\partial\varphi}{\partial\bar z}\right| \geqslant |\varphi|^2 \tag{1.4}$$

at any point where Du is not zero.

$$\frac{\partial}{\partial z}\,\frac{|\varphi|^2 \pm i\sqrt{4|\partial\varphi/\partial\bar z|^2 - |\varphi|^4}}{\partial\bar\varphi/\partial z} + \varphi = 0 \tag{1.5}$$

at any point where Du & $\partial\bar\varphi/\partial z$ are not zero.

As usual, we have set

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right). \tag{1.6}$$

One aim of the present paper is to investigate the geometry of solutions to special partial differential equations via theorem 1. Theorem 1 tells us that k and h are automatically constrained by a system of two partial differential equations of the second order, whatever u is (provided u is smooth enough and has no critical points in the region of interest). On the other hand, partial differential equations, of a suitably restricted type, should constrain further, in a readable way, the curvatures of the level lines and/or the lines of steepest descent of solutions. As we show later, this is true for first order partial differential equations for instance, or for Laplace equation. Thus we believe that coupling theorem 1 with suitable partial differential equations may lead to significant conclusions about solutions to those equations.

Although some comments are made in section 3 about harmonic functions, we implement the above ideas in the case of the following partial differential equation:

$$u_y(u_x + \lambda u_y)u_{xx} - (2\lambda u_x u_y + u_x^2 - u_y^2)u_{xy} + u_x(\lambda u_x - u_y)u_{yy} = 0\,, \tag{1.7}$$

which is a very convenient test for our purposes.

Here λ is a constant. Equation (1.7) reads

$$h = \lambda k\,,$$

so that it characterizes those functions whose lines of steepest descent bend proportionally to level lines; in case $\lambda = 0$, equation (1.7) characterizes functions whose lines of steepest discent are straight. (1.7) is a quasi-linear equation of the second order with quadratic nonlinearities; note that the equation is *hyperbolic*, since

$$\begin{vmatrix} a & b \\ b & c \end{vmatrix} = -\frac{1}{4}\,|Du|^4$$

if a, b, c denote the coefficients of u_{xx}, u_{xy}, u_{yy}. (1.7) belongs to the following class of equations:

$$(1.8)\qquad \begin{cases} a(u_x, u_y)u_{xx} + 2b(u_x, u_y)u_{xy} + c(u_x, u_y)u_{yy} = 0 \\ a(p, q)\ \&\ c(p, q) \qquad \text{homogeneous of the same degree}, \\ 2b(p, q) = -\,a(p, q)\dfrac{p}{q} - c(p, q)\dfrac{q}{p}\,. \end{cases}$$

As is easy to see, any equation of the form (1.8) enjoys the property

$$v \text{ solution} \Rightarrow u = A(v) \text{ solution},$$

whatever A is. In section 5 we sketch some considerations about equations (1.8). Incidentally, (1.7) is the Euler equation of the following integral:

$$\iint\left\{\exp\left[\lambda \arctan\frac{2u_x u_y}{u_x^2 - u_y^2}\right] - 1\right\}\frac{dx\,dy}{\lambda}$$

(in the case $\lambda = 0$, the integrand is simply arctan).

In section 4 we prove

THEOREM 2. *Let u be a solution to equation* (1.7). *Suppose u has no critical points*; *suppose further u is smooth near a bounded closed set E. Then the largest r such that u is twice continuously differentiable in*

$$E + \{(x, y): x^2 + y^2 \leqslant r^2\}$$

cannot exceed

$$[\sqrt{1+\lambda^2}\max\{|k(x, y)|: (x, y) \in E\}]^{-1}.$$

Theorem 2 tells us that solutions to equation (1.7), which are not plane waves, either have critical points or develop singularities. Our proof is based on a priori estimates of integral type, see subsections 43 and 44.

Theorem 2 is sharp. In fact, we can show solutions, whose singularities are exactly as predicted by theorem 2. A convenient example is the distance from a point set. Specifically, let E be a smooth arc without inflection points and let

$$x = a(s) \qquad y = b(s) \qquad s = \text{arclength}$$

be a parametric representation of E; the formula

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a(v) \\ b(v) \end{pmatrix} + u \begin{pmatrix} -b'(v) \\ a'(v) \end{pmatrix}$$

defines a system of smooth curvilinear coordinates u, v in a neighbourhood of E. Clearly, the value of $|u|$ at any point (x, y) sufficiently close to E is the distance of (x, y) from E. We have for the functional *matrix* of the derivatives

$$\begin{pmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{pmatrix} = \frac{\partial(u, v)}{\partial(x, y)} = \left[\frac{\partial(x, y)}{\partial(u, v)}\right]^{-1}$$
$$= \frac{-1}{1 - K(v)u} \begin{pmatrix} b'(v) + a''(v)u & -a'(v) + b''(v)u \\ -a'(v) & -b'(v) \end{pmatrix},$$

where $K = a'b'' - a''b'$ is the curvature of E. Hence

$$u_x v_x + u_y v_y = 0$$

$$[x - a(v)]v_x + [y - b(v)]v_y = 0;$$

furthermore u is a solution to

$$u_x^2 + u_y^2 = 1\,, \tag{1.9}$$

the equation of geometrical optics. The lines of steepest descent of a twice continuously differentiable solution to equation (1.9) must be straight (see e.g. [1], or section 6). Thus u is a solution to the following equation:

$$h = 0\,,$$

a special case of equation (1.7). The above formula for the jacobian of u, v and the following formulas:

$$\frac{\partial(x, y)}{\partial(u, v)} = K(v)u - 1\,, \qquad x_v^2 + y_v^2 = \big(K(v)u - 1\big)^2$$

tell us that u and the level lines of u cannot be smooth beyond the curve

$$u = 1/K(v)\,,$$

the evolute of E. The same conclusion is drawn from theorem 2.

2. – Proof of theorem 1.

From (1.1) and (1.2) we get

$$k = -\operatorname{div}\frac{Du}{|Du|}\,, \tag{2.1}$$

$$h = -\operatorname{div}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\frac{Du}{|Du|}\,. \tag{2.2}$$

By the way, we have also

$$h = \frac{\partial(u, 1/|Du|)}{\partial(x, y)}\,.$$

Formulas (2.1) and (2.2) can be rewritten this way

$$k = -2\,\mathrm{Re}\frac{\partial}{\partial z}\frac{u_x + iu_y}{|Du|}\,, \qquad h = -2\,\mathrm{Im}\frac{\partial}{\partial z}\frac{u_x + iu_y}{|Du|}\,,$$

if notations (1.6) are used. Thus (1.3) yields

$$\varphi = -2\frac{\partial}{\partial z}\frac{u_x + iu_y}{|Du|}\,. \tag{2.3}$$

Let us denote by ω the angle between Du and a fixed direction; specifically

$$\cos\omega = \frac{u_x}{|Du|}\,, \qquad \sin\omega = \frac{u_y}{|Du|}\,. \tag{2.4}$$

From (2.3) and (2.4) we obtain

$$\varphi = -2\frac{\partial}{\partial z}e^{i\omega}\,, \tag{2.5}$$

hence

$$\varphi = -\,2i\,e^{i\omega}\frac{\partial\omega}{\partial z}\,. \tag{2.6}$$

Formula (2.6) has an obvious geometrical meaning. In fact, from (2.6) we infer

$$-\,k = \begin{vmatrix} u_x/|Du| & u_y/|Du| \\ \partial/\partial x & \partial/\partial y \end{vmatrix}\,\omega\,, \tag{2.6i}$$

the derivative of ω (with respect to the arclength) along the level lines of u. Similarly, (2.6) implies

$$\left(\frac{Du}{|Du|}\times D\right)\omega = -\,h\,. \tag{2.6ii}$$

Since ω is real-valued, (2.6) gives

$$\frac{\partial\omega}{\partial z} = \frac{i}{2}\,e^{-i\omega}\varphi\,, \qquad \frac{\partial\omega}{\partial\bar z} = -\,\frac{i}{2}\,e^{i\omega}\bar\varphi\,,$$

equivalently

$$d\omega = \frac{i}{2}\left(e^{-i\omega}\varphi\,dz - e^{i\omega}\bar\varphi\,d\bar z\right). \tag{2.7}$$

Using (2.7) twice gives

$$2i\,d(d\omega) = \left(e^{i\omega}\frac{\partial\bar\varphi}{\partial z} + e^{-i\omega}\frac{\partial\varphi}{\partial\bar z} - |\varphi|^2\right)dz\wedge d\bar z\,.$$

Obviously

$$d(d\omega) = 0\,.$$

Then

$$e^{i\omega}\frac{\partial\bar\varphi}{\partial z} + e^{-i\omega}\frac{\partial\varphi}{\partial\bar z} - |\varphi|^2 = 0\,. \tag{2.8}$$

Equation (2.8) implies (1.4) trivially. Equation (2.8) is quadratic in $e^{i\omega}$. Thus (2.8) gives

$$e^{i\omega} = \frac{|\varphi|^2 \pm i\sqrt{4\,(\partial\bar\varphi/\partial z)\,(\partial\varphi/\partial\bar z) - |\varphi|^2}}{2\,(\partial\bar\varphi/\partial z)}\,,$$

provided $\partial\bar\varphi/\partial z \neq 0$. The last equation and (2.5) give (1.5).

3. – Harmonic functions.

THEOREM 3. *Suppose u is harmonic and has no critical points. Then:*

(i) $\dfrac{k}{|Du|}, \dfrac{h}{|Du|}$ *are conjugate harmonic functions*;

(ii) $\ln \dfrac{1}{|k|}$ & $\ln \dfrac{1}{|h|}$ *are subharmonic*;

(iii) $4\left|\dfrac{\partial \varphi}{\partial \bar{z}}\right|^2 - |\varphi|^4 = 0$;

(iv) $\mathfrak{A}\varphi \equiv \varphi \Delta \varphi - \varphi_x^2 - \varphi_y^2 = 0$;

(v) $$\begin{cases} (k^2 + h^2)\Delta k = k(|Dk|^2 - |Dh|^2) + 2hDk\cdot Dh \\ (k^2 + h^2)\Delta h = h(|Dh|^2 - |Dk|^2) + 2kDk\cdot Dh , \end{cases}$$

where $Dk\cdot Dh = k_x h_x + k_y h_y$;

(vi) $\mathfrak{A}^2\, k = \mathfrak{A}^2\, h = 0$.

REMARKS. Property (ii) implies a *minimum principle* for the curvature k of level lines of a harmonic function u: the minimum of k, in any region where Du has no zeros, is attained on the boundary. A somewhat related statement is in [3]. Statement (vi) tells us that the curvature of level lines of a harmonic function obeys a partial differential equation of the fourth order. Analogous remarks hold for h, of course. Recall that the lines of steepest descent of a harmonic function are level lines of the harmonic conjugate function.

PROOF. Let us introduce the complex gradient

$$f = u_x - iu_y$$

of u. By hypothesis, f has no zeros in the region of interest. From formula (2.3) we have $\varphi = -2(\partial/\partial z)(\bar{f}/|f|)$. Since u is harmonic, f is a holomorphic function of the complex variable $z = x + iy$. Hence

$$\varphi = |f|^{-3}(\bar{f})^2 f' , \tag{3.1}$$

where $'$ denotes differentiation with respect to z.

Formula (3.1) yields

$$\frac{\varphi}{|Du|} = \left(-\frac{1}{f}\right)' , \tag{3.2}$$

a holomorphic function. Assertion (i) follows. Assertion (ii) follows from (i), since $\ln 1/|k| = \ln |Du| - \ln |k|/|Du|$, $\ln |Du|$ is harmonic and the logarithm of a harmonic function is superharmonic. From (3.1) we get $\partial\varphi/\partial\bar{z} = |f'|^2/2f|f|$, hence (iii) follows. Assertion (iv) is an easy consequence of (iii) and theorem 1 (in agreement with theorem 1, one might check that any function φ, having the form (3.1), satisfies (iv) whenever f is holomorphic). Note that

$$\frac{1}{4}\mathfrak{A}\varphi = \varphi\frac{\partial^2\varphi}{\partial z\partial\bar{z}} - \frac{\partial\varphi}{\partial z}\frac{\partial\varphi}{\partial\bar{z}}.$$

Multiplying both sides of $\varphi\Delta\varphi = \varphi_x^2 + \varphi_y^2$ by $\bar{\varphi}$, then splitting into real and imaginary parts, give (v).

From (v) we have

$$(k^2 + h^2)(k\Delta k - k_x^2 - k_y^2) = -|hDk - kDh|^2.$$

On the other hand

$$|hDk - kDh| = 2\left|h\frac{\partial k}{\partial\bar{z}} - k\frac{\partial h}{\partial\bar{z}}\right| = 2\left|h\frac{\partial}{\partial\bar{z}}\frac{k}{|Du|} - k\frac{\partial}{\partial\bar{z}}\frac{h}{|Du|}\right| = |Du|\left|\frac{\partial}{\partial\bar{z}}\frac{|\varphi|^2}{|Du|^2}\right|,$$

since (i) gives $(\partial/\partial\bar{z})(k/|Du|) = -i(\partial/\partial\bar{z})(h/|Du)|$. The last quantity equals

$$|\varphi|\left|\frac{\partial}{\partial z}\frac{\varphi}{|Du|}\right| = |\varphi|\left|\left(\frac{1}{f}\right)''\right|,$$

since $\varphi/|Du|$ is a holomorphic function and (3.2) holds. We conclude that

$$\mathfrak{A}k \equiv k\Delta k - k_x^2 - k_y^2 = -\left|\left(\frac{1}{f}\right)''\right|^2, \tag{3.3}$$

for the zeros of φ are isolated.

Formula (3.3) implies (vi), for $\mathfrak{A}|g|^2 = 0$ whenever g is a holomorphic function.

4. – Proof of theorem 2.

41. PRELIMINARIES. Equation (1.7) yields

$$h = \lambda k, \tag{4.1}$$

as long as Du is different from zero. We assume throughout this section that u is a smooth solution to equation (1.7) and u has no critical point. Equation (4.1) and theorem 1 give at once results (i) (ii) (iii) below.

(i) The following inequality holds:

$$|Dk| \geqslant \sqrt{1+\lambda^2}\, k^2 . \tag{4.2}$$

(ii) Either

$$\frac{\partial}{\partial x}\frac{\sqrt{1+\lambda^2}\,k^2 k_y + k_x\sqrt{|Dk|^2-(1+\lambda^2)k^4}}{|Dk|^2} + \frac{\partial}{\partial y}\frac{-\sqrt{1+\lambda^2}\,k^2 k_x + k_y\sqrt{|Dk|^2-(1+\lambda^2)k^4}}{|Dk|^2} = 0 \tag{4.3}$$

or

$$\frac{\partial}{\partial x}\frac{\sqrt{1+\lambda^2}\,k^2 k_y - k_x\sqrt{|Dk|^2-(1+\lambda^2)k^4}}{|Dk|^2} + \frac{\partial}{\partial y}\frac{-\sqrt{1+\lambda^2}\,k^2 k_x - k_y\sqrt{|Dk|^2-(1+\lambda^2)k^4}}{|Dk|^2} = 0$$

at any point where $Dk \neq 0$.

(iii) The following equation:

$$\frac{\partial}{\partial x}\frac{\sqrt{1+\lambda^2}\,k^2 k_x \mp k_y\sqrt{|Dk|^2-(1+\lambda^2)k^4}}{|Dk|^2} + \frac{\partial}{\partial y}\frac{\sqrt{1+\lambda^2}\,k^2 k_y \pm k_x\sqrt{|Dk|^2-(1+\lambda^2)k^4}}{|Dk|^2} + \sqrt{1+\lambda^2}\,k = 0 \tag{4.4}$$

holds at any point where $Dk \neq 0$.

42. Remarks on equation (4.3). Equation (4.3) and inequality (4.2) yield

$$\frac{\partial}{\partial x}\frac{\partial j}{\partial p}(k^{-2}k_x,\, k^{-2}k_y) + \frac{\partial}{\partial y}\frac{\partial j}{\partial q}(k^{-2}k_x,\, k^{-2}k_y) = 0 \tag{4.5}$$

at any point where $k \neq 0$. Equation (4.5) is the Euler equation of the following integral:

$$\iint j(k^{-2}k_x,\, k^{-2}k_y)\, dx\, dy$$

Here

$$\begin{cases} \dfrac{\partial j}{\partial p} = \dfrac{\sqrt{1+\lambda^2}\,q + p\sqrt{p^2+q^2-1-\lambda^2}}{p^2+q^2} \\[2ex] \dfrac{\partial j}{\partial q} = \dfrac{-\sqrt{1+\lambda^2}\,p + q\sqrt{p^2+q^2-1-\lambda^2}}{p^2+q^2}, \end{cases}$$

and

$$j(p,q) = \sqrt{p^2+q^2-1-\lambda^2} + \sqrt{1+\lambda^2}\arctan\frac{p\sqrt{p^2+q^2-1-\lambda^2}+q\sqrt{1+\lambda^2}}{q\sqrt{p^2+q^2-1-\lambda^2}-p\sqrt{1+\lambda^2}}, \tag{4.6}$$

is a solution to the equation of geometrical optics

$$\left(\frac{\partial j}{\partial p}\right)^2 + \left(\frac{\partial j}{\partial q}\right)^2 = 1$$

whose lines of steepest descent are tangent rays to the circle $p^2+q^2=1$. Clearly

$$\frac{\partial^2 j}{\partial p^2}\frac{\partial^2 j}{\partial q^2} - \left(\frac{\partial^2 j}{\partial p\,\partial q}\right)^2 = 0,$$

so that equation (4.5) should be qualified as parabolic.

43. Lemma. The following lemma is the key to a proof of theorem 2 and will be derived from equation (4.4).

Lemma. *Let p be any exponent >2 and let G be any bounded open set with a rectifiable boundary. Then*

$$\int_G |k|^p\,dx\,dy \leqslant \frac{(1+\lambda^2)^{-\frac12}}{p-2}\int_{\partial G} |k|^{p-1}\sqrt{dx^2+dy^2}\,. \tag{4.7}$$

44. Conclusive steps. Set

$$d(x,y;E) = \textit{distance of } (x,y) \textit{ from } E\,,$$

and

$$U_p(t) = \left[\int_{\{(x,y):\,d(x,y;E)\leqslant t\}} |k|^p\,dx\,dy\right]^{1/p} \tag{4.8}$$

for every sufficiently small $t>0$. As is well-known and easy to see,

$$|Dd(x,y;E)| = 1$$

at almost every point (x, y) out of E. Consequently, Federer's coarea formula (see [2], section 3.2) ensures that U_p is absolutely continuous and

$$U_p'(t) = \frac{1}{p} U_p(t)^{1-p} \int\limits_{\{(x,y):\, d(x,y;\,E)=t\}} |k|^p \sqrt{dx^2 + dy^2}$$

for almost every t.

Analogously, if we set

$$m(t) = \text{area of } \{(x, y) \colon d(x, y; E) \leqslant t\} ,$$

we have

$$m'(t) = \text{length of } \{(x, y) \colon d(x, y; E) = t\} .$$

Schwarz inequality yields

$$U_p'(t) \geqslant \frac{1}{p} m'(t)^{1-q} U_p(t)^{1-p} \Bigg[\int\limits_{\{(x,y):\, d(x,y;\,E)=t\}} |k|^{p-1} \sqrt{dx^2 + dy^2} \Bigg]^q ,$$

where $q = p/(p-1)$. The right-hand side of the last inequality can be estimated from below with the help of (4.7). Thus we obtain

$$U_p'(t) \geqslant \frac{1}{p} \left[(p-2)\sqrt{1+\lambda^2}\right]^q m'(t)^{1-q} U_p(t)^{1+q} , \tag{4.9}$$

a differential inequality for U_p.

From (4.9) we get

$$U_p(\varepsilon)^{-q} - U_p(t)^{-q} \geqslant (q-1)\left[(p-2)\sqrt{1+\lambda^2}\right]^q \int_\varepsilon^t m'(s)^{1-q} ds ,$$

if $0 < \varepsilon < t$. Since $U_p(t) \geqslant 0$ and

$$\int_\varepsilon^t m'(s)^{1-q} ds \geqslant (t-\varepsilon)^q \cdot [m(t) - m(\varepsilon)]^{1-q} ,$$

we see that

$$\sqrt{1+\lambda^2}\,(t-\varepsilon) \leqslant \frac{(p-1)^{1-1/p}}{p-2} \left[\text{area } \{(x, y) \colon \varepsilon \leqslant d(x, y; E) \leqslant t\}\right]^{1/p} \frac{1}{U_p(\varepsilon)} . \tag{4.10}$$

Letting $p \to +\infty$ and $\varepsilon \to 0$, we infer from (4.10) that if k is bounded in

$$\{(x, y) \colon d(x, y; E) \leqslant t\} ,$$

then

$$\sqrt{1+\lambda^2}\, t \leqslant [\text{maximum of } |k| \text{ in } E]^{-1}.$$

Theorem 2 is proved. Note incidentally that the solutions to equation (1.7) obey the following inequality:

$$\sqrt{1+2\lambda^2}\, |k| \leqslant |Du|^{-1}(u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2)^{\frac{1}{2}}.$$

45. PROOF OF THE LEMMA. Equation (4.4) reads

$$\frac{\partial}{\partial x}\{kA(k^{-2}k_x, k^{-2}k_y)\} + \frac{\partial}{\partial y}\{kB(k^{-2}k_x, k^{-2}k_y)\} = 0\,, \tag{4.11}$$

where $A(p, q)$ and $B(p, q)$ are solutions of the following system:

$$A(p, q)^2 + B(p, q)^2 = 1 \quad pA(p, q) + qB(p, q) = \sqrt{1+\lambda^2}\,. \tag{4.12}$$

Inequality (4.2) ensures that (4.11) holds at every point where $k \neq 0$. Consider

$$\{(x, y) \in G\colon |k(x, y)| > t\}\,, \tag{4.13}$$

a level set of k. Here $t > 0$. Inequality (4.2) ensures that (4.13) is free from critical points of k. Hence (4.13) must reach the boundary of G; on the other hand, that part of the boundary of (4.13), which is in the interior of G, is exactly $\{(x, y) \in G\colon |k(x, y)| = t\}$, a collection of smooth arcs.

Integrating both sides of equation (4.11) over level set (4.13) gives

$$\begin{aligned} &\int_{\{(x,y)\in G\colon |k(x,y)|=t\}} |k| \left\{A(k^{-2}k_x, k^{-2}k_y)\frac{k_x}{|Dk|} + B(k^{-2}k_x, k^{-2}k_y)\frac{k_y}{|Dk|}\right\}\sqrt{dx^2+dy^2} \\ &\qquad = \int_{\{(x,y)\in\partial G\colon |k(x,y)|\geqslant t\}} k\{A(\ldots)\nu_1 + B(\ldots)\nu_2\}\sqrt{dx^2+dy^2}\,, \end{aligned} \tag{4.14}$$

where (ν_1, ν_2) is the exterior normal to ∂G.

Thanks to (4.12), the integrand at the right-hand side of (4.14) $\leqslant |k|$, and the integrand at the left-hand side equals $\sqrt{1+\lambda^2}(|k|^3/|Dk|$. Hence (4.14) gives

$$\sqrt{1+\lambda^2}\, t^3[-\mu'(t)] \leqslant \int_{\{(x,y)\in\partial G\colon |k(x,y)|\leqslant t\}} |k|\sqrt{dx^2+dy^2}\,, \tag{4.15}$$

where

$$\mu(t) = \text{area of } \{(x, y) \in G \colon |k(x, y)| > t\}$$

is the distribution function of k. In fact, Federer's coarea formula ([2], section 3.2) yields

$$-\mu'(t) = \int\limits_{\{(x,y)\in G\colon |k(x,y)|=t\}} \frac{1}{|Dk|}\sqrt{dx^2+dy^2}\,.$$

From (4.15) we deduce:

$$\begin{aligned}\sqrt{1+\lambda^2}\int\limits_G |k|^p\,dx\,dy &= \sqrt{1+\lambda^2}\int\limits_0^\infty t^p[-\,d\mu(t)]\\ &\leqslant \int\limits_0^\infty t^{p-3}dt \int\limits_{\{(x,y)\in\partial G\colon |k(x,y)|\geqslant t\}} |k|\sqrt{dx^2+dy^2}\\ &= \int\limits_{\partial G}|k|\sqrt{dx^2+dy^2}\int\limits_0^{|k|} t^{p-3}\,dt = \frac{1}{p-2}\int\limits_{\partial G}|k|^{p-1}\sqrt{dx^2+dy^2}\,.\end{aligned}$$

The lemma is proved.

5. – Remarks on equations (1.8).

51. Consider first

$$u_x u_y(u_{xx} - u_{yy}) - (u_x^2 - u_y^2)u_{xy} = 0\,, \tag{5.1}$$

the $\lambda = 0$ case of equation (1.7). Let u be a solution without critical points. By (1.2) and (2.2) we have

$$\frac{\partial}{\partial x}\frac{u_y}{|Du|} - \frac{\partial}{\partial y}\frac{u_x}{|Du|} = 0\,,$$

so that

$$\frac{du}{|Du|} \equiv \frac{u_x}{|Du|}\,dx + \frac{u_y}{|Du|}\,dy$$

is an exact differential. Let v be defined by

$$v_x = \frac{u_x}{|Du|} \qquad v_y = \frac{u_y}{|Du|}\,.$$

Clearly

$$v_x^2 + v_y^2 = 1 \tag{5.2}$$

and

$$\frac{\partial(u, v)}{\partial(x, y)} = 0 .$$

Thus the following proposition is proved: if u satisfies equation (5.1) and has no critical point, then a solution v to the equation of geometrical optics (5.2) exists such that u and v are functions of each other (*i.e.* interdependent).

Recall that equation (5.1) characterizes functions whose lines of steepest descent are straight.

52. Consider

$$u_y^2 u_{xx} - 2u_y u_x u_{xy} + u_x^2 u_{yy} = 0 , \tag{5.3}$$

the $(1/\lambda) = 0$ case of equation (1.7), and

$$\frac{\partial}{\partial x}(\cos\omega) + \frac{\partial}{\partial y}(\sin\omega) = 0 , \tag{5.4}$$

a first-order partial differential equation having the form of a conservation law. According to formula (1.1), equation (5.3) characterizes functions u whose level lines are straight. Note that (5.4) becomes

$$v_y = v v_x ,$$

if $v = \tan\omega$.

The following proposition holds: if u satisfies (5.3) and has no critical point, then a solution ω to equation (5.4) exists such that u and ω are functions of each other.

PROOF. Let ω be defined by (2.4), i.e. the angle between Du and the x-axis. Since u satisfies (5.3), formulas (1.1) and (2.6i) tell us that

$$\frac{\partial(u, \omega)}{\partial(x, y)} = 0 .$$

On the other hand, we learn from (1.3) and (2.7) that $\omega_x = -h\cos\omega$, $\omega_y = -h\sin\omega$; thus

$$-\omega_x \sin\omega + \omega_y \cos\omega = 0 .$$

53. Consider now any

equation of the form (1.8) .

Since the $(p/q)\,a - (q/p)\,c \equiv 0$ case is covered in subsection 52, we assume here

$$a(p, q)\,\frac{p}{q} - c(p, q)\,\frac{q}{p} \not\equiv 0\,.$$

Let $f(p, q)$ be defined by

$$d(\ln f) = \frac{a}{q}\left(a\,\frac{p}{q} - c\,\frac{q}{p}\right)^{-1} dp + \frac{-c}{p}\left(a\,\frac{p}{q} - c\,\frac{q}{p}\right)^{-1} dq\,. \tag{5.5}$$

Note that the right-hand side of (5.5) is an exact differential (provided the underlying region does not wind round the origin). In fact

$$d\left[\frac{a}{q}\left(a\,\frac{p}{q} - c\,\frac{q}{p}\right)^{-1} dp - \frac{c}{p}\left(a\,\frac{p}{q} - c\,\frac{q}{p}\right)^{-1} dq\right]$$
$$= -\frac{1}{pq}\left(a\,\frac{p}{q} - c\,\frac{q}{p}\right)^{-2} \begin{vmatrix} a & c \\ pa_p + qa_q & pc_p + qc_q \end{vmatrix} dp \wedge dq = 0\,,$$

for a and b are homogeneous of the same degree.

In the case of equation (1.7), one has

$$f(p, q) = \varrho e^{\lambda\omega}\,,$$

where ϱ, ω are polar coordinates in the pq-plane ($p = \varrho \cos \omega$, $q = \varrho \sin \omega$).

The following propositions hold:

(i) Any (sufficiently smooth) solution u to the first-order partial differential equation $f(u_x, u_y) = 1$ is a solution to equation (1.8).

(ii) If u is a solution to equation (1.8), such that Du ranges over the domain of f and avoids the zeros of f, then $f(v_x, v_y) = 1$ has a solution v such that u and v are functions of each other.

PROOF. From (5.5) we get

$$\frac{qf_p}{a(p, q)} = \frac{-pf_q}{c(p, q)} \tag{5.5i}$$

and

$$pf_p + qf_q = f\,. \tag{5.5ii}$$

By virtue of (5.5i), equation (1.8) can be rewritten this way:

$$u_y \frac{\partial f}{\partial u_x}(u_x, u_y)u_{xx} - \left[u_x \frac{\partial f}{\partial u_x}(u_x, u_y) - u_y \frac{\partial f}{\partial u_y}(u_x, u_y)\right]u_{xy} - u_x \frac{\partial f}{\partial u_y}(u_x, u_y)u_{yy} = 0\,.$$

The left-hand side of the last equation is exactly the negative of the jacobian of u and $f(u_x, u_y)$. Thus equation (1.8) reads

$$\frac{\partial(u, f(u_x, u_y))}{\partial(x, y)} = 0\,. \tag{5.6}$$

Proposition (i) follows.

Let u be as in (ii). As

$$d\left[\frac{du}{f(u_x, u_y)}\right] = f(u_x, u_y)^{-2}\frac{\partial(u, f(u_x, u_y))}{\partial(x, y)}\, dx \wedge dy$$

and (5.6) holds,

$$\frac{du}{f(u_x, u_y)}$$

is an exact differential. Let v be defined by

$$Dv = \frac{Du}{f(Du)}\,.$$

Obviously, the jacobian of u and v vanishes. On the other hand, equation (5.5ii) tells us that f is homogeneous of degree 1. Thus $f(v_x, v_y) = 1$.

Proposition (ii) is proved.

6. – Further comments on theorem 3.

If u is a twice continuously differentiable solution to a first-order partial differential equation of the form

$$f(u, u_x, u_y) = \text{Constant}\,, \tag{6.1}$$

the lines of steepest descent and the level lines of u are constrained by

$$\left[u_x \frac{\partial f}{\partial u_x}(u, u_x, u_y) + u_y \frac{\partial f}{\partial u_y}(u, u_x, u_y)\right] h - \left[u_y \frac{\partial f}{\partial u_x}(u, u_x, u_y) - u_x \frac{\partial f}{\partial u_y}(u, u_x, u_y)\right] k = 0 . \tag{6.2}$$

In fact, the left-hand side of (6.2) is the negative of

$$\begin{vmatrix} u_x/|Du| & u_y/|Du| \\ \partial/\partial x & \partial/\partial y \end{vmatrix} f(u, u_x, u_y) ,$$

the derivative of $f(u, u_x, u_y)$ along the level lines of u.

More precisely, differentiations of the left-hand side of (6.1), and formulas (1.1) and (1.2), give

$$\begin{cases} f_p u_{xx} + f_q u_{xy} & = -f_u p \\ \qquad f_p u_{xy} + f_q u_{yy} & = -f_u q \\ q^2 u_{xx} - 2qp u_{xy} + p^2 u_{yy} & = -k(p^2 + q^2)^{\frac{3}{2}} \\ pq u_{xx} - (p^2 - q^2) u_{xy} - pq u_{yy} & = h(p^2 + q^2)^{\frac{3}{2}} , \end{cases}$$

a linear system in the second-order derivatives of u. Here f_p, f_q, f_u stand for $(\partial f/\partial u_x)(u, u_x, u_y)$, etc. Solving with respect to u_{xx} and u_{yy}, then forming Δu, yields

$$\begin{aligned} &k : \left[u_x \frac{\partial f}{\partial u_x}(u, u_x, u_y) + u_y \frac{\partial f}{\partial u_y}(u, u_x, u_y)\right] \\ &= h : \left[u_y \frac{\partial f}{\partial u_x}(u, u_x, u_y) - u_x \frac{\partial f}{\partial u_y}(u, u_x, u_y)\right] \\ &= -|Du|^{-3} \frac{\left[u_x \dfrac{\partial f}{\partial u_x}(u, u_x, u_y) + u_y \dfrac{\partial f}{\partial u_y}(u, u_x, u_y)\right] \Delta u + \dfrac{\partial f}{\partial u}(u, u_x, u_y)|Du|^2}{\left[\dfrac{\partial f}{\partial u_x}(u, u_x, u_y)\right]^2 + \left[\dfrac{\partial f}{\partial u_y}(u, u_x, u_y)\right]^2} . \end{aligned} \tag{6.3}$$

Consider for instance the equation of geometrical optics

$$u_x^2 + u_y^2 = 1 . \tag{6.4}$$

In this case, formulas (6.2) and (6.3) give

$$h = 0 \qquad k = -\Delta u .$$

Furthermore $(\Delta u)^2 = u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2$, for the hessian $u_{xx}u_{yy} - u_{xy}^2$ vanishes (recall that the graph of a twice continuously differentiable solution to (6.1) is a developable surface, if $\partial f/\partial u \equiv 0$). Thus the following proposition is comprised in theorem 2. *Suppose u is a solution to* (6.4) *and u is smooth near a bounded closed set E. Then u cannot be twice continuously differentiable in the whole of $E + \{(x, y): x^2 + y^2 \leqslant r^2\}$ unless*

$$r \times \text{maximum of } (u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2)^{\frac{1}{2}} \text{ in } E \leqslant 1 .$$

Similar remarks can be made for the conservation law

$$a(u)u_x + b(u)u_y = 0 . \tag{6.5}$$

For (6.2) and (6.3) tells us that the twice continuously differentiable solutions to (6.5) obey

$$k = 0 , \quad h = \frac{-|Du|}{a(u)^2 + b(u)^2} \begin{vmatrix} a(u) & b(u) \\ a'(u) & b'(u) \end{vmatrix} .$$

On the other hand, theorem 2 has the following corollary. *Let u satisfy*

$$u_y^2 u_{xx} - 2u_y u_x u_{xy} + u_x^2 u_{yy} = 0 \tag{6.6}$$

and let u be smooth near a bounded closed set E. If u has no critical point, then u cannot be smooth in $E + \{(x, y): x^2 + y^2 \leqslant r^2\}$ unless

$$r \times \text{maximum of } |h| \text{ in } E \leqslant 1 .$$

The above statements about equations (6.4), (6.5) and (6.6) can be alternatively derived via a straightforward geometric argument (we are indebted to professor J. C. C. Nitsche for remarks on this matter). They are quoted here both for testing the sharpness of theorem 2 and for showing how our method might work for first-order partial differential equations.

For instance, let u be a solution to equation (6.4) and suppose that u is twice continuously differentiable in the disk $x^2 + y^2 < r^2$. An elementary argument shows that the perpendiculars to a smooth arc, whose foots have an infinitesimal distance from each other, meet at points of the evolute. On the other hand, a perpendicular to a level line of our solution u is a line of steepest descent of u. Thus the evolute of $u^{-1}(u(0, 0))$, the level line of u passing through the origin, must lie out of the disk $x^2 + y^2 < r^2$. In particular we must have $|k(0, 0)^{-1}| \leqslant 1$, that is

$$u_{xx}^2(0, 0) + 2u_{xy}^2(0, 0) + u_{yy}^2(0, 0) \leqslant r^{-2} .$$