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The \oplus_c -Topology is Not Completable.

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1. - Introduction.

G. D'Este [5] introduced and studied an interesting and difficult functorial topology defined on the category of abelian groups: Let \oplus_c be the class of all direct sums of cyclic p -groups. For each group A let $\mathcal{U}_A = \{U \leq A : A/U \in \oplus_c\}$. Then \mathcal{U}_A is a neighborhood basis at 0, a « local basis » for short, for some topology on A which makes A into a topological group. We write $A[\mathcal{U}_A] = A[\oplus_c]$ for this topological group. Every homomorphism $f: A \rightarrow B$ is then a continuous map $f: A[\oplus_c] \rightarrow B[\oplus_c]$. In the terminology of Boyer-Mader [2], \oplus_c is a discrete class and $A \rightarrow A[\oplus_c]$, $f \rightarrow f$ is the corresponding minimal functorial topology. This minimal functorial topology as well as the associated topology on an individual group is called the \oplus_c -topology. Every group $A[\oplus_c]$ has a (Hausdorff) completion \check{A} and if the completion topology of \check{A} is the \oplus_c -topology then A is called *completable*; if every A is completable then the \oplus_c -topology is *completable*. A crucial result in [5], Theorem 1.4, states that the \oplus_c -topology is indeed completable. In this note we disprove this claim. This is achieved by noting that separable $p^{\omega+1}$ -projective p -groups are either \oplus_c -complete or not completable. We then construct such groups which are \oplus_c -incomplete as well as some which are \oplus_c -complete. Unfortunately, the error invalidates most of D'Este's results, and as it stands very little is known about the \oplus_c -topology.

In Section 2 we summarize what is known about the \oplus_c -topology. Section 3 contains our examples.

All groups in this paper are abelian. The notation is standard and follows Fuchs [6]. The background on linear functorial topologies can be found in Mader [9]. Unless indicated otherwise a topological group carries the \oplus_c -topology. \check{A} denotes the \oplus_c -completion of A , and \hat{A} the p -adic

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completion. The explicit construction of the completion of a group with linear topology can be found in [6; Vol. I, pp. 68/69] as well as the definition of the appropriate topology which is called the completion topology. Suppose T is a functorial topology so that, for every abelian group A , we obtain the topological group TA with A as the underlying group. Every subgroup of A then has two topologies: its own functorial topology and the topology induced by the topology of TA . The subgroup is called T -concordant if these two topologies coincide. Maps are written on the right.

I owe thanks to Ray Mines with whom I began studying the paper of G. D'Este and who first noted the likely errors.

3. - Properties of the \oplus_c -topology.

Most of the results in this section are due to D'Este [5]. We indicate how the results follow from the general considerations of Mader [9].

The first observation follows from the fact that the class \oplus_c is closed under arbitrary direct sums ([9; 3.21 and 4.1c]).

(2.1) $(\oplus_i A_i)^\vee = \oplus_i \check{A}_i$. In particular, ([5; 2.1]) any direct sum of \oplus_c -complete groups is \oplus_c -complete. \square

The following fact is true for any minimal functorial topology and follows from [9; 3.21 and 4.1c].

(2.2) ([5; Lemma 1.3]). A direct summand of a \oplus_c -complete group is \oplus_c -complete. \square

The next result essentially follows from the fact that an extension of a direct sum of cyclic groups by a bounded group is a direct sum of cyclic groups ([6; 18.3]).

(2.3) ([5; Lemma 2.3]). If B is a subgroup of A such that A/B is a bounded p -group then B is a \oplus_c -concordant open and hence closed subgroup of A . Thus A is \oplus_c -complete if and only if B is \oplus_c -complete. \square

It is helpful to compare the \oplus_c -topology with better understood topologies. If A/U is a bounded p -group then $A/U \in \oplus_c$. Hence the p -adic topology is weaker than the \oplus_c -topology. On the other hand it is easy to see that each subgroup U with $A/U \in \oplus_c$ is closed in the p -adic topology. Hence [9; 4.11] applies:

(2.4) *The natural map $\check{A} \rightarrow \hat{A}$, where \hat{A} denotes the p -adic completion of A , is injective. \square*

If A is a p -group and U is a large subgroup of A then $A/U \in \oplus_c$ by [6; 67.4]. Hence the large subgroup or inductive topology is weaker than the \oplus_c -topology. It is well-known ([3; 3.9] or [4; 2.8]) that the completion of a p -group A in the large subgroup topology is its torsion-completion \bar{A} , i.e. the maximal torsion subgroup of \hat{A} . Also the p -adic topology is weaker than the large subgroup topology. Hence there are natural maps $\check{A} \rightarrow \bar{A} \rightarrow \hat{A}$. The following fact now follows from (2.4).

(2.5) ([5; Lemma 1.2]). *For a p -group A the \oplus_c -completion \check{A} is naturally imbedded in the torsion-completion \bar{A} . In particular, \check{A} is again p -primary. \square*

If \check{A} is purely imbedded in \bar{A} then \bar{A} is also the torsion completion of \check{A} and by (2.5) we have $(\check{A})^\check{}$ imbedded in \bar{A} . Thus, if \check{A} were always purely imbedded in \bar{A} then $(\check{A})^\check{}$ and all transfinitely iterated \oplus_c -completions would be contained in \bar{A} and hence the chain of iterated completions would have to become stationary. It will be shown below that for minimal functorial topologies in general, the chain of iterated completions of a group A becomes stationary if and only if it is constant, i.e. A is completable.

This is D'Este's idea. It fails because \check{A} need not be pure in \bar{A} , and the error is made in the middle of page 244 by equating two distinct imbeddings in \bar{A} .

(2.6) ITERATED COMPLETIONS. *Let T be a minimal functorial topology on the category of abelian groups. Let LA be the completion of TA as an abstract group and let $\varepsilon_A: A \rightarrow LA$ be the natural map. For simplicity assume that TA is Hausdorff. Define $L^0A = A$, $L^1A = LA$, $\varepsilon_{01}: L^0A \rightarrow L^1A: \varepsilon_{01} = \varepsilon_A$ and let $\varepsilon_{ii}: L^iA \rightarrow L^iA: \varepsilon_{ii} = 1$. Suppose $L^\alpha A$ and maps $\varepsilon_{\alpha\beta}: L^\alpha A \rightarrow L^\beta A$ have been defined for $\alpha \leq \beta < \lambda$ satisfying $\varepsilon_{\alpha\beta} \circ \varepsilon_{\beta\gamma} = \varepsilon_{\alpha\gamma}$ for $\alpha \leq \beta < \gamma < \lambda$. If $\lambda - 1$ exists let $L^\lambda A = L(L^{\lambda-1}A)$ and $\varepsilon_{\alpha\lambda} = \varepsilon_{\alpha\lambda-1} \circ \varepsilon_{L^{\lambda-1}A}$; if λ is a limit ordinal let $L^\lambda A = \varinjlim \{L^\alpha A: \alpha < \lambda\}$ and $\varepsilon_{\alpha\lambda} = \varinjlim \{\varepsilon_{\alpha\beta}: \alpha \leq \beta < \lambda\}$. In any case let $\varepsilon_{\lambda\lambda} = 1$. Then, clearly, each $\varepsilon_{\alpha\beta}$ is injective and $\varepsilon_{\alpha\beta} \circ \varepsilon_{\beta\gamma} = \varepsilon_{\alpha\gamma}$ for $\alpha \leq \beta < \gamma$. Furthermore, if some $\varepsilon_{\alpha\beta}$ with $\alpha < \beta$ is bijective then A is completable, and if so the whole chain of iterated completions is constant.*

PROOF. Suppose $\varepsilon_{\lambda\beta}$ is bijective for $\lambda < \beta$. Then $\varepsilon_{\lambda\lambda+1}: L^\lambda A \rightarrow L(L^\lambda A)$ is bijective, i.e. $TL^\lambda A$ is complete. We identify all $L^\alpha A$, $\alpha < \lambda$, with their images in $L^\lambda A$. By [9; 5.7] $L^2A = LA \oplus K_1$. Suppose that K_α , $\alpha < \mu < \lambda$, has been found such that $L^\alpha A = LA \oplus K_\alpha$ and $K_\alpha \leq K_\beta$ for $\alpha < \beta < \mu$. If

$\mu-1$ exists then $L^\mu A = L(L^{\mu-1}A) = L^2A \oplus LK_\alpha = LA \oplus (K_1 \oplus LK_\alpha)$ and we let $K_\mu = K_1 \oplus LK_\alpha$. If μ is a limit ordinal then $L^\mu A = \bigcup_{\alpha < \mu} L^\alpha A = LA \oplus \bigcup_{\alpha < \mu} K_\alpha$ and we let $K_\mu = \bigcup_{\alpha < \mu} K_\alpha$. Hence, by induction, $L^\lambda A = LA \oplus K_\lambda$ and TLA is complete as a direct summand of a complete group. \square

Megibben [10] called a p -group A *thick* if $A/U \in \oplus_e$ implies that U contains a large subgroup of A .

(2.7) ([5; 1.1]). *A p -group A is thick if and only if the \oplus_e -topology on A coincides with the large subgroup topology. The completion of a thick group A is its torsion-completion \bar{A} and every thick group is \oplus_e -completable.*

PROOF. It has been mentioned earlier that \bar{A} is the completion of A in the large subgroup topology. Since \bar{A}/A is divisible, it is \oplus_e -indiscrete and it follows from the completability criterion of Mines-Oxford (see [9; 5.10 (6)]) that A is completable. \square

The next result follows immediately from (2.7) and (2.1).

(2.8) ([5; 2.2]). *Direct sums of torsion-complete p -groups are \oplus_e -complete.* \square

A little more can be asserted.

(2.9) *If A is the direct sum of thick groups A_i , then $\check{A} = \bigoplus_i \bar{A}_i$, and A is completable although usually not thick.*

PROOF. $\check{A} = \bigoplus_i \check{A}_i$ and $\check{A}_i = \bar{A}_i$ since A_i is thick. Completeness follows since $\check{A}/A \cong \bigoplus_i \bar{A}_i/A_i$ is divisible. If $A = \bigoplus_i A_i$ and if A is thick then $\bar{A} = \bigoplus_i \bar{A}_i$. By [6; 71.3] there is m such that $p^m \bar{A}_i = 0$ for almost all i . Hence A is usually not thick. \square

(2.10) REMARK. *We just showed: If $A = \bigoplus_i A_i$ is thick then, for some positive integer m , $p^m A_i = 0$ for almost all i .* \square

There is also a large class of groups for which the \oplus_e -topology coincides with the p -adic topology. This is trivially the case for torsion-free groups of finite p -rank ($= \dim A/pA$).

(2.11) *The \oplus_e -completion of a direct sum of torsion-free groups of finite rank is the free p -adic module with the same p -rank. Such a group is \oplus_e -completable.* \square

More interesting examples are provided by the theory of Howard [7]. If a group A is of second category in its p -adic topology then every reduced p -primary epimorphic image of A is bounded hence the p -adic topology and the \oplus_c -topology on A coincide. Examples of second category groups are the p -adically complete groups, but ([7; 4.3]) there are others as well, a situation very much reminiscent of thick groups. In [8; 4.6] it was shown that every reduced p -primary epimorphic image of a group K is bounded if and only if K is not the union of an ascending sequence of p -adically nowhere dense subgroups. Thus such groups are \oplus_c -completable and their completions are just the p -adic completions.

3. – Groups which are not completable.

It appears to be rather difficult to determine the \oplus_c -completions in general. As far as completability is concerned $p^{\omega+1}$ -projective p -groups are particularly simple since they are either complete or not completable as we will show first. Recall that a $p^{\omega+1}$ -projective group is an extension of an elementary p -group by a direct sum of cyclic p -groups (*). Thus a $p^{\omega+1}$ -projective group contains an open subgroup which is elementary. This fact is exploited in the first lemma.

(3.1) LEMMA. *Let A be a separable p -group having a subsocle T with $A/T \in \oplus_c$. Then $\check{A}/A \cong T^\sharp/T$ where T^\sharp is both the topological closure of T in \check{A} and the completion of T when T has the topology induced by the \oplus_c -topology of A . Hence \check{A}/A is p -bounded and A is completable if and only if A is complete.*

PROOF. We have ([9; 4.5]) the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 E: & 0 & \rightarrow & T & \rightarrow & A & \rightarrow & A/T & \rightarrow & 0 \\
 & & & \downarrow & & \downarrow & & \parallel & & \\
 & & & \eta & & & & & & \\
 \eta E: & 0 & \rightarrow & T^\sharp & \rightarrow & \check{A} & \rightarrow & A/T & \rightarrow & 0
 \end{array}$$

A diagram chase yields $\check{A}/A \cong T^\sharp/T$. Thus \check{A}/A is p -elementary. By the Completeness Criterion [9; 5.10 (6)] A is completable if and only if \check{A}/A is p -divisible, i.e. if and only if $\check{A} = A$. \square

(*) R. NUNKE, *Purity and subfunctors of the identity, Topics in Abelian Groups*, Scott, Foresman & Co, Chicago, 1963, pp. 121-171.

The problem is now reduced to deciding whether or not T is complete with the induced topology. In general it is neither clear what this induced topology might look like nor what the completion is. Fortunately, a method due to Benabdallah-Irwin [1] permits to construct a group A such that the induced topology on T is the topology induced by the p -adic topology on A , and this case can be handled.

We first need a special case of a theorem by Benabdallah-Irwin [1; Theorem 2.2].

(3.2) LEMMA. *If G is a p -group and K a pure subgroup of G such that $G/K[p] \in \oplus_c$ then K is a direct summand of G .*

Starting with any p -group G the method of Benabdallah-Irwin [1; pp. 326-327] yields a $p^{\omega+1}$ -projective group A whose properties are related to those of G .

(3.3) CONSTRUCTION. *Let G be a given p -group. Let $\tilde{G} = \bigoplus \{ \langle \tilde{g} \rangle : g \in G \}$ where $\langle \tilde{g} \rangle \cong \langle g \rangle$, and let $\varepsilon: \tilde{G} \rightarrow G: \tilde{g}\varepsilon = g$. It is well-known that $K = \text{Ker } \varepsilon$ is pure in \tilde{G} . Put $A = \tilde{G}/K[p]$. Then $T = \tilde{G}[p]/K[p]$ is a subsocle of A with $A/T \cong \tilde{G}/\tilde{G}[p] \cong p\tilde{G} \in \oplus_c$. Hence A is $p^{\omega+1}$ -projective. Furthermore by 3.2, $A \in \oplus_c$ if and only if $G \in \oplus_c$.*

In the following we always refer to this situation placing stronger and stronger conditions on G .

(3.4) *Let G be separable. Then A is separable.*

PROOF. G separable implies that K is p -adically closed in \tilde{G} . So is $\tilde{G}[p]$, and hence $K[p] = K \cap \tilde{G}[p]$. Thus $A = \tilde{G}/K[p]$ is separable. \square

(3.5) *Let G be pure-complete. Then for any subsocle S with $K[p] \leq S \leq \tilde{G}[p]$ there exists a pure subgroup L of \tilde{G} containing K with $L[p] = S$.*

PROOF. Since $S\varepsilon \leq G[p]$ and G is pure-complete there is a pure subgroup M of G with $M[p] = S\varepsilon$. Let $L = M\varepsilon^{-1}$. Then L is pure in \tilde{G} and contains K . It is easily checked that $L[p] = S$. \square

(3.6) *Let G be quasi-complete. If $K[p] \leq S \leq \tilde{G}[p]$ and $\tilde{G}/S \in \oplus_c$ then $\tilde{G} = L \oplus M$ with $L[p] = S$ and M bounded.*

PROOF. By [6; 74.2] G is pure-complete. Hence, by (3.5) and (3.2), there exist groups L and M such that $\tilde{G} = L \oplus M$, $K \leq L$ and $L[p] = S$,

Now $G \cong (L/K) \oplus M$ and $M \in \oplus_c$. If G is torsion-complete then so is M and hence M is bounded. If G is not torsion-complete then by [6; 74.6] either L/K or M is bounded. But $M \in \oplus_c$ cannot be unbounded in view of [6; 74.6]. \square

(3.7) *Let G be quasi-complete. The topology induced on $T = \tilde{G}[p]/K[p]$ by the \oplus_c -topology of $A = \tilde{G}/K[p]$ has the local basis $\{(p^n \tilde{G}[p] + K[p])/K[p] : n \in \omega\}$. Thus the \oplus_c -topology and the p -adic topology on A induce the same topology on T .*

PROOF. It is clear that each $(p^n \tilde{G}[p] + K[p])/K[p]$ is open in T . Suppose U is an open subgroup of $A[\oplus_c]$. Then so is $U \cap T$ and hence there exists a subgroup S of \tilde{G} such that $S/K[p] \leq U \cap T$ and $\tilde{G}/S \in \oplus_c$. By (3.6) there exists n such that $(p^n \tilde{G})[p] \subset S$ hence

$$(p^n \tilde{G}[p] + K[p])/K[p] \leq U \cap T. \quad \square$$

We now relate the topological group T to the socle of G .

(3.8) *For any group G , the groups $G[p]$ and $T = \tilde{G}[p]/K[p]$ are isomorphic as topological groups with topologies induced by the p -adic topologies on G and \tilde{G} respectively.*

PROOF. Clearly $\varepsilon: \tilde{G} \rightarrow G$ induces an isomorphism $\varepsilon: T \rightarrow G[p]$ with

$$((p^n \tilde{G}[p] + K[p])/K[p])\varepsilon = p^n G[p]. \quad \square$$

(3.9) THEOREM. *Let G be quasi-complete, $0 \rightarrow K \rightarrow \tilde{G} \rightarrow G \rightarrow 0$ the standard pure-projective resolution of G and $A = \tilde{G}/K[p]$. Then A is a separable $p^{\omega+1}$ -projective group and $A[\oplus_c]$ is complete if and only if G is torsion-complete.*

PROOF. By the construction (3.3) we have that A is $p^{\omega+1}$ -projective, and A is separable by (3.4). Since G is quasi-complete the p -adic and the \oplus_c -topologies on A induce the same topology on $T = \tilde{G}[p]/K[p]$ by (3.7). By (3.1), A is complete if and only if T is complete. But T and $G[p]$ are isomorphic topological groups by (3.8) where $G[p]$ has the topology induced by the p -adic topology on G . By [6; 70.6] $G[p]$ is complete if and only if G is torsion-complete. Thus A is complete if and only if G is torsion-complete. \square

(3.10) COROLLARY. *The \oplus_c -topology is not completable.*

PROOF. There exist quasi-complete groups which are not torsion-complete ([6], Vol. II, p. 48). Results (3.9) and (3.1) complete the proof. \square

Thus a $p^{\omega+1}$ -projective group may or may not be complete. The class of \oplus_c -complete group is smaller than it appeared in [5], and many of the theorems of [5] now became open questions, e.g. are \oplus_c -complete p -groups determined by their valuated socles?

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