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Diffusion on Viscous Fluids. Existence and Asymptotic Properties of Solutions.

H. BEIRÃO DA VEIGA

Summary. – *In this paper we consider the motion of a continuous medium consisting of two components, for example, water and a dissolved salt, with a diffusion effect obeying Fick's law. We denote by* $v, w, \varrho, \pi, \mu, \lambda$ *the mean-volume velocity, mean-mass velocity, density, pressure, viscosity and diffusion constant, respectively. By using Fick's law we eliminate* w *from the equations and we obtain* (1.1), *where* p *is the modified pressure; see section* 1 *and references* [2], [4], [5], [6]. *The initial boundary conditions are given by equation* (1.2). *Kazhikhov and Smagulov* [5], [6] *consider equation* (1.1) *for a small diffusion coefficient* λ. *More precisely they assume that condition* (1.3) *holds, and they omit the* λ^2 *term in equation* $(1.1)_1$. *Under these conditions they prove the existence of a unique local solution for the* 3-*dimensional motion (in the two-dimensional case, solutions are global). In our paper we consider the full equation* (1.1), *without assumption* (1.3), *and we prove:* (i) *the existence of a (unique) local solution;* (ii) *the existence of a global solution in time for small initial velocities and external forces, and for initial densities that are almost constant;* (iii) *the exponential decay (when* $t \to +\infty$*) of the solution* (ϱ, v) *to the equilibrium solution* $(\hat{\varrho}, 0)$, *if* $f \equiv 0$. *See Theorem A, section* 1.

Main notations.

Ω:	an open bounded set in $\mathbb{R}^3$, locally situated on one side of its boundary Γ, a regular (say C^4) manifold.
$n = n(x)$:	unit outward normal to Γ.
D_i, D_{ij}, D_t:	$\partial/\partial x_j$, $\partial^2/\partial x_i\, \partial x_j$, $\partial/\partial t$.
$\| \ \|$, $(\ , \ )$:	norm and scalar product in $L^2(\Omega)$.

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H^k: Sobolev space $H^{k,2}(\Omega)$ with norm

$$\|\sigma\|_k^2 \equiv \sum_{l=0}^{k} \|D^l\sigma\|^2 ,$$

where

$$\|D^l\sigma\|^2 \equiv \sum_{|\alpha|=l} \|D^\alpha\sigma\|^2 .$$

Further,

$$\|D^l\sigma\|_m^2 \equiv \sum_{|\alpha|=l} \|D^\alpha\sigma\|_m^2 .$$

H_0^1: closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$.

$\| \ \|_\infty$: norm in $L^\infty(\Omega)$.

$\boldsymbol{L}^2, \boldsymbol{H}^k, \boldsymbol{H}_0^1$: Hilbert spaces of vector $v = (v_1, v_2, v_3)$ such that $v_i \in L^2$, $v_i \in H^k$, $v_i \in H_0^1$ $(i = 1, 2, 3)$ respectively. Corresponding notation is used for other spaces of vector fields. Norms are defined in the natural way, and denoted by the symbols used for the scalar fields.

Let us introduce the following functional spaces (see for instance [7], [8], [12] for their properties):

$$H_N^k \equiv \left\{\sigma \in H^k : \frac{\partial\sigma}{\partial n} = 0 \text{ on } \Gamma \text{ and} \int_\Omega \sigma(x)\, dx = 0\right\}, \qquad k \geqslant 2 .$$

$$\mathcal{V} \equiv \{v \in \mathrm{C}_0^\infty(\Omega) : \operatorname{div} v = 0 \text{ in } \Omega\},$$

$$\boldsymbol{H} = \{v \in \boldsymbol{L}^2 : \operatorname{div} v = 0 \text{ in } \Omega,\ v \cdot n = 0 \text{ on } \Gamma\},$$

$$\boldsymbol{V} = \{v \in \boldsymbol{H}_0^1 : \operatorname{div} v = 0 \text{ on } \Omega\}.$$

$\boldsymbol{H}$ and $\boldsymbol{V}$ are the closures of $\mathcal{V}$ in $\boldsymbol{L}^2(\Omega)$ and $\boldsymbol{H}_0^1(\Omega)$ respectively. Moreover $\boldsymbol{L}^2 = \boldsymbol{H} + G$, where $G \equiv \{\nabla p : p \in H^1(\Omega)\}$. Denoting by P the orthogonal projection of $\boldsymbol{L}^2$ onto $\boldsymbol{H}$, we define the operator $A \equiv -P\Delta$ on $D(A) \equiv \boldsymbol{H}^2 \cap \boldsymbol{V}$. One has

$$(Au, v) = ((u, v)) \equiv \sum_{i,j} (D_i u_j, D_i v_j), \qquad \forall u \in D(A),\ v \in \boldsymbol{V}.$$

The norms $\|\sigma\|_2$, $\|\Delta\sigma\|$ are equivalent in H_N^2, $\|\sigma\|_3$, $\|\nabla\Delta\sigma\|$ are equivalent in H_N^3 and $\|v\|_2$, $\|Av\|$ are equivalent in $D(A)$. We define $\|v\|_V^2 \equiv ((v, v))$; the norms $\|v\|_V$, $\|v\|_1$ are equivalent in $\boldsymbol{V}$.

$L^2(0, T; X)$: Banach space of strongly measurable functions defined in $]0, T[$ with values in (a Banach space) X, for which

$$\|z\|^2_{L^2(0,T;X)} \equiv \int_0^T \|z(t)\|^2_X \, dt < +\infty .$$

$C(0, T; X)$: Banach space of X-vector valued continuous functions on $[0, T]$ endowed with the usual norm $\|z\|_{C(0,T;X)}$.

μ: viscosity (a positive constant).

λ: diffusion coefficient (a positive constant).

$v(t, x)$, $v_0(x)$: mean-volume velocity. Initial m.v. velocity.

$\varrho(t, x)$, $\varrho_0(x)$: density of the mixture. Initial density. Further

$$m \equiv \inf_{x\in\Omega} \varrho_0(x) , \qquad M \equiv \sup_{x\in\Omega} \varrho_0(x) ,$$
$$\hat{\varrho} \equiv \frac{1}{|\Omega|} \int_\Omega \varrho_0(x) \, dx .$$

We assume that $m > 0$.

$\pi(t, x)$, $p(t, x)$: pressure. Modified pressure

$$p = \pi + \lambda v \cdot \nabla \varrho - \lambda^2 \Delta \varrho + \lambda (2\mu + \mu') \Delta \log \varrho .$$

$f(t, x)$: external mass-force.

We denote by $c, \bar{c}, c_0, c_1, c_2, \ldots$ positive constants depending at most on Ω and on the parameters μ, λ, m, M, $\hat{\varrho}$. It is easy to derive, at any stage of the proofs, the explicit dependence of the constants on the parameters.

For convenience we sometimes denote different constants by the same symbol c. Otherwise, we utilize the symbols $\bar{c}$, c_k, $k \in \mathbb{N}$.

1. – Main results.

In this paper we consider the motion of a viscous fluid consisting of two components, for instance, saturated salt water and water. The equations of the model are obtained, for example, in [2], [4], [5], [6]. Let us give a brief sketch. Let ϱ_1, ϱ_2 be the characteristic densities (constants) of the two components, $v^{(1)}(t, x)$ and $v^{(2)}(t, x)$ their velocities and $e(t, x)$, $d(t, x)$ the mass and volume concentration of the first fluid. We define the density $\varrho(t, x) \equiv d\varrho_1 + (1-d)\varrho_2$, and the mean-volume and mean-mass velocities $v \equiv dv^{(1)} + (1-d)v^{(2)}$, $w \equiv ev^{(1)} + (1-e)v^{(2)}$. Then the equations of motion are given by

$$\begin{cases} \varrho[D_t w + (w\cdot\nabla)w - f] - \mu\, \Delta w - (\mu + \mu')\nabla \operatorname{div} w = -\nabla\pi\,, \\ \operatorname{div} v = 0\,, \\ D_t\varrho + \operatorname{div}(\varrho w) = 0\,. \end{cases}$$

On the other hand, Fick's diffusion law (see [2]) gives $w = v - \lambda\varrho^{-1}\nabla\varrho$. By eliminating of w in the preceeding equation one gets, after some calculations,

$$\text{(1.1)}\quad \begin{cases} \varrho(D_t v + (v\cdot\nabla)v) - \mu\, \Delta v - \lambda[(v\cdot\nabla)\nabla\varrho + (\nabla\varrho\cdot\nabla)v] \\ \qquad + \dfrac{\lambda^2}{\varrho}\Big[(\nabla\varrho\cdot\nabla)\nabla\varrho - \dfrac{1}{\varrho}(\nabla\varrho\cdot\nabla\varrho)\nabla\varrho + \Delta\varrho\,\nabla\varrho\Big] = -\nabla p + \varrho f\,, \\ D_t\varrho + v\cdot\nabla\varrho - \lambda\,\Delta\varrho = 0\,, \\ \operatorname{div} v = 0\,. \end{cases}$$

We want to solve system (1.1) in $Q_T \equiv\,]0, T[\times\Omega$. Here p is the modified pressure. We add to system (1.1) the following initial boundary-value conditions

$$\text{(1.2)}\quad \begin{cases} v = 0 & \text{on }]0, T[\times\Gamma\,, \\ \dfrac{\partial\varrho}{\partial n} = 0 & \text{on }]0, T[\times\Gamma\,, \\ v|_{t=0} = v_0(x) & \text{in } \Omega\,, \\ \varrho|_{t=0} = \varrho_0(x) & \text{in } \Omega\,. \end{cases}$$

The first two conditions mean that there is no flux through the boundary.

In [5], [6] Kazhikhov and Smagulov consider the simplified system obtained from $(1.1)_1$ by omitting the term containing λ^2. Moreover they assume that

$$\text{(1.3)}\qquad \lambda \geqslant \frac{2\mu}{M-m}\,.$$

Under these conditions Kazhikhov and Smagulov state the existence of a local solution in time (global in the two dimensional case).

In our paper we take into account the full equation (1.1) and omit the condition (1.3). For this more general case we prove: (i) the existence of a (unique) local solution for arbitrary initial data and external force field; (ii) the existence of a unique global strong solution for small initial data and external force field. Moreover, if $f\equiv 0$, the solution (ϱ, v) decays exponentially to the equilibrium solution $(\hat{\varrho}, 0)$. More precisely we prove the following result:

THEOREM A. *Let $v_0\in \boldsymbol{V}$, $\varrho_0\in H^2_N$, $f\in L^2(0, T; \boldsymbol{L}^2)$. Then there exists $T_1 \in]0, T]$ such that problem* (1.1), (1.2) *is uniquely solvable in Q_{T_1}. Moreover $v\in L^2(0, T_1; \boldsymbol{H}^2)\cap C(0, T_1; \boldsymbol{V})$, $D_t v\in L^2(0, T_1; \boldsymbol{H})$, $\varrho\in L^2(0, T_1; H^3_N)\cap C(0, T_1; H^2_N)$, $D_t\varrho\in L^2(0, T_1; H^1)$ and $m\leqslant\varrho(t, x)\leqslant M$.*

Moreover, there exist positive constants k_1, k_2, k_3 depending at most on Ω, μ, λ and on the mean density $\hat{\varrho}$(1) *such that if*

$$\|v_0\|_1+\|\varrho_0-\hat{\varrho}\|_2\leqslant k_1\,, \tag{1.4}$$

and

$$\|f\|_{L^\infty(0,+\infty;\boldsymbol{L}^2)}\leqslant k_2\,, \tag{1.5}$$

then the solution is global in time. If $f\equiv 0$ the solution (ϱ, v) decays expotentially to the equilibrium solution $(\hat{\varrho}, 0)$, i.e.

$$\|v(t)\|_1+\|\varrho(t)-\hat{\varrho}\|_2\leqslant(\|v_0\|_1+\|\varrho_0-\hat{\varrho}\|_2)\exp[-k_3 t]\,, \tag{1.6}$$

for every $t\geqslant 0$.

Theorem A also holds for coefficients μ, λ regularly dependent on ϱ, v, provided they are strictly positive and bounded in a neighborhood of the range of values of the initial data $\varrho_0(x)$, $v_0(x)$. This generalization can be done without any difficulty. Moreover, with standard techniques, one can prove that the solutions have more regularity (up to C^∞) if the data are sufficiently regular and the usual compatibility conditions hold.

Local existence in the general case (i.e. with the λ^2 term, and without (1.3)) was proved in the *inviscid* case by Beirão da Veiga, Serapioni, and Valli in [1]. A similar result, in the viscous case and for $\Omega=\mathbb{R}^3$, was proved by Secchi [11]. For another kind of approach (concerning Graffi's model) see [10].

(1) Or, equivalently, depending on the total amount of mass $|\Omega|\hat{\varrho}=\int_\Omega\varrho_0(x)\,dx$.

2. – The linearized equations.

We start by proving the following theorem:

THEOREM 2.1. *Let $\varrho(t,x)$ be a measurable function satisfying*

$$0 < m \leqslant \varrho(t,x) \leqslant M\,, \qquad a.e.\ in\ Q_T\,, \tag{2.1}$$

and let $F \in L^2(0,T;\boldsymbol{H})$ and $v_0 \in \boldsymbol{V}$. Then there exists a (unique) strong solution v of problem

$$\begin{cases} \varrho D_t v - \mu\,\Delta v = -\nabla p + F & in\ Q_T\,,\\ \operatorname{div} v = 0 & in\ Q_T\,,\\ v = 0 & on\ ]0,T[\times\Gamma\,,\\ v|_{t=0} = v_0(x) & in\ \Omega\,. \end{cases} \tag{2.2}$$

Moreover $v \in L^2(0,T;D(A)) \cap C(0,T;\boldsymbol{V})$, $D_t v \in L^2(0,T;\boldsymbol{H})$ and

$$\begin{aligned}\mu\|v\|^2_{C(0,T;\boldsymbol{V})} + m\|v'\|^2_{L^2(0,T;\boldsymbol{H})} + \frac{m\mu^2}{4M^2}\|Av\|^2_{L^2(0,T;\boldsymbol{H})}&\\ \leqslant \mu\|v_0\|^2_{\boldsymbol{V}} + \left(\frac{2}{m} + \frac{m}{2M^2}\right)\|F\|^2_{L^2(0,T;\boldsymbol{H})}&\,.\end{aligned} \tag{2.3}$$

PROOF. Let us write equation (2.2) in the equivalent form

$$P(\varrho D_t v) + \mu A v = F\,, \qquad v|_{t=0} = v_0(x)\,. \tag{2.4}$$

For brevity let us put

$$\mathfrak{X} = \{v\colon v \in L^2(0,T;D(A)),\ v' \in L^2(0,T;\boldsymbol{H})\}\,.$$

From well known results (see [9], Vol. I, chap. I: theorem 3.1 with $Y = \boldsymbol{H}$, $X = D(A)$, $j = 0$; and (2.42) proposition 2.1) it follows that $\mathfrak{X} \hookrightarrow C(0,T;\boldsymbol{V})$.

We start by proving the apriori bound (2.3); an essential device is to introduce a parameter ε_0 in order to conveniently balance the estimates. In $\boldsymbol{H}$ take the inner product of (2.4) with $D_t v + \varepsilon_0 A v$, $\varepsilon_0 > 0$. Since

$(v', Av) = 2^{-1}D_t\|v\|_V^2$ one gets

$$(2.5)\qquad m\|D_tv\|^2 + \frac{\mu}{2}\frac{d}{dt}\|v\|_V^2 + \varepsilon_0\mu\|Av\|^2 \leqslant \|F\|\|D_tv\| + \varepsilon_0\|F\|\|Av\| + \varepsilon_0 M\|D_tv\|\|Av\| .$$

By using the inequalities $\|F\|\|D_tv\| \leqslant 4^{-1}m\|D_tv\|^2 + m^{-1}\|F\|^2$, $\|F\|\|Av\| \leqslant 4^{-1}\mu\|Av\|^2 + \mu^{-1}\|F\|^2$ and $\|D_tv\|\|Av\| \leqslant (4M)^{-1}\mu\|Av\|^2 + \mu^{-1}M\|D_tv\|^2$ one gets

$$(2.6)\qquad \frac{3}{4}m\|D_tv\|^2 + \frac{\mu}{2}\frac{d}{dt}\|v\|_V^2 + \varepsilon_0\frac{\mu}{2}\|Av\|^2 \leqslant \left(\frac{1}{m} + \frac{\varepsilon_0}{\mu}\right)\|F\|^2 + \frac{\varepsilon_0 M^2}{\mu}\|D_tv\|^2 .$$

Now fix $\varepsilon_0 = (4M^2)^{-1}m\mu$ and integral equation (2.6) on $(0, T)$. This gives the apriori bound (2.3).

Define $\|v\|_{\mathfrak{X}}^2 \equiv$ «left hand side of equation (2.3)», $\mathfrak{Y} \equiv L^2(0, T; \boldsymbol{H}) \times \boldsymbol{V}$ and $\|(F, v_0)\|^2 \equiv$ «right hand side of (2.3)».

We solve (2.4) by the continuity method. Define $\varrho_\alpha \equiv (1-\alpha)\hat{\varrho} + \alpha\varrho$, $\alpha \in [0, 1]$. Clearly ϱ_α verifies condition (2.1), for any α. Define $T_\alpha \equiv (1-\alpha)\overline{T} + \alpha T$, where

$$Tv \equiv \big(P(\varrho D_tv) - \mu Av, v|_{t=0}\big) \in \mathfrak{Y} .$$

$$\overline{T}v \equiv \big(P(\bar{\varrho} D_tv) - \mu Av, v|_{t=0}\big) \in \mathfrak{Y} .$$

Finally consider problem (2.2) with ϱ replaced by ϱ_α, i.e. problem $T_\alpha v = (F, v_0)$. Denote by γ the set of values $\alpha \in [0, 1]$ for which that problem is solvable in $\mathfrak{X}$ for every pair $(F, v_0) \in \mathfrak{Y}$. Clearly $0 \in \gamma$, because for this value of the parameter equation (2.6) becomes the linearized Navier-Stokes equation. Let us verify that γ is open and closed.

γ is open. Let $\alpha_0 \in \gamma$ and denote by $G(F, v_0) \equiv v$ the solution v of problem $T_{\alpha_0}v = (F, v_0)$. From (2.3) one gets $G \in L(\mathfrak{Y}, \mathfrak{X})$ (2), with $\|G\|_{\mathfrak{Y},\mathfrak{X}} \leqslant 1$. Equation $T_{\alpha_0+\varepsilon}v = (F, v_0)$ can be written in the form

$$(2.7)\qquad [1 - \varepsilon G(\overline{T} - T)]v = G(F, v_0) .$$

Since $\|G(\overline{T} - T)\|_{\mathfrak{X},\mathfrak{X}} \leqslant \|\overline{T} - T\|_{\mathfrak{X},\mathfrak{Y}}$, equation (2.7) is solvable for $|\varepsilon| < \|\overline{T} - T\|_{\mathfrak{X},\mathfrak{Y}}^{-1}$ (by a Neumann expansion).

γ is closed. Let $\alpha_n \in \gamma$, $\alpha_n \to \alpha_0$, and let v_n be the solution of $T_{\alpha_n}v_n = (F, v_0)$. From (2.3) one has $\|v_n\|_{\mathfrak{X}} \leqslant \|(F, v_0)\|_{\mathfrak{Y}}$. Since $\mathfrak{X}$ is an Hilbert space,

(2) The Banach space of linear continuous operators from $\mathfrak{Y}$ into $\mathfrak{X}$, with norm $\| \ \|_{\mathfrak{Y},\mathfrak{X}}$.

there exists a subsequence $v_\nu \to v \in \mathfrak{X}$, weakly in $\mathfrak{X}$. From $T, \overline{T} \in L(\mathfrak{X}; \mathfrak{Y})$ one has $\overline{T}v_\nu \to \overline{T}v$, $Tv_\nu \to Tv$ weakly in $\mathfrak{Y}$. Hence $T_{\alpha_\nu}v_\nu \to T_{\alpha_0}v$, i.e. $T_{\alpha_0}v = (F, v_0)$. ■

Let us now return to problem (1.1). Define

$$F(\varrho, v) \equiv P\Big\{-\varrho(v\cdot\nabla)v + \lambda[(v\cdot\nabla)\nabla\varrho + (\nabla\varrho\cdot\nabla)v] + \\ + \frac{\lambda^2}{\varrho^2}\Big[(\nabla\varrho\cdot\nabla)\nabla\varrho - \frac{1}{\varrho}(\nabla\varrho\cdot\nabla\varrho)\nabla\varrho + \Delta\varrho\cdot\nabla\varrho\Big] + \varrho f\Big\}. \tag{2.8}$$

For convenience we will use in the sequel the translation

$$\varrho = \hat{\varrho} + \sigma. \tag{2.9}$$

Recall that $\hat{\varrho}$ is a given constant. To solve problem (1.1), (1.2) in our functional framework is equivalent to finding $v \in L^2(0, T; D(A))$, $v' \in L^2(0, T; \boldsymbol{H})$ and $\sigma \in L^2(0, T; H^3_N)$, $\sigma' \in L^2(0, T; H^1)$ such that

$$\begin{cases} P(\varrho D_t v) + \mu A v = F(\varrho, v)\,, \\ v|_{t=0} = v_0(x)\,, \\ D_t\sigma - \lambda\,\Delta\sigma = v\cdot\nabla\sigma\,, \\ \sigma|_{t=0} = \sigma_0(x)\,, \end{cases} \tag{2.10}$$

where $v_0 \in \boldsymbol{V}$ and $\sigma_0(x) \equiv \varrho_0(x) - \hat{\varrho} \in H^2_N(\Omega)$ are given. Note that from the above conditions on σ it follows that $\sigma \in C(0, T; H^2_N)$.

We solve (2.10) by considering the linearized problem

$$\begin{cases} P(\bar{\varrho}\, D_t v) + \mu A v = F(\bar{\varrho}, \bar{v}) \equiv \overline{F}\,, \\ v|_{t=0} = v_0(x)\,, \\ D_t\sigma - \lambda\,\Delta\sigma = -\bar{v}\cdot\nabla\bar{\sigma}\,, \\ \sigma|_{t=0} = \sigma_0(x)\,, \end{cases} \tag{2.11}$$

and by proving the existence of a fixed point $(\bar{\varrho}, \bar{v}) = (\varrho, v)$ for the map $(\bar{\varrho}, \bar{v}) \to (\varrho, v)$ defined by (2.11).

In order to get a sufficiently strong estimate for the linearized equation $(2.11)_3$ we take in account the particular form of the data $\bar{v}\cdot\nabla\bar{\varrho}$. As for estimate (2.3) we will introduce a balance parameter $\varepsilon > 0$.

THEOREM 2.2. *Assume that* $\bar{v} \in L^2(0, T; \boldsymbol{H}^2) \cap C(0, T; \boldsymbol{H}_0^1)$ *and that* $\bar{\sigma} \in L^2(0, T; H_N^3) \cap C(0, T; H_N^2)$. *Then the solution* $\sigma \in L^2(0, T; H_N^3)$, $\sigma' \in L^2(0, T; H^1)$ *of problem* $(2.11)_3$, $(2.11)_4$ *verifies the estimate*

$$(2.12) \qquad \|\sigma\|^2_{C(0,T;H^2_N)} + \|\sigma\|^2_{L^2(0,T;H^3_N)} \leqslant c_2\|\sigma_0\|_2^2 + c_3\varepsilon^{-3}T[\|\bar{v}\|^6_{C(0,T;\boldsymbol{H}^1)} + \|\nabla\bar{\sigma}\|^6_{C(0,T;\boldsymbol{H}^1)}] + c_3\varepsilon[\|\bar{v}\|^2_{L^2(0,T;\boldsymbol{H}^2)} + \|\nabla\bar{\sigma}\|^2_{L^2(0,T;\boldsymbol{H}^2)}],$$

for every positive ε *satisfying*

$$(2.13) \qquad \varepsilon \leqslant \frac{\lambda}{2c_1},$$

where c_1 *is the constant in* (2.16). *Here* c_1, c_2, c_3 *are positive constants depending only on* Ω.

PROOF. The existence of a solution σ in the required space follows from standard techniques using the apriori bound (2.12) or using [9], vol. II, chap. 4, theorem 5.2, with $\mathscr{H} = H^1$. Let us prove (2.12).

By applying of the operator Δ to both sides of $(2.11)_3$, then multiplying by $\Delta\sigma$, and finally integrating over Ω, one gets $\frac{1}{2}D_t\|\Delta\sigma\|^2 + (\nabla(\lambda\,\Delta\sigma - \bar{v}\cdot\nabla\bar{\sigma}), \nabla\,\Delta\sigma) = 0$. Note that $\partial/\partial n(\lambda\,\Delta\sigma - \bar{v}\cdot\nabla\bar{\sigma}) = 0$ on Γ. Hence

$$(2.14) \qquad \frac{1}{2}\frac{d}{dt}\|\Delta\sigma\|^2 + \lambda\|\nabla\,\Delta\sigma\|^2 \leqslant (\|D\bar{v}\,D\bar{\sigma}\| + \|\bar{v}\,D^2\bar{\sigma}\|)\|\nabla\,\Delta\sigma\|.$$

Using Sobolev's embedding theorem $H^1 \hookrightarrow L^6$ and Hölder's inequality we obtain

$$(2.15) \qquad \frac{1}{2}\frac{d}{dt}\|\Delta\sigma\|^2 + \|\nabla\,\Delta\sigma\|^2 \leqslant c_1(\|D\bar{v}\|^{\frac{1}{2}}\|D\bar{v}\|_1^{\frac{1}{2}}\|\nabla\bar{\sigma}\|_1 + \|\bar{v}\|_1\|\nabla\bar{\sigma}\|_1^{\frac{1}{2}}\|\nabla\,\Delta\bar{\sigma}\|^{\frac{1}{2}})\|\nabla\,\Delta\sigma\|.$$

A utilization of $abc \leqslant (8\varepsilon^3)^{-1}a^4 + (\varepsilon/2)b^2 + (\varepsilon/2)c^4$, $\varepsilon > 0$, leads to

$$(2.16) \qquad \frac{1}{2}\frac{d}{dt}\|\Delta\sigma\|^2 + \lambda\|\nabla\,\Delta\sigma\|^2 \leqslant \frac{c_1\varepsilon}{2}\|D\bar{v}\|_1^2 + c_1\varepsilon\|\nabla\,\Delta\sigma\|^2 + \frac{c}{\varepsilon^3}\|\bar{v}\|_1^4\|\nabla\bar{\sigma}\|_1^2 + \frac{c_1\varepsilon}{2}\|\nabla\,\Delta\bar{\sigma}\|^2 + \frac{c}{\varepsilon^3}\|\bar{v}\|_1^2\|\nabla\bar{\sigma}\|_1^4.$$

Hence for ε satisfying (2.13) one has

$$(2.17) \qquad \frac{d}{dt}\|\Delta\sigma\|^2 + \lambda\|\nabla\,\Delta\sigma\|^2 \leqslant \frac{c}{\varepsilon^3}(\|\bar{v}\|_1^2\|\nabla\bar{\sigma}\|_1^4 + \|\bar{v}\|_1^4\|\nabla\bar{\sigma}\|_1^2) + c_1\varepsilon(\|D\bar{v}\|_1^2 + \|\nabla\,\Delta\bar{\sigma}\|^2),$$

where the constants c depend only on Ω. This proves inequality (2.12). Recall that $\|\sigma\|_2 \leqslant c\|\Delta\sigma\|$, $\|\sigma\|_3 \leqslant c\|\nabla \Delta\sigma\|$. ■

REMARK. One could also consider the linearized equation $D_t\sigma + \bar{v}\cdot\nabla\sigma - \lambda\Delta\sigma = 0$ instead of $(2.11)_3$; then estimate (2.17) holds with $\bar{\sigma}$ replaced by σ and without the term $c_9\varepsilon\|\nabla\Delta\bar{\sigma}\|^2$. In this case the solution σ of the linearized problem satisfies the maximum principle (which doesn't hold for the solution of $(2.11)_3$). However, the linearization $(2.11)_3$ seems to be more in keeping with the linearization $(2.11)_1$. Besides, the maximum principle will be recovered for the solution of the full nonlinear problem $(2.10)_3$.

3. – The nonlinear problem. Local existence.

We will not take care of the explicit dependence of contants on μ, λ, m, M; some of the constants c, c_k, depend on these fixed quantities. In order to simplify the equations, we denote by K_0, K_1, K_2, ... constants depending on the norms of the initial data $\|v_0\|_V$ and $\|\sigma_0\|_2$.

In this section we solve (2.10) by proving the existence of a fixed point $(\varrho, v) = (\bar{\varrho}, \bar{v})$ for system (2.11). Define

$$\mathcal{K}_1 \equiv \left\{\bar{v}\colon \bar{v}|_{t=0} = v_0(x),\ \|\bar{v}\|^2_{L^2(0,T;H^2)} + \|\bar{v}\|^2_{C(0,T;V)} + \|\bar{v}'\|^2_{L^2(0,T;H^1)} \leqslant 2c_4\|v_0\|^2_V\right\},$$

$$\mathcal{K}_2 \equiv \left\{\bar{\sigma}\colon \bar{\sigma}|_{t=0} = \sigma_0(x),\ \|\bar{\sigma}\|^2_{L^2(0,T;H^3_N)} + \|\bar{\sigma}\|^2_{C(0,T;H^2_N)} \leqslant 2c_2\|\sigma_0\|^2_2,\ \|\sigma'\|^2_{L^2(0,T;H^1)} \leqslant K_0,\ \|\bar{\sigma} - \sigma_0\|_{C(\bar{Q}_T)} \leqslant \frac{m}{2}\right\},$$

where $c_4 \equiv \mu[\min\{\mu, \bar{M}, (\bar{m}\mu^2)/(4\bar{M}^2)\}]^{-1}$ (see (3.2) below) and $K_0 \equiv \lambda\sqrt{2}c_2\|\sigma_0\|_2 + \bar{c}\sqrt{4c_2c_4}\|v_0\|_V\|\sigma_0\|_2$. Here $\bar{c} = \bar{c}(\Omega)$ is a positive constant such that

$$\|\bar{v}\cdot\bar{w}\|_1 \leqslant \bar{c}\|\bar{v}\|_V\|\bar{w}\|_2\,, \qquad \forall \bar{v} \in \boldsymbol{V},\ \bar{w} \in \boldsymbol{H}^2\,. \tag{3.1}$$

Note that for every $\bar{\sigma} \in \mathcal{K}_2$ one has in Q_T

$$\bar{m} \equiv \frac{m}{2} \leqslant \bar{\varrho}(t, x) \leqslant M + \frac{m}{2} \equiv \bar{M}\,. \tag{3.2}$$

We now evaluate the $\boldsymbol{L}^2$ norm of $\bar{F} \equiv F(\bar{\varrho}, \bar{v})$. By using Sobolev's embeding theorem $H^1 \hookrightarrow L^6$ and Hölder's inequality one easily gets

$$\begin{aligned}\|F(\bar{\varrho}, \bar{v})\|^2 \leqslant{}& c\|\bar{v}\|^3_1\|D\bar{v}\|_1 + c\|\bar{v}\|^2_1\|D^2\sigma\|\,\|D^2\bar{\sigma}\|_1 \\ &+ c\|\bar{v}\|_1\|D\bar{v}\|_1\|D\bar{\sigma}\|^2_1 + c\|D\bar{\sigma}\|^3_1\|D^2\bar{\sigma}\|_1 + c\|D\bar{\sigma}\|^6_1 + c\|f\|^2\,.\end{aligned} \tag{3.3}$$

Consequently

$$(3.4)\qquad \|F(\bar\varrho, \bar v)\|^2_{L^2(0,T;H)} \leqslant c(\|v_0\|_V + \|\sigma_0\|_2)^4 T^{\frac12} + c\|\sigma_0\|_2^6 T + c\|f\|^2_{L^2(0,T;H)}\,.$$

Hence, by using (2.3), it follows that the solution v of $(2.11)_1$, $(2.11)_2$ verifies

$$(3.5)\qquad \|v\|^2_{C(0,T;V)} + \|v\|^2_{L^2(0,T;H^2)} + \|v'\|^2_{L^2(0,T;L^2)} \leqslant c_4\|v_0\|^2_V + K_1(\sqrt{T} + T) + c\|f\|^2_{L^2(0,T;H)}\,.$$

On the other hand, from (2.12),

$$(3.6)\qquad \|\sigma\|^2_{C(0,T;H^2_N)} + \|\sigma\|^2_{L^2(0,T;H^3_N)} \leqslant c_2\|\sigma_0\|^2_2 + K_3\varepsilon^{-3}T + K_4\varepsilon\,.$$

Now we fix $\varepsilon > 0$ such that $K_4\varepsilon \leqslant 2^{-1}c_2\|\sigma_0\|^2_2$. Finally, by choosing $T > 0$ sufficiently small, it follows that $v \in \mathcal{K}_1$, $\sigma \in \mathcal{K}_2$. The estimate for $D_t\sigma$ follows by using equation $(2.11)_3$ and (3.1). The estimate for the sup norm of $\sigma - \sigma_0$ in $\bar Q_T$ is proved as follows. Clearly,

$$\|\sigma(t) - \sigma_0\|_1 \leqslant \int_t^0 \|\sigma'(s)\|_1\, ds \leqslant K_0 T^{\frac12}\,.$$

On the other hand $\|\sigma(t) - \sigma_0\|_{C(\bar\Omega)} \leqslant c_5\|\sigma(t) - \sigma_0\|_1^{\frac13}\|\sigma(t) - \sigma_0\|_2^{\frac23}$, where c_5 depends only on Ω; recall that $H^{5/3}(\Omega) \hookrightarrow C(\bar\Omega)$. Consequently

$$\|\sigma - \sigma_0\|_{C(\bar Q_T)} \leqslant c_5 K_0^{\frac13} T^{\frac16}(\sqrt{2c_2}\|\sigma_0\|_2 + \|\sigma_0\|_2)^{\frac23}\,.$$

Hence, by choosing (if necessary) a smaller value for T, one gets $\|\sigma - \sigma_0\|_{C(\bar Q_T)} \leqslant m/2$.

Now we utilize Schauder's fixed point theorem. Clearly $\mathcal{K} \equiv \mathcal{K}_1 \times \mathcal{K}_2$ is a convex, compact set in $L^2(0, T; \boldsymbol{L}^2) \times L^2(0, T; L^2)$. Let us denote by Φ the map $\Phi(\bar\varrho, \bar v) = (\varrho, v)$, defined by (2.11). Since $\Phi(\mathcal{K}) \subset \mathcal{K}$, it is sufficient to prove that $\Phi\colon \mathcal{K} \to \mathcal{K}$ is continuous in the L^2 topology. If $\bar v_n \to \bar v$ in $\boldsymbol{L}^2(Q_T)$, $\bar\varrho_n \to \bar\varrho$ in $L^2(Q_T)$, it then follows by compactness arguments that $\bar v_n \to \bar v$ weakly in $L^2(0, T; \boldsymbol{H}^2)$ and in $H^1(0, T; \boldsymbol{L}^2)$, and $\nabla\bar\varrho_n \to \nabla\bar\varrho$ weakly in $L^2(0, T; H^2)$ and in $H^1(0, T; L^2)$. In particular $\bar\varrho_n$ is a bounded sequence in $H^{(\frac12)+\varepsilon_1}(0, T; H^{2-\varepsilon_2}) \hookrightarrow C^{0,\alpha}(\bar Q_T)$ (3), for suitable positive ε_1, ε_2, α. Hence $\bar\varrho_n \to \bar\varrho$ uniformly in $\bar Q_T$. Moreover, $\bar v_n$ and $\nabla\bar\varrho_n$ are bounded in $H^{\frac12}(0, T; \boldsymbol{H}^1)$

(3) α-Hölder continuous functions in $\bar Q_T$.

and in $H^{\frac{1}{2}}(0, T; \boldsymbol{H}^1)$ respectively. Hence $\bar{v}_n \to \bar{v}$ and $\nabla\bar{\varrho}_n \to \nabla\bar{\varrho}$ strongly in the L^4 topology. It follows from (2.8) that $F(\bar{\varrho}_n, \bar{v}_n) \to F(\bar{\varrho}, \bar{v})$ as a distribution in Q_T. Consequently $F(\bar{\varrho}_n, \bar{v}_n) \to F(\bar{\varrho}, \bar{v})$ weakly in $\boldsymbol{L}^2(Q_T)$ because $F(\bar{\varrho}_n, \bar{v}_n)$ is a bounded sequence in this space. Analogously, $\bar{v}_n \cdot \nabla\bar{\varrho}_n \to \bar{v}\cdot\nabla\bar{\varrho}$ strongly in $L^2(Q_T)$. It follows from the linear equations (2.1) that $v_n \to v$ and $\varrho_n \to \varrho$ in $\boldsymbol{L}^2(Q_T)$ and $L^2(Q_T)$ respectively. Hence Φ is continuous. This finishes the proof of the existence of a local solution. Uniqueness will be proved in section 5.

4. – Global solutions. Asymptotic behavior.

In this section the constants c_k depend at most on Ω and on the quantities μ, λ and $\hat{\varrho}$ i.e. on the total amount of mass $|\Omega|\hat{\varrho}$. We assume that

$$\|\sigma_0\|_2 \leqslant (2c_6)^{-1}\hat{\varrho}\,, \tag{4.1}$$

where $c_6 = c_6(\Omega)$ is a positive constant, such that $\|\sigma\|_{C(\bar{\Omega})} \leqslant c_6\|\sigma\|_2$, for all $\sigma \in H^2_N(\Omega)$. Hence $\hat{\varrho}/2 \leqslant m \leqslant M \leqslant 3\hat{\varrho}/2$. Let (ϱ, v) be a solution of (1.1). From (2.6) for $\varepsilon_0 = (4M^2)^{-1}m\mu$ and from (3.3) one gets

$$\begin{aligned}\frac{m}{2}\|D_t v\|^2 + \frac{\mu}{2}\frac{d}{dt}\|v\|_V^2 + \frac{m\mu^2}{8M^2}\|Av\|^2 \\ \leqslant c(\|v\|_1^3 + \|\sigma\|_2^3)(\|v\|_2 + \|\sigma\|_3) + c\|\sigma\|_2^6 + c\|f\|^2\,,\end{aligned} \tag{4.2}$$

where c depends only on Ω, μ, $\hat{\varrho}$. On the other hand, from (2.17) for $\varepsilon = (2c_9)^{-1}\lambda$, one obtains

$$\frac{d}{dt}\|\Delta\sigma\|^2 + \frac{\lambda}{2}\|\nabla\Delta\sigma\|^2 \leqslant c(\|v\|_1^6 + \|\sigma\|_2^6)\,. \tag{4.3}$$

From (4.2) and (4.3) it follows easily that

$$\begin{aligned}\frac{d}{dt}\left(\frac{\mu}{2}\|v\|_V^2 + \|\Delta\sigma\|^2\right) + \frac{m}{2}\|D_t v\|^2 + \frac{m\mu^2}{16M^2}\|Av\|^2 + \frac{\lambda}{4}\|\nabla\Delta\sigma\|^2 \\ \leqslant c(\|v\|_V^6 + \|\Delta\sigma\|^6 + \|f\|^2)\,.\end{aligned}$$

In particular, since $\|Av\| \geqslant c\|v\|_V$ and $\|\nabla\Delta\sigma\| \geqslant c\|\Delta\sigma\|$, one has

$$\frac{d}{dt}(\|v\|_V^2 + \|\Delta\sigma\|^2) \leqslant -[c_{10} - c_{11}(\|v\|_V^2 + \|\Delta\sigma\|^2)^2]\cdot(\|v\|_V^2 + \|\Delta\sigma\|)^2 + c_{12}\|f\|^2. \tag{4.4}$$

Then $c_{11}(\|v(t)\|_V^2 + \|\Delta\sigma(t)\|^2)^2 < c_{10}/2$ holds for every $t \in [0, +\infty[$ provided

$$(4.5)\qquad \begin{cases} c_{11}(\|v_0\|^2 + \|\Delta\sigma_0\|^2)^2 < \dfrac{c_{10}}{2}, \\[2ex] c_{12}\|f\|^2_{L^\infty(0,+\infty;H)} < \dfrac{c_{10}}{2}\sqrt{\dfrac{c_{10}}{2c_{11}}}. \end{cases}$$

In fact, if $c_{11}(\|v(t)\|_V^2 + \Delta\sigma(t)\|^2)^2 = c_{10}/2$ it must be, from (4.4), that $d/dt\,(\|v(t)\|_V^2 + \|\Delta\sigma(t)\|^2) < 0$.

Let us now prove the last assertion in theorem A. Under the hypothesis $(4.5)_1$ it follows from (4.4) that

$$\frac{d}{dt}(\|v\|_V^2 + \|\Delta\sigma\|^2) \leqslant -\frac{c_{10}}{2}(\|v\|_V^2 + \|\Delta\sigma\|^2).$$

This proves (1.6). ■

5. – Uniqueness.

We prove that the solution (ϱ, v) of problem (1.1) is unique in the class for which existence was proved; see theorem A. We remark that more careful calculations lead to uniqueness in a larger class.

Let (ϱ, v), $(\bar\varrho, \bar v)$ be two solutions of problem (1.1), (1.2) and put $u = v - \bar v$, $\eta = \varrho - \bar\varrho$. By subtracting the equations $(2.10)_1$ written for (ϱ, v) and $(\bar\varrho, \bar v)$ respectively, and by taking the inner product with u in $\boldsymbol{H}$ one gets

$$\frac{1}{2}\frac{d}{dt}(\varrho u, u) + \mu\|u\|_V^2 = -\frac{1}{2}(v\cdot\nabla\varrho, u^2) + \frac{\lambda}{2}(\Delta\varrho, u^2) - (\eta, D_t\bar v\cdot u) + (F - \bar F, u).$$

By using $(\Delta\varrho, u^2) = -(\nabla\varrho, \nabla u^2)$, we show that

$$(5.1)\qquad \begin{aligned} \frac{1}{2}\frac{d}{dt}(\varrho u, u) + \frac{\mu}{2}\|u\|_V^2 \leqslant{}& \frac{1}{2}\|v\|_\infty\|\nabla\varrho\|_\infty\|u\|^2 \\ &+ \frac{\lambda^2}{\mu}\|\nabla\varrho\|_\infty^2\|u\|^2 + c\|D_t\bar v\|^2\|u\|^2 \\ &+ \frac{\lambda}{4}\|\Delta\eta\|^2 + (F - \bar F, u). \end{aligned}$$

On the other hand, by subtracting equations $(2.10)_3$ written for (ϱ, v)

and for $(\bar{\varrho}, \bar{v})$ respectively, and by taking the inner product with $\Delta\eta$ in $L^2(\Omega)$ one gets

$$\frac{1}{2}\frac{d}{dt}\|\nabla\eta\|^2 + \frac{\lambda}{2}\|\Delta\eta\|^2 \leqslant c\|\nabla\bar{\varrho}\|_\infty^2\|u\|^2 + c\|v\|_\infty^2\|\nabla\eta\|^2 . \tag{5.2}$$

By adding (5.1) and (5.2) it follows that

$$\frac{d}{dt}[(\varrho u, u) + \|\nabla\eta\|^2] + \mu\|u\|_V^2 + \frac{\lambda}{2}\|\Delta\eta\|^2 \leqslant \theta_1(t)(\|u\|^2 + \|\nabla\eta\|^2) + (F - \bar{F}, u) , \tag{5.3}$$

where $\theta_1(t)$ is a real integrable function on $[0, T]$.

On the other hand, by using Sobolev's embedding theorems and Hölder's inequality (and also $ab \leqslant \varepsilon a^2 + \varepsilon^{-1}b^2$) the reader easily verifies that given $\varepsilon > 0$ there exists an integrable real function $\theta_2(t)$ (dependent on $\varrho, \bar{\varrho}, v, \bar{v}$ and on ε) such that

$$|(F - \bar{F}, u)| \leqslant \theta_2(t)\|u\|^2 + \varepsilon(\|\eta\|_2^2 + \|u\|_1^2) . \tag{5.4}$$

From $\|u\|^2 \leqslant m^{-1}(\varrho u, u)$, (5.3) and (5.4) it follows that

$$\frac{d}{dt}[(\varrho u, u) + \|\nabla\eta\|^2] \leqslant (\theta_1(t) + \theta_2(t))[(\varrho u, u) + \|\nabla\eta\|^2] .$$

Uniqueness follows now from Gronwall's lemma and from $u|_{t=0} = 0$, $\eta|_{t=0} = 0$. ■

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