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On inseparable descent

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On Inseparable Descent.

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Introduction.

Let $k$ be a perfect field of characteristic $p \neq 0$. Put $R = k[[x]] = \text{the ring of formal power series in } x \text{ with coefficients in } k$, $R' = k[[x^{1/p^\infty}]] = \bigcup_{n=0}^{\infty} k[[x^{1/p^n}]]$, and $Q$, $Q'$ the quotient fields of $R$, $R'$, respectively. We also use the notation $W(\cdot)$ to denote the ring of infinite Witt vectors (relative to the prime number $p$) with components in $\cdot$, and put $K = W(k)$. Let $A$ denote the ring $K[[X]]$ of formal power series in $X$ with coefficients in $K$, and let $B$ denote the $p$-adic completion of the ring $A[1/X]$. We will define in section 3 embeddings of rings $A \to W(R')$ and $B \to W(Q')$.

The purpose of this paper is to give a manageable expression for descent data on modules relatively to the extensions $R \to R'$, $Q \to Q'$ and on $p$-adically separated and complete modules relatively to the extensions $A \to W(R')$, $B \to W(Q')$.

The simple form of the results obtained, say in the case $R \to R'$, depends on the following fact. Let $S = \text{Spec } R$, $X = \text{Spec } R'$, $G = \text{the affine } S\text{-group Cartier dual to } (Q_p/Z_p)_S$ (the standard étale $p$-divisible group of height 1, viewed over $S$). Then it is possible to define a morphism of schemes: $G \times_S X \to X$, making $X \to S$ into a principal homogeneous space under $G$. We do not pursue in the present paper this geometric viewpoint: our aim here is not towards greatest generality but towards a complete understanding of the extensions of rings mentioned above.

We will apply the results obtained here in subsequent papers to give a generalization of Dieudonné theory for $p$-divisible groups defined over $R$ or $Q$.

This paper is essentially self-contained: we send to the references only for the proof of two theorems. Some computations are however left to the reader.

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In writing this paper we have been strongly influenced by the work of Barsotti: some of the constructions we use are due to him and others are direct generalizations of the former worked out in the same spirit.

1. - In this paper the word « ring » means « commutative ring with 1 »; a morphism of rings always sends 1 to 1 and « module » means « unitary module ». If k is a ring, a k-algebra will always be associative with a right and left identity element 1, and a morphism of k-algebras (a representation) will always send 1 to 1. If A, B are k-algebras, an antirepresentation f: A → B is a representation of the opposite k-algebra A* of A in B.

If A is a linearly topologized (l.t.) ring and M, N are linearly topologized (l.t.) A-modules, the usual topology of Hom_A (M, N) will be the topology of simple convergence on the elements of M. The usual topology of Hom_A (M, N) is A-linear and a fundamental system of open submodules of Hom_A (M, N) in that topology is given by the set of the

\[ \mathcal{U}(S, V) = \{ f \in \text{Hom}_A (M, N) : f(S) \subseteq V \} \]

as S varies among the finite subsets of M and V among the open submodules of N. Unless otherwise specified A, M, N will be equipped with the discrete topology and Hom_A (M, N) with the usual topology.

If f: A → B is a morphism of rings and g: M → N a morphism of A-modules, we denote by \( g_f : M \otimes B \to N \otimes B \) the morphism of B-modules obtained by the scalar extension f.

(1.1) DEFINITION. Let k be a l.t. ring, separated and complete. A linearly topologized (l.t.) k-hyperalgebra is a structure (A, i, 1', P, e, g), that we usually denote simply by A where:

(1.1.i) A is a separated and complete l.t. k-module;

(1.1.ii) \( \mu: A \otimes_k A \to A \) (« product ») and \( i: k \to A \) (« structural morphism »)

are continuous morphisms of k-modules satisfying:

(1.1.ii.1) \( \mu(\text{id}_A \otimes \mu) = \mu(\mu \otimes \text{id}_A) \) (associativity),

(1.1.ii.2) \( \mu(i \otimes \text{id}_A) = \mu(\text{id}_A \otimes i) = \text{id}_A \) (existence of 1_A);

therefore \((A, i, \mu)\) is a l.t. k-algebra;
are continuous morphisms of $k$-algebras such that:

Let $x : A \otimes_k A \to A (\otimes_k A).$ be the continuous $k$-linear map defined by

\[ x(a \otimes b) = b \otimes a. \]

Then the l.t. $k$-hyperalgebra $A$ is commutative if $x = \mu x,$ and it is cocommutative if $x^{\prime} = P.$

Morphisms of l.t. $k$-hyperalgebras are defined in the obvious way. The kernel of $x : A \to k$ will be denoted by $A^+$ and will be called the augmentation ideal of $A.$

If $k$ is a ring, a $k$-hyperalgebra is a discrete l.t. $k$-hyperalgebra, where $k$ is equipped with the discrete topology.

(1.2) DEFINITION. Let $k$ be a morphism of rings and $M$ be an $A$-module. A descent datum on $M$ relatively to $k \to A$ is a homomorphism of $A \otimes_k A$-modules:

\[ \Theta : M \otimes_k A \to A \otimes_k M \]

such that:

\[ \Theta(\mu_0) = \text{id}_M; \]

\[ \Theta_{(\mu_0)} = \Theta; \]

\[ \Theta = \Theta, \]  
\[ \Theta_{(\mu_0)} = \Theta; \]

and $\Theta_{(\mu_0)},$ the diagram:

\[ \begin{array}{ccc}
M \otimes A \otimes A & \longrightarrow & A \otimes A \otimes M \\
\downarrow \Theta & & \downarrow \Theta \\
A \otimes M \otimes A & & \\
\end{array} \]

is commutative.

It follows from (1.2.1), (1.2.2) that:

\[ \Theta \] is an isomorphism.
Let us prove (1.2.3). Let \( \mu_{13} : A \otimes A \otimes A \to A \otimes A, \mu_{13}(a \otimes b \otimes c) = ac \otimes b \) and \( \Theta_{ij} = (\Theta_{ij})_{(\mu_{13})} \). Then: \( \Theta_{13} = \text{id}_{M \otimes A}, \Theta_{13} = \Theta, \Theta_{33} = \kappa \Theta \kappa^{-1}, \Theta_{13} \); therefore \( \Theta_{13} = \text{id}_{M \otimes A} \). Analogously \( \Theta_{\kappa} = \text{id}_{A \otimes M} \). Q.E.D.

(1.3) Grothendieck's descent theorem. Let \( k \to A \) be a morphism of rings and \( M_0 \) be a \( k \)-module. Then \( M = M_0 \otimes_k A \) is automatically equipped with a descent datum \( \Theta = \Theta_{M_0} \), relative to \( k \to A \), namely:

\[
\Theta_{M_0} : (M_0 \otimes_k A) \otimes_k A \to A \otimes_k (M_0 \otimes_k A)
\]

\[
(m \otimes a) \otimes b \mapsto a \otimes (m \otimes b)
\]

for \( m \in M_0, a, b \in A \).

Assume that \( k \to A \) is faithfully flat and let \( M, \Theta \) be as in (1.2). Then,

\[
(1.3.2)
M_0 = \{ m \in M | \Theta(m \otimes 1) = 1 \otimes m \}
\]

is a \( k \)-submodule of \( M, M = M_0 \otimes_k A \), and \( \Theta = \Theta_{M_0} \). Moreover, if \( (M, \Theta), (M', \Theta') \) are two data as in (1.2), if \( M_0 \) is given by (1.3.2) and \( \Theta_M \) is given analogously, then an \( A \)-linear morphism \( f : M \to M' \) is obtained by the scalar extension \( k \to A \) from a \( k \)-linear morphism \( \phi : M_0 \to M'_0 \) if and only if the following diagram commutes:

\[
\begin{array}{ccc}
M \otimes_k A & \xrightarrow{f \otimes \text{id}_A} & M' \otimes_k A \\
\phi & \downarrow & \phi' \\
A \otimes_k M & \xrightarrow{\text{id}_A \otimes f} & A \otimes_k M'.
\end{array}
\]

(See [2] for the proof).

Let \( k \) be a ring, \( A \) a commutative faithfully flat \( k \)-algebra and \( D \) a commutative (but not necessarily cocommutative) \( k \)-hyperalgebra. Let

\[
(1.4) u : A \to D \otimes_k A
\]

be a \( k \)-algebra morphism such that:

\[
(1.4.1) (P_D \otimes \text{id}_A) u = (\text{id}_D \otimes u) u;
\]

\[
(1.4.2) \text{if } p_A : A \to D \otimes A \text{ is defined by } p_A(a) = 1 \otimes a, \text{ then}
\]

\[
\chi = u \otimes p_A : A \otimes A \to D \otimes A
\]

is a \( k \)-algebra isomorphism.
From (1.4.1) and (1.4.2) it follows that

\[(1.4.3) \quad (e_p \otimes \text{id}_A)u = \text{id}_A.\]

Let us prove (1.4.3). Let \( f = (e_p \otimes \text{id}_A)u; \) we have: \( uf = (e_p \otimes u)u = (e_p \otimes \text{id}_D \otimes \text{id}_A)(\text{id}_B \otimes u)(e_p \otimes \text{id}_D)u = u; \) but \( u \) is injective. Q.E.D.

Let \( \tau \) denote the \( \chi \)-linear isomorphism:

\[(1.4.4) \quad \tau: A \otimes_k M \to D \otimes_k M \]
\[a \otimes m \mapsto u(a)(1 \otimes m).\]

Suppose we are given \( M, \Theta \) as in (1.2). Let us put:

\[(1.5) \quad \varphi: M \to D \otimes_k M \]
\[m \mapsto \tau(\Theta(m \otimes 1)).\]

Then:

\[(1.5.1) \quad \varphi(am) = \tau((a \otimes 1)\Theta(m \otimes 1)) = u(a)\varphi(m);\]

\[(1.5.2) \quad \text{Let } m \in M \text{ and } \Theta(m \otimes 1) = \sum_i a_i \otimes m_i. \text{ Then: } (e_p \otimes \text{id}_M)\varphi(m) = \sum_i (e_p \otimes \text{id}_M)\tau(a_i \otimes m_i) = \sum_i (e_p \otimes \text{id}_M)(u(a_i)(1 \otimes m_i)) = \sum_i (e_p \otimes \text{id}_A)(u(a_i))m_i = \sum_i a_i m_i = m, \quad (\text{the last equality follows from (1.2.1)});\]

\[(1.5.3) \quad \text{Let } m \in M, \Theta(m \otimes 1) = \sum_i a_i \otimes m_i, \Theta(m_i \otimes 1) = \sum_i a_{ij} \otimes m_{ij}.\]

According to (1.2.2) we have:

\[(1.5.3.1) \quad \sum_i a_i \otimes 1 \otimes m_i = \sum_{i,j} a_i \otimes a_{ij} \otimes m_{ij}.\]

Therefore:

\[(P_p \otimes \text{id}_M)\varphi(m) = (P_p \otimes \text{id}_M) \sum_i u(a_i)(1 \otimes m_i) = \]
\[\sum_i ((P_p \otimes \text{id}_A)(u(a_i))(1 \otimes 1 \otimes m_i) = \sum_i ((\text{id}_D \otimes u)u(a_i))(1 \otimes 1 \otimes m_i) = \]
\[(\text{id}_D \otimes \tau) \sum_i u(a_i) \otimes m_i = (\text{id}_D \otimes \tau) \sum_i (\chi \otimes \text{id}_M)(a_i \otimes 1 \otimes m_i) = \]
We conclude from the computations (1.5) that from a descent datum \( \Theta \) on \( M \) relatively to \( k \to A \), if the morphism \( u: A \to D \otimes_k A \), as in (1.4), is given, we obtain a \( k \)-linear morphism:

\[
\varphi: M \to D \otimes_k M
\]

satisfying:

\[
\begin{align*}
(1.6.1) \quad & \varphi(am) = u(a)\varphi(m), \quad \text{for } a \text{ in } A \text{ and } m \text{ in } M; \\
(1.6.2) \quad & (e_D \otimes \text{id}_M)\varphi = \text{id}_M; \\
(1.6.3) \quad & (P_D \otimes \text{id}_M)\varphi = (\text{id}_D \otimes \varphi)\varphi.
\end{align*}
\]

Conversely, let \( \varphi \) as in (1.6) be given, and define \( \Theta: M \otimes A \to A \otimes M \), by:

\[
(1.7) \quad \Theta(m \otimes a) = \tau^{-1}(1 \otimes a)\varphi(m).
\]

Then \( \Theta \) is \( A \otimes A \)-linear, and:

\[
(1.7.1) \quad \text{if } m \in M \text{ and } \Theta(m \otimes 1) = \tau^{-1}\varphi(m) = \sum_i a_i \otimes m_i, \text{ then } \varphi(m) = \sum_i u(a_i)(1 \otimes m_i) \text{ so that } \sum_i a_i m_i = (e_D \otimes \text{id}_M) \sum_i u(a_i)(1 \otimes m_i) =
\]

\[
(e_D \otimes \text{id}_M)\varphi(m) = m;
\]

\[
(1.7.2) \quad \text{if } \Theta(m \otimes 1) = \sum_i a_i \otimes m_i \text{ and } \Theta(m_i \otimes 1) = \sum_i a_{ii} \otimes m_{ii}, \text{ by inverting the reasoning used in (1.5.3) one proves that } \sum_i a_i \otimes 1 \otimes m_i = \sum_{i,j} a_{ij} \otimes m_{ij}, \text{ and therefore (1.2.2) for this } \Theta.
\]

It follows from Grothendieck's theory of descent (1.3), that if \( M, \varphi \) are as in (1.6) and one puts:

\[
(1.8) \quad M_0 = \{ m \in M / \varphi(m) = 1 \otimes m \}
\]
then \( M = A \otimes_k M_a \) and \( \varphi(a \otimes m) = u(a)(1 \otimes m) = u(a) \otimes m \). If now we denote \( \varphi \) by \( \varphi_M \) and let \((N, \varphi_N)\) be a datum analogous to \((M, \varphi_M)\), a \( A \)-linear homomorphism \( f: M \to N \) is the extension by \( A \)-linearity of a \( k \)-linear homomorphism \( f_a: M_a \to N_a \) iff the following diagram commutes:

\[
\begin{array}{ccc}
M & \xrightarrow{f} & N \\
\downarrow{\varphi_M} & & \downarrow{\varphi_N} \\
D \otimes_k M & \xrightarrow{\text{id} \otimes f} & D \otimes_k N.
\end{array}
\]

(1.8.1)

We will assume in the rest of this section that \( D \), as a \( k \)-module, is the direct limit of a direct system of finite locally free \( k \)-modules, say \( D = \lim \frac{D_n}{\xrightarrow{\alpha}} \). Then if \( k \) is given the discrete topology and the \( k \)-modules \( C_n = \text{Hom}_k(D_n, k) \), \( C = \text{Hom}_k(D, k) \) are given the usual topology, \( C = \lim \frac{C_n}{\xrightarrow{\alpha}} \), with the inverse limit topology (the topology of \( C_n \) is the discrete). Besides \( \text{Hom}_k(D \otimes D_n, k) = C_n \otimes_k C_n \) and \( \text{Hom}_k(D \otimes D_n, k) \), with the usual topology, equals \( \lim \frac{C_n \otimes C_n}{\xrightarrow{\alpha}} \). Under our assumptions, \( C \), endowed with the operations dualizing those of \( D \), is a l.t. \( k \)-hyperalgebra (commutative but not necessarily commutative) called the Cartier dual of \( D \). Let us put:

\[
S: C \to \text{End}_k D
\]

\[
c \mapsto S_c , \quad \text{where}
\]

\[
S_c(d) = (c \otimes \text{id}_D) P_d d , \quad \text{for} \ d \in D ,
\]

a continuous injective antirepresentation of \( k \)-algebras (the topology of \( D \) is the discrete). If \( c \in C \) and \( d \in D \) we will denote \( c(d) \in k \) by \( c \circ d \) and \( S_c(d) \in D \) by \( cd \); the previous formula then reads:

\[
(1.9.1) \quad cd = (c \otimes \text{id}_D) P_d d .
\]

One can prove that:

\[
P_d(cd) = (c \otimes \text{id}_D) P_d d
\]

\[
c(dd') = \mu_D((P_d c)(d \otimes d'))
\]

(1.9.2)

\[
(c' d) = c'(cd)
\]

\[
c \circ d = \varepsilon_D(cd)
\]
for $c, c' \in C$ and $d, d' \in D$. The second formula in (1.9.2), as many similar formulas to come, should be interpreted as follows. Suppose $P_c d = \sum_{i,h} c_i \otimes c_h$ (a converging sum in $C \otimes_k C$); then $(P_c d)(d \otimes d') = \sum_{i,h} c_i d \otimes c_h d' = \sum S_{c_i}(d) \otimes S_{c_h}(d')$ (a finite sum in $D \otimes_k D$). Therefore $c(d d') = \sum_{i,h} (c_i d)(c_h d')$.

Furthermore, if $D$ is free with basis $\{d_i, i \in I\}$ over $k$ and if $\{c_i, i \in I\}$ denotes the dual topological $k$-basis of $C$, we have:

\begin{equation}
P_D d = \sum_{i \in I} d_i \otimes c_i d.
\end{equation}

Given $u: A \to D \otimes A$ as in (1.4), let us put:

\begin{equation}
T: C \to \text{End}_k A,
\end{equation}

where $T_c(a) = (c \otimes \text{id}_A)u(a)$, for $a \in A$.

Again, $T$ is a continuous antirepresentation of $k$-algebras (the topology of $A$ being the discrete). If $T_c(a)$ is denoted by $ca$, one has, for $c \in C$ and $a, a' \in A$:

\begin{equation}
u(ea) = (c \otimes \text{id}_A)u(a)\end{equation}

\begin{equation}c(aa') = \mu_A((P_c e)(a \otimes a')).\end{equation}

The second formula in (1.12) means that $c(aa') = \sum_{i,h} (c_i a)(c_h a')$, if $P_c e = \sum_{i,h} c_i \otimes c_h$.

If $D$ is free we have again:

\begin{equation}u(a) = \sum_{i \in I} d_i \otimes c_i a.\end{equation}

Analogously, given $\varphi: M \to D \otimes M$ as in (1.6), we define:

\begin{equation}
U: C \to \text{End}_k M,
\end{equation}

where $U_c(m) = (c \otimes \text{id}_M)\varphi(m)$, for $m \in M$.

Once again, $U$ is a continuous antirepresentation of $k$-algebras (the topology of $M$ being the discrete) and, after writing $em$ for $U_c(m)$, it satisfies:

\begin{equation}\varphi(em) = (c \otimes \text{id}_M)\varphi(m)\end{equation}

\begin{equation}c(am) = \mu_M((P_c e)(a \otimes m)).\end{equation}
for any \( c \in C, a \in A, m \in M \), where \( \mu_{\sigma} : A \otimes_k M \to M \) denotes the scalar products. So \( c(am) = \sum_{j,h} (c_j a)(c_h m) = \sum_{j,h} T_{c_j}(a) U_{c_h}(m) \), if \( P_c a = \sum_{j,h} c_j \otimes c_h \).

If \( D \) is free, we have again:

\[
q(m) = \sum_{i \in I} d_i \otimes c_i m.
\]

Conversely, let \( U : C \to \text{End}_k M \), for a (discrete) \( A \)-module \( M \), be a continuous antirepresentation of \( k \)-algebras satisfying the second formula in (1.15); assume that \( D \) is free. Then \( q : M \to D \otimes M \) defined by (1.16) satisfies (1.6.1, 2, 3). Formula (1.7) now becomes:

\[
M_q = \{ m \in M \mid cm = 0, \text{ for all } c \in C^+ \}.
\]

Clearly:

\[
M_q = \{ m \in M \mid cm = (e_c c)m, \text{ for all } c \in C \},
\]

and, if \( c \in C, m = m_0 \otimes a \in M = M_0 \otimes_k A \), then:

\[
cm = U_c(m) = m_0 \otimes T_c(a) = m_0 \otimes ca.
\]

As a consequence of the considerations above we will say that the map \( U \) of (1.14) is a descent datum on \( M \) relatively to \( k \to A \). Let now \( M \) and \( N \) be two \( A \)-modules with descent data relatively to \( k \to A \). An \( A \)-linear map \( f : M \to N \) is the extension by \( A \)-linearity of a \( k \)-linear map \( f_0 : M_0 \to N_0 \) iff:

\[
f(cm) = cf(m), \quad \text{for all } c \in C \text{ and } m \in M.
\]

One verifies immediately that the induced descent data on \( M \otimes_A N \) and \( \text{Hom}_A(M, N) \) can be respectively expressed as follows:

\[
c(m \otimes n) = (P_c)(m \otimes n),
\]

\[
(cf)(m) = \sum_{i \in I} c_i \left( f((c_{i,c} c_{i,n}) m) \right),
\]

if \( c \in C, P_c = \sum_{i,h} c_i \otimes c_h \) (a converging series in \( C \otimes C \)), \( m \in M, n \in N \), \( f \in \text{Hom}_A(M, N) \). The right-hand term in (1.21) is to be interpreted in the following way. Let \( t : M \otimes_k N \to M \otimes_A N \) be the canonical map. Then \( c(m \otimes n) = c(m \otimes_A n) = t((P_c)(m \otimes_k n)) = \sum_{i,h} (c_i m)(c_h n) \) (a finite sum in \( M \otimes_A N \)). Notice that this is a good definition.
2. Let $k$ be a perfect field of characteristic $p \neq 0$ and, for $n \in \mathbb{N}$, let $K_n = W_n(k)$ be the ring of Witt vectors (relative to the prime number $p$) of length $n$ with components in $k$; in particular $K_1 = k$. Let $K_n[x]$ be the affine algebra of the standard multiplicative formal group $G_{K_n}$ over $K_n$: $K_n[x]$ is endowed with the $(x)$-adic topology and it is the l.t. $K_n$-hyperalgebra ($K_n$ is discrete) whose coproduct $\Delta$ and augmentation $\varepsilon$ are given by:

\[(2.1)\]  

$\Delta(x) = 1 \otimes x + x \otimes 1 + x \otimes x, \quad \varepsilon(x) = 0.$

For any $m$ in $\mathbb{N}$, the multiplication by $p^m$ of $G_{K_n}$ (in additive notation) is expressed by the continuous morphism of $K_n$-algebras:

\[(2.2)\]  

$P_m: K_n[x] \to K_n[x]$  

$x \mapsto (1 + x)^{p^m} - 1.$

In fact $P_m$ is an injective homomorphism of l.t. $K_n$-hyperalgebras. Let us regard $P_m$ as an embedding of $K_n[x]$ in another copy of itself that we denote by $K_n[x_m]$; namely we put:

\[(2.3)\]  

$P_m: K_n[x] \hookrightarrow K_n[x_m]$  

$P_m(x) = x = (1 + x_m)^{p^m} - 1.$

One immediately checks that $K_n[x_m]$ is freely generated as a $K_n[x]$-module by $\{1, x_m, x_m^2, \ldots, x_m^{p^m-1}\}$. Let us denote by $K_n[p^{-m}\mathbb{Z}_p/\mathbb{Z}_p]$ the group hyperalgebra of the group $p^{-m}\mathbb{Z}_p/\mathbb{Z}_p$ (that is the Cartier dual of its affine algebra). Explicitly we have: $K_n[p^{-m}\mathbb{Z}_p/\mathbb{Z}_p]$ is the free $K_n$-module generated by the symbols $\{Y_g, g \in p^{-m}\mathbb{Z}_p/\mathbb{Z}_p\}$, and:

\[(2.4)\]  

$Y_g Y_h = Y_{g+h}$  

$P Y_g = Y_g \otimes Y_g$  

$\varepsilon Y_g = 1$

for any $g, h$ in $p^{-m}\mathbb{Z}_p/\mathbb{Z}_p$. It is clear that the $K_n$-module homomorphism:

\[(2.5)\]  

$K_n[x_m] \otimes_{K_n[p^{-m}\mathbb{Z}_p/\mathbb{Z}_p]} K_n \to K_n[p^{-m}\mathbb{Z}_p/\mathbb{Z}_p]$  

$(1 \otimes x_m) \otimes 1 \mapsto Y_{\bar{x}_m \otimes 1}$
is an isomorphism of $K_n$-hyperalgebras (notice that $K_n[x_m] \otimes_{K_n} K_n$ is naturally a $K_n$-hyperalgebra). We deduce from (2.5) a surjective morphism of l.t. $K_n$-hyperalgebras ($K_n[p^{-m}Z_p/Z_p]$ is discrete), with kernel $xK_n[x_m]$:

$$\sigma_m: K_n[x_m] \to K_n[p^{-m}Z_p/Z_p]$$

$$x_m \mapsto Y_{p^{-m}Z_p} - 1.$$

If we denote by $\varphi^n G_{K_n}$ the finite multiplicative $K_n$-group whose affine algebra is $K_n[p^{-m}Z_p/Z_p]$, we have proved above that the sequence:

$$0 \to \varphi^n G_{K_n} \to G_{K_n} \xrightarrow{\varphi} G_{K_n} \to 0$$

is exact (in the category of (faithfully) flat sheaves of abelian groups on finite $K_n$-algebras).

Let us define now:

$$u_m: K_n[x_m] \to K_n[p^{-m}Z_p/Z_p] \otimes_{K_n} K_n[x_m]$$

$$f \mapsto (\sigma_m \otimes \text{id}) f$$

(notice that, since $K_n[p^{-m}Z_p/Z_p]$ is a finite $K_n$-module and $K_n[x_m]$ is complete, we could replace $z$ by $\otimes$ in (2.8)). Clearly, $u_m$ is a $K_n[x]$-algebra morphism and it is determined, as a $K_n[x]$-linear map, by:

$$(2.8.1) \quad u_m(1 + x_m)^a = Y_{a p^{-m}Z_p} \otimes (1 + x_m)^a, \quad \text{for } a \in \mathbb{N}.$$

We would like to prove that $u_m$ satisfies to the properties required for $u$ in (1.4), with the following replacements: (in the left-hand column of (2.9) find the symbols of section 1 while in the right-hand one find the symbols replacing them)

$$k \quad , \quad K_n[x]$$
$$A \quad , \quad K_n[x_m]$$
$$D \quad , \quad K_n[x] \otimes_{K_n} K_n[p^{-m}Z_p/Z_p]$$
$$D \otimes A \quad , \quad K_n[p^{-m}Z_p/Z_p] \otimes_{K_n} K_n[x_m].$$

In the first place since $K_n[x] \hookrightarrow K_n[x_m]$ is free, it is also faithfully flat. (1.4.1) is obvious. Let us check (1.4.2). We observe first that:

$$l: K_n[x_m] \otimes_{K_n} K_n[x_m] \to K_n[x_m] \otimes_{K_n} K_n[x_m]$$

$$f \otimes g \to (f)(1 \otimes g)$$

(2.10)
is a continuous \( K_n \)-algebra isomorphism. The inverse of \( t \) is:

(2.11) \[ r: f \otimes g \rightarrow ((\text{id} \otimes q) \mathcal{P} f) (1 \otimes g). \]

Let us now consider the diagram:

\[
\begin{array}{ccc}
K_n[x_m] \otimes_{K_n} K_n[x_m] & \xrightarrow{\text{can}} & K_n[x_m] \\
\downarrow{\text{can.}} & & \downarrow{1} \\
K_n[x_m] \otimes_{K_n[x]} K_n[x_m] & \xrightarrow{1} & K_n[p^{-n}Z_p/Z_p] \otimes_{K_n} K_n[x_m]
\end{array}
\]

(2.12)

If we prove that in (2.12) the barred arrows exist in such a way that the resulting diagram is commutative, (1.4.2) will follow for \( u_m \). We would have in fact then \( \bar{t} (f \otimes g) = u_m(f)(1 \otimes g) \) and \( \bar{t} \bar{t} = \text{id}_{K_n[x]} \otimes_{K_n[x]} K_n[x_m] \). To show the existence of \( \bar{t} \) it is enough to prove that \( (\sigma_m \otimes \text{id}) \bar{t} (f \otimes g) = (\sigma_m \otimes \text{id}) f (g \otimes h) \), for any \( f \) in \( K_n[x] \) and \( g, h \) in \( K_n[x_m] \). Now \( \bar{t} \) is right \( K_n[x_m] \)-linear so that we can put \( h = 1 \). We have to prove that: \( (\sigma_m \otimes \text{id})(f) (g \otimes h) = (1 \otimes f) (\sigma_m \otimes \text{id}) g \), if \( f \in K_n[x] \) and \( g \in K_n[x_m] \). This follows from the fact that the kernel of \( \sigma_m \) is \( xK_n[x_m] \).

For the existence of \( \bar{r} \), it is enough to prove that \( r(x \otimes 1) \) has zero image in \( K_n[x_m] \otimes_{K_n[x]} K_n[x_m] \). Now we have \( r(x \otimes 1) = (\text{id} \otimes q) \mathcal{P} x \in K_n[x] \otimes_{K_n[x]} K_n[x] \); its image in \( K_n[x_m] \otimes_{K_n[x]} K_n[x_m] \) coincides with \( \mu_{K_n[x]}(\text{id} \otimes q) \mathcal{P} x = ix = 0 \) (\( i = i_{K_n[x]} \) is as usual the structural morphism of \( K_n[x] \)).

We conclude that the map \( u_m \) defined in (2.8) is a \( K_n[x] \)-algebra homomorphism:

(2.13) \[ u_m: K_n[x_m] \rightarrow K_n[p^{-n}Z_p/Z_p] \otimes_{K_n} K_n[x_m] \]

such that:

(2.13.1) \[ (\text{id} \otimes u_m) u_m = (\mathcal{P} \otimes \text{id}) u_m \]

and

(2.13.2) \[ \text{the map: } K_n[x_m] \otimes_{K_n[x]} K_n[x_m] \rightarrow K_n[p^{-n}Z_p/Z_p] \otimes_{K_n} K_n[x_m] \]

\[ f \otimes g \rightarrow u_m(f)(1 \otimes g) \]

is an isomorphism of \( K_n[x] \)-algebras.

We are exactly in the situation of section 1 with the substitutions indicated in (2.9).
At this point we need to make a typographical specification: the \( x_m \) belonging to \( K_n[x_m] \) will be denoted by \( x_m^{(n)} \); the symbol \( x_m \) will denote only \( x_m^{(1)} \), and we also write \( x \) for \( x_0 \), so that \( x_m = x^{m}_m \) for \( m \) in \( N \). We also put \( x_0^{(n)} = x^{(n)} \). Let \( A_n = \bigcup_{m=0}^{+\infty} K_n[x_m^{(n)}] \). We want to prove that \( A_n = W_n(A_1) \), where \( A_1 \) obviously coincides with the perfectionate \( k[[x^{m}_m]] \) of \( k[x] \). Let us denote by \( \varphi_{n,m} \) the continuous ring homomorphism \( K_{n+1}[x_m^{(n+1)}] \to K_n[x_m^{(n)}] \) extending the natural map (reduction modulo \( p^n \)) \( K_{n+1} \to K_n \), such that \( \varphi_{n,m}(x_m^{(n+1)}) = x_m^{(n)} \). Let then \( \varphi_n: A_{n+1} \to A_n \) be defined by \( \varphi_n(a) = \varphi_{n,m}(a) \) if \( a \in K_{n+1}[x_m^{(n+1)}] \). Let us regard \( A_n \) as a discrete l.t. ring. Then \( A = \lim \limits_{\to n} (A_n, \varphi_n) \) is a strict \( p \)-ring in the sense of [1], chap. II, sect. 5, and it coincides then with \( W(A_1) \). It follows that \( A_n = W_n(A_1) \). If the symbol \( [a] \) denotes the multiplicative representative in \( W_n(A_1) \) of \( a \in A_1 \), we have \( [1 + x] = 1 + x^{(n)} \) and, in general, \( [1 + x^{p^m}] = 1 + x^{(n)}_m \), in the identification above. A word of caution on the embedding of \( K_n[x_m^{(n)}] \) in \( W_n(A_1) \). Let us ( provisionally ) topologize \( A_1 \) with the \((x)\)-adic topology and \( W_n(A_1) \) (in \( 1 - 1 \) correspondence with \( A_1^{\star} \)) with the product topology (notice that this topology coincides with the \( ([x])\)-adic). Then the embedding above is characterized as a continuous \( K_n \)-algebra morphism \( (K_n[x_m^{(n)}], \varphi_{n,m}^{(n)}) \) being endowed with the \((x)\)-adic topology by the assignment \( x_m^{(n)} \mapsto [1 + x^{p^m}] - 1 \). In the sequel, no topology will be given to \( K_n[x_m^{(n)}] \) or to \( W_n(A_1) \), but by \( * \) the embedding \( x_m^{(n)} \mapsto [1 + x^{p^m}] - 1 \), we will always mean the one described above.

Let us fix \( n \) and put \( x_m^{(n)} = x_m \), \( x_0 = x \). By taking direct limits in (2.13) we get a morphism of \( K_n[X] \)-algebras:

\[
(2.14) \quad u: W_n(A_1) \to K_n[Q_n/Z_n] \otimes_{K_n} W_n(A_1),
\]

where \( K_n[Q_n/Z_n] \) is the free \( K_n \)-module generated by \( \{Y_p, g \in Q_n/Z_n\} \), endowed with the hyperalgebra operations defined by formulas (2.4). The morphism of \( K_n[X] \)-algebras \( u \) is determined by the relations:

\[
(2.14.0) \quad u([1 + x^{p^m}]) = Y_{p^m} \otimes [1 + x^{p^m}],
\]

for \( m \in N \). It satisfies:

\[
(2.14.1) \quad (P \otimes \text{id}) u = (\text{id} \otimes u) u;
\]

\[
(2.14.2) \quad W_n(A_1) \otimes_{K_n} W_n(A_1) \to K_n[Q_n/Z_n] \otimes_{K_n} W_n(A_1),
\]

\[
f \otimes g \mapsto u(f)(1 \otimes g),
\]

is an isomorphism of \( K_n[X] \)-algebras.
Moreover the morphism $K_n[[X]] \rightarrow W_n(A_i)$ is faithfully flat. (The facts just stated follow from the properties of direct limits). We are then in a position to apply the theory of section 1 to get information on the descent relatively to $K_n[[X]] \rightarrow W_n(k[[x^{p^n\infty}]])$, $X \mapsto [1 + x] - 1$.

Let us carry out in detail the constructions of section 1 for the present data. The next diagram indicates the replacements to be operated (as before the right-hand column replaces the left-hand one):

\[
\begin{align*}
k & \rightarrow A, \quad K_n[[X]] \\
A & \rightarrow W_n(k[[x^{p^n\infty}]]) \\
D & \rightarrow K_n[[X]] \otimes_{K_n} K_n[Q_0/Z_p] \\
D \otimes A & \rightarrow K_n[Q_0/Z_p] \otimes_{K_n} W_n(k[[x^{p^n\infty}]])
\end{align*}
\]

(2.15)

Let $F_n = \text{Hom}_{K_n}(K_n[Q_0/Z_p], K_n)$ be the l.t. $K_n$-hyperalgebra Cartier dual to $K_n[Q_0/Z_p]$. We can obviously identify $F_n$ with the l.t. $K_n$-hyperalgebra of functions defined on the group $Q_0/Z_p$ taking values in $K_n$, endowed with the topology of simple convergence on $Q_0/Z_p$ with respect to the discrete topology of $K_n$. (We recall that $P: F_n \rightarrow F_n \otimes K_n$ is defined by identifying $F_n \otimes K_n$ with the $K_n$-algebra of functions defined on the group $Q_0/Z_p$ taking values in $K_n$, endowed with the topology of simple convergence, and by putting $(Pf)(a, b) = f(a + b)$, for $f$ in $F_n$ and $a, b$ in $Q_0/Z_p$, and $ef = 0$.)

Such an identification is obtained by interpreting $f: Q_0/Z_p \rightarrow K_n$ as the $K_n$-linear map: $\sum_{i \in Q_0/Z_p} a_i Y_i \mapsto \sum_{i \in Q_0/Z_p} a_i f(i)$, from $K_n[Q_0/Z_p]$ to $K_n$. Notice that $F_n$ can naturally be identified with $W_n(F_1)$ as a $K_n$-algebra: $f \in F_n$ is identified with $(f_0, ..., f_{n-1}) \in W_n(F_1)$ if $f_i = c_i f$, where for $i = 0, ..., n - 1$, $c_i: K_n \rightarrow K_n(k) \rightarrow k$ is the function « $i$-th component » of a Witt vector. The topology of $F_n$ corresponds then to the product topology of the topology of $F_1$, in the natural bijection $W_n(F_1) \leftrightarrow F_n$. We can also identify $W_n(F_1 \otimes K_n)$ with $W_n(F_1) \otimes_{K_n} W_n(F_1)$, since they are both isomorphic to the $K_n$-algebra of functions $f: Q_0/Z_p \times Q_0/Z_p \rightarrow K_n$ endowed with the topology of simple convergence. The coproduct of $F_n$ then corresponds to the map:

\[
W_n(P_{F_1}): W_n(F_1) \rightarrow W_n(F_1) \otimes_{K_n} W_n(F_1) = W_n(F_1) \otimes_{K_n} W_n(F_1)
\]

\[
(f_0, ..., f_{n-1}) \mapsto (P_{F_1}f_0, ..., P_{F_1}f_{n-1})
\]
To pursue the correspondence with section 1, we see that the l.t. $k$-hyperalgebra $C$ is now replaced by $K_n[X] \otimes_{E_n} F_n$ (or $K_n[X] \otimes_{E_n} W_n(F_1)$) where $\otimes$ is taken with respect to the discrete topology of $K_n[X]$. The representation $S$ of (1.9) is now the extension by $K_n[X]$-linearity of the $K_n$-linear continuous representation:

$$S': F_n \to \text{End}_{K_n} K_n[Q_0/Z_0]$$

(2.17) \hspace{1cm} f \mapsto S'_f, \quad \text{where}

$$S'_f(Y_g) = f(g) Y_g, \quad \text{for } g \in Q_0/Z_0.$$

Similarly the $T$ of (1.11) is the extension by $K_n[X]$-linearity of the $K_n$-linear continuous representation:

$$T': F_n \to \text{End}_{K_n[x]} W_n(k[[x^{-m}]])$$

(2.18) \hspace{1cm} f \mapsto T'_f, \quad \text{where}

$$T'_f((1 + X)^{a^n}) = f(a^m + Z_0)(1 + X)^{a^n}, \quad \text{for } a, m \in Z.$$

We leave to the reader the verification of the formulas in (2.17) and (2.18). We then conclude from section 1, that a descent datum on $\text{relatively to } \Gamma P_n \{ xP \} X \{ [1 + x] - 1,}$ is equivalent to a continuous representation of $K_n$-algebras:

$$U': F_n \to \text{End}_{K_n} M$$

(2.19) \hspace{1cm} f \mapsto U'_f

such that (after skipping the symbols $U'$, $T'$ and denoting by

$$\mu_m: W_n(k[[x^{-m}]]) \otimes_{E_n} M \to M$$

the scalar product):

$$d(rm) = \mu_m \left((\text{ad})r \otimes_{E_n} m\right),$$

(2.20) \hspace{1cm} \\

for any $d$ in $F_n$, $r$ in $W_n(k[[x^{-m}]])$, $m$ in $M$. Notice that each $U'_f$, for $f$ in $F_n$, is then in fact $K_n[X]$-linear.

Since $K_n[X] \hookrightarrow W_n(k[[x^{-m}]]))$ is faithfully flat, all descent data with respect to it are effective and therefore if one puts:

$$M_0 = \{ m \in M \cap \langle d \rangle = 0 \text{ for } d \in F_n^+ \},$$

(2.21) \hspace{1cm}
one concludes that $M_0$ is a $K_n[[X]]$-module, that $M = W_n(k[[x^{p^{-m}}]]) \otimes_{K_n[[X]]} M_0$, and that $d(r \otimes m) = d r \otimes m$ for each $d$ in $F_n$, $r$ in $W_n(k[[x^{p^{-m}}]])$, and $m$ in $M_0$.

Since $u_m$ in (2.13) is a $K_n[[X]]$-algebra morphism, we can extend it by $K_n[[X]][1/X]$-linearity to:

\[ u'_m : K_n[[X]][1/X] \to K_n[p^{-m}Z_p/Z_p] \otimes_{\mathbb{Q}_p} K_n[[X]][1/X] \]

satisfying:

\[ (id \otimes u'_m) u'_m = (P \otimes id) u'_m ; \]

\[ (2.22.2) \quad \text{the map:} \]

\[ K_n[[X]][1/X] \otimes_{K_n[[X]][1/X]} K_n[[X]][1/X] \to K_n[p^{-m}Z_p/Z_p] \otimes_{\mathbb{Q}_p} K_n[[X]][1/X] \]

\[ f \otimes g \mapsto u'_m(f)(1 \otimes g), \]

is an isomorphism of $K_n[[X]][1/X]$-algebras.

Moreover $K_n[[X]][1/X] \to K_n[[X]][1/X]$ is free and therefore faithfully flat. Notice that $1/X_n = (1 + (1 + X_n) + ... + (1 + X_n)^{p^{-m}-1})/X$ so that $K_n[[X]][1/X] = K_n[[X]][1/X]$. By passing to the direct limit for $m$ going to infinity, we get: $\lim_m K_n[[X]][1/X] = A_n[1/X] = W_n(k((x^{p^{-m}})))$, where $k((x^{p^{-m}}))$ denotes the perfect closure of the field $k((x))$. We obtain again a morphism of $K_n[[X]][1/X]$-algebras:

\[ u' : W_n(k((x^{p^{-m}}))) \to K_n[Q_p/Z_p] \otimes_{\mathbb{Q}_p} W_n(k((x^{p^{-m}}))) ; \]

given by:

\[ (2.23.0) \quad u'(1 + x^{p^{-m}}) = Y_{p^{-m}Z_p} \otimes [1 + x^{p^{-m}}] \quad \text{for } m \in \mathbb{N}, \]

and satisfying:

\[ (2.23.1) \quad (P \otimes id) u' = (id \otimes u') u' ; \]

\[ (2.23.2) \quad \text{the map:} \]

\[ W_n(k((x^{p^{-m}}))) \otimes_{K_n[[X]][1/X]} W_n(k((x^{p^{-m}}))) \to K_n[Q_p/Z_p] \otimes_{\mathbb{Q}_p} W_n(k((x^{p^{-m}}))), \]

\[ f \otimes g \mapsto u'(f)(1 \otimes g), \]

is an isomorphism of $K_n[[X]][1/X]$-algebras.
Moreover $K_n[X][1/X] \hookrightarrow W_n(k((x^{a^m})))$ is faithfully flat. We can therefore apply to the descent relatively to that extension the same criteria we proved for $K_n[X] \hookrightarrow W_n(k[[x^{a^m}]])$.

3. We keep the notation of section 2. Let us denote by $R$ the ring $k[[x]]$ and by $R'$ its perfectionate $k[[x^{a^m}]]$. Let $Q, Q'$ denote the quotient fields of $R, R'$, respectively. We also put $K = W(k) =$ the ring of infinite Witt vectors with components in $k$. Let $K[X]$ be the ring of formal power series in $X$ with coefficients in $K$; there is a unique morphism of $K$-algebras:

\[(3.1) \quad K[X] \rightarrow W(R') \]

sending $X$ to $[1 + x] - 1$, which is continuous for, say, the $(p, X)$-adic topology in $K[X]$ and the $(p, [x])$-adic one in $W(R')$. We will always regard $K[X]$ as embedded in $W(R')$ by means of (3.1). The embedding (3.1) can obviously be uniquely extended, as a ring homomorphism, to give an embedding:

\[(3.2) \quad K[X][1/X] \rightarrow W(Q'), \]

that we will always use in the sequel. Notice that (3.2) can again be extended by $p$-adic continuity, to an embedding of the $p$-adic completion $B$ of $K[X][1/X]$ in $W(Q')$:

\[(3.3) \quad B \rightarrow W(Q'). \]

The embeddings (3.1) and (3.3) reduce modulo $p^n$, to the embeddings used in section 2, for which we were able to give simple descent criteria. Let us restate those results in a more manageable form.

Formula (2.18) provides us with a map:

\[(3.4) \quad \begin{cases} F_n \times W_n(R') \rightarrow W_n(R') \\ (d, r) \mapsto T_d(r) = dr. \end{cases} \]

Analogously, using (2.22), we get a map (that extends (3.4)):

\[(3.5) \quad \begin{cases} F_n \times W_n(Q') \rightarrow W_n(Q') \\ (d, r) \mapsto dr. \end{cases} \]

The map (3.5) can be characterized by the properties:

\begin{align*}
(3.5.1) & \quad r \mapsto dr \quad \text{is} \quad K_n[X][1/X]-\text{linear;} \\
(3.5.2) & \quad d[(1 + x^{a^m})] = d(ap^m + Z_p)[(1 + x^{a^m})], \quad \text{for} \; a, m \in \mathbb{Z}.
\end{align*}
Moreover (3.5) makes $W_n(Q')$, endowed with the discrete topology, into a l.t. $F_n$-module and satisfies:

$$ (3.5.3) \quad d(rr') = \mu((P_d)(r \otimes r')) , \quad \text{for } d \in F_n, \ r, r' \in W_n(Q') . $$

The right-hand term in (3.5.3) is to be interpreted in the following way. Suppose $\mathbf{P}d = \sum_{i,j} d_i \otimes d_j$, a converging sum in $F_n \otimes_{\mathbb{K}_n} F_n$, then: $d(rr') = \sum_{i,j} (d_i r)(d_j r')$, a finite sum in $W_n(Q')$. A descent datum on a $W_n(R')$-(resp. $W_n(Q')$-) module $M$, relatively to $K_n[[X]] \hookrightarrow W_n(R')$ (resp. $K_n[[X]] \cdot [1/X] \hookrightarrow W_n(Q')$) is equivalent to a $\mathbb{K}_n$-bilinear map:

$$ (3.6) \quad \begin{align*}
F_n \times M & \rightarrow M \\
(d, m) & \mapsto d m
\end{align*} $$

making $M$, endowed with the discrete topology, into a topological $F_n$-module and satisfying:

$$ (3.7) \quad d(rm) = \mu_{\mathbb{K}_n}((\mathbf{P}d)(r \otimes m)) , $$

for $d \in F_n, r \in W_n(R')$ (resp. $W_n(Q')$), $m \in M$. Here, as usual, $\mu_{\mathbb{K}_n}: W_n(R') \otimes_{\mathbb{K}_n} M \rightarrow M$ (resp. $W_n(Q') \otimes_{\mathbb{K}_n} M \rightarrow M$) is the scalar product, and, if $\mathbf{P}d = \sum_{i,j} d_i \otimes d_j$, a converging sum in $F_n \otimes_{\mathbb{K}_n} F_n$, the right-hand term of (3.7) is to be interpreted as $\sum_{i,j} (d_i r)(d_j m)$ (a finite sum in $M$), through (3.5) and (3.6). Notice that $m \mapsto d m$ is then automatically $\mathbb{K}_n[[X]]$-(resp. $\mathbb{K}_n[[X]][1/X]$-) linear.

Let $F$ be the l.t. $K$-hyperalgebra ($K$ being endowed with the $p$-adic topology) of functions from $\mathbb{Q}_p/\mathbb{Z}_p$ to $K$, with the topology of simple convergence. A fundamental system of open $K$-submodules (ideals, in fact) of $F$ is given by the

$$ U_{m,n} = \{ f \in F | f(p^{-m}\mathbb{Z}_p/\mathbb{Z}_p) \subseteq p^n K \} , $$

as $m, n$ vary in $\mathbb{N}$. Clearly, $F = \varprojlim F_n$, as a topological ring. The identification $F_n \cong W_n(F_1)$ of section 2, now carries over to an identification $F \cong W(F_1)$, the last being equipped with the product topology of the topology of $F_1$. 

By taking inverse limits for \( n \to +\infty \) in (3.5), we obtain a map:

\[
\begin{align*}
E \times W(Q') & \to W(Q') \\
(d, r) & \mapsto \tilde{d}r
\end{align*}
\]

that can be characterized by the following properties (3.8.1) and (3.8.2):

(3.8.1) \( r \mapsto \tilde{d}r \) is \( B \)-linear;

(3.8.2) \( \tilde{d}((1 + x^{p^m})) = \tilde{d}(ap^m + Z_p[(1 + x)^{p^m}]) \), for \( a, m \in \mathbb{Z} \).

Moreover, the map (3.8) makes \( W(Q') \), endowed with the \( p \)-adic topology, into a topological \( F \)-module and satisfies:

\[
(3.8.3) \quad \tilde{d}(rr') = \mu((Pd)(r \otimes r')) , \quad \text{for } d \in F, \ r, r' \in W(Q').
\]

The right-hand term of (3.8.3) should be interpreted as follows. Let \( Pd = \sum_{i,j} d_i \otimes d_j \) (a converging sum in \( F \otimes K \)); then \( d(rr') = \sum_{i,j} (d_i r)(d_j r') \) (a \( p \)-adically convergent sum in \( W(Q') \)).

Let \( M \) be a \( W(R') \)-\( (\text{resp. } W(Q')-) \) module, \( p \)-adically separated and complete. Let

\[
(3.9) \quad \begin{align*}
F \times M & \to M \\
(d, m) & \mapsto dm
\end{align*}
\]

be a \( K \)-bilinear map, making \( M \), endowed with the \( p \)-adic topology, into a topological \( F \)-module, and satisfying:

\[
(3.10) \quad \tilde{d}(am) = \mu_{sc}((Pd)(a \otimes K m))
\]

for \( d \in F, \ a \in W(R') \) (resp. \( W(Q') \)), \( m \in M \). Here \( W(R') \otimes K M \) (resp. \( W(Q') \otimes K M \)) denotes the \( p \)-adic completion of \( W(R') \otimes K M \) (resp. \( W(Q') \otimes K M \)), \( \mu_{sc} : W(R') \otimes K M \to M \) (resp. \( W(Q') \otimes K M \to M \)) denotes the scalar product, and, if \( Pd = \sum_{i,j} d_i \otimes d_j \) (a converging sum in \( F \otimes K \)) the right-hand member of (3.10) is to be interpreted as \( \sum_{i,j} (d_i a)(d_j m) \) (a \( p \)-adically convergent sum in \( M \)) through (3.8) and (3.9).

It is clear that, by reduction modulo \( p^n \), the datum (3.9) satisfying (3.10), provides a series of compatible data on \( M/p^n M \) of the type (3.7).
We then easily conclude from the previous section that if we put:

$$M_0 = \{ m \in M/ \hat{d}m = 0 \text{ if } \hat{d}(0) = 0 \},$$

$M_0$ is a $K[X]$- (resp. $B$-) submodule of $M$, $M_0$ is $p$-adically separated and complete, $M = M_0 \hat{\otimes}_R W(Q')$ (resp. $M_0 \hat{\otimes}_R W(Q')$) where $\hat{\otimes}$ means $p$-adic completion of $\otimes$, and $\hat{d}(m \otimes a) = m \otimes \hat{d}a$ for $d \in F$, $m \in M_s$, $a \in W(F^*)$ (resp. $W(Q')$). Analogous results hold for the descent of morphisms of modules.

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