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# A Class of Pseudo Differential Operators with Multiple Non-Involutive Characteristics.

MARIA MASCARELLO RODINO - LUIGI RODINO

## Introduction.

Let  $\Omega$  be an open subset of  $\mathbf{R}^n$ , let  $\Sigma_1, \Sigma_2$  be conic submanifolds of codimension 1 of  $\Omega \times \mathbf{R}^n$  and let  $\rho = (x_0, \xi_0)$ ,  $\xi_0 \neq 0$ , be a point of  $\Sigma = \Sigma_1 \cap \Sigma_2 \neq \emptyset$ . Assume that  $\Sigma$  is non-involutive at  $\rho$ ; precisely speaking: if  $\Sigma_1, \Sigma_2$  are locally given by  $u_1(x, \xi) = 0$ ,  $u_2(x, \xi) = 0$ , where  $u_1, u_2$  are smooth real valued positively homogeneous, then we have  $u_1(\rho) = 0$ ,  $u_2(\rho) = 0$  and

$$(0.1) \quad \{u_1, u_2\}(\rho) = \sum_{i=1}^n \left( \frac{\partial u_1}{\partial \xi_i}(\rho) \frac{\partial u_2}{\partial x_i}(\rho) - \frac{\partial u_1}{\partial x_i}(\rho) \frac{\partial u_2}{\partial \xi_i}(\rho) \right) \neq 0.$$

Let  $M \geq 0, k \geq 1$  be fixed integers. For  $m \in \mathbf{R}$  we define  $A_k^{m, M}(\Sigma_1, \Sigma_2, \rho)$  to be the class of (germs of) symbols  $p(x, \xi) \in S^m(\Omega \times \mathbf{R}^n)$  at  $\rho$ ,

$$(0.2) \quad p(x, \xi) \sim \sum_{j=0}^{\infty} p_{m-j}(x, \xi),$$

$p_{m-j}$  positively homogeneous of degree  $m-j$ , having the property that in some conic neighborhood  $\Gamma$  of  $\rho$

$$(0.3) \quad |p_{m-j}(x, \xi)| / |\xi|^{m-j} \leq C(d_{\Sigma_1}(x, \xi) + d_{\Sigma_2}^k(x, \xi))^{M-j(k+1)/k},$$

$$0 \leq j \leq Mk/(k+1),$$

$$(0.4) \quad |p_m(x, \xi)| / |\xi|^m \geq C^{-1}(d_{\Sigma_1}(x, \xi) + d_{\Sigma_2}^k(x, \xi))^M.$$

Here  $d_{\Sigma_1}, d_{\Sigma_2}$  are the distances from  $(x, \xi/|\xi|)$  to  $\Sigma_1, \Sigma_2$ , respectively, and we understand  $|\xi| > 1$ ;  $C$  is a suitable real constant which depends only

on  $\Gamma$ . We have in particular that in  $\Gamma$  the characteristic manifold of  $p_m$  is given by  $\Sigma = \Sigma_1 \cap \Sigma_2$ , and  $p_m$  vanishes on  $\Sigma \cap (\Omega \times \mathbf{S}^{n-1})$  exactly as  $(d_{\Sigma_1} + d_{\Sigma_2}^k)^M$ .

Observe that  $A_k^{m,M}(\Sigma_1, \Sigma_2, \varrho) = A_k^{m,M}(\Sigma'_1, \Sigma'_2, \varrho)$  whenever  $\Sigma = \Sigma'_1 \cap \Sigma'_2 = \Sigma_1 \cap \Sigma_2$  is non-involutive at  $\varrho$  and in a conic neighborhood of  $\varrho$   $\Sigma'_1, \Sigma'_2$  have on  $\Sigma$  an osculation of degree  $k - 1$ . For  $k = 1$  one can simply replace the sum  $d_{\Sigma_1} + d_{\Sigma_2}$  in (0.3), (0.4) with  $d_{\Sigma}$ , the distance from  $(x, \xi/|\xi|)$  to  $\Sigma$ . However, to argue on a fixed choice of  $\Sigma_1, \Sigma_2$  will make things easier in the following (see Helffer - Nourrigat [24] for other classes in a similar geometrical framework).

We want to give explicit necessary and sufficient conditions for the hypoellipticity with loss of  $Mk/(k + 1)$  derivatives at  $\varrho$  of an arbitrary properly supported pseudo differential operator  $P = p(x, D)$  with symbol  $p(x, \xi) \in A_k^{m,M}(\Sigma_1, \Sigma_2, \varrho)$ . Namely, we shall discuss for  $P$  the property:

(0.5) *there are  $\varphi(x, \xi), \varphi'(x, \xi) \in \mathcal{S}^0(\Omega \times \mathbf{R}^n)$ , with  $\varphi = 1$  in a conic neighborhood of  $\varrho, \varphi' \neq 0$  on the support of  $\varphi$ , and the support of  $\varphi'$  arbitrarily close to the support of  $\varphi$ , such that the estimates*

$$\begin{aligned} \|\varphi(x, D)f\|_{s+m-Mk/(k+1)} &\leq \\ &\leq C_s \{ \|\varphi'(x, D)Pf\|_s + \|f\|_{s+m-Mk/(k+1)-\varepsilon} \}, \quad f \in C_0^\infty(\Omega), \end{aligned}$$

*are valid, for some  $\varepsilon > 0$ .*

As we shall see in Theorem 2.1, for every  $P$  with symbol in  $A_k^{m,M}(\Sigma_1, \Sigma_2, \varrho)$  the property (0.5) implies the existence of a left parametrix and the hypoellipticity at  $\varrho$ .

In the case  $M = 1$ , (0.5) is a microlocalized version of the so-called subelliptic estimates of Egorov [4], Trèves [21], Hörmander [10]; actually, the symbols of a suitable class of subelliptic pseudo differential operators are in  $A_k^{m,1}(\Sigma_1, \Sigma_2, \varrho)$  at every point  $\varrho$  of their characteristic manifold, for an appropriate choice of  $k, \Sigma_1, \Sigma_2$  (see Corollary 6.2). Operators with symbol in  $A_k^{m,2}(\Sigma_1, \Sigma_2, \varrho)$  are studied in Menikoff [14], Boutet de Monvel-Trèves [3], Yamamoto [23]. The case when  $k = 1$  and  $M$  is arbitrary is considered in Sjöstrand [19], Rodino [17], Mascarello-Rodino [13], Helffer - Rodino [8], [9] and, under general hypotheses on  $\Sigma$ , in Boutet de Monvel - Grigis - Helffer [2].

An implicit result for  $k > 1, M > 2$  is known when  $\Sigma_1, \Sigma_2, \Sigma$  are « flat »:

$$(0.6) \quad \Sigma_1 = \{\xi_n = 0\}, \quad \Sigma_2 = \{x_n = 0\}, \quad \Sigma = \{x_n = \xi_n = 0\};$$

see for example Grušin [6], Nourrigat [15], [25], Parenti - Rodino [16]. In particular, in [16] general classes of anisotropic symbols are considered with a flat symplectic characteristic manifold  $\Sigma$  of arbitrary codimension (« symplectic » is equivalent to « non-involutive ») and the validity of (0.5) is related to the injectivity on the Schwartz space  $\mathcal{S}$  of an auxiliary operator  $\mathcal{L}_p$  with polynomial coefficients; moreover, parametrices are constructed and the wave front set of the related kernel is estimated.

With respect to [16] here we shall go forward in two directions. First in Chapter I (Sections 1, 2) we shall extend the implicit result to operators with symbol in  $A_k^{m,M}(\Sigma_1, \Sigma_2, \varrho)$ , where  $\Sigma_1, \Sigma_2$  are not flat, in general. This will be an easy application of the Fourier integral operators of Hörmander [11]; in fact, our very definition in (0.3), (0.4) suggests the choice of new canonical coordinates. Then in Chapter II (Sections 3, 4, 5) we shall give necessary and sufficient conditions for the injectivity of  $\mathcal{L}_p$ ; actually, for symbols in  $A_k^{m,M}(\Sigma_1, \Sigma_2, \varrho)$ ,  $\mathcal{L}_p$  comes down to an ordinary differential operator, which we shall study by means of the theory of the Meijer's  $\mathcal{G}$ -functions and the classical methods of the asymptotic integration. In Chapter III (Sections 6, 7) combining the results of Chapter I and II we shall obtain the desired explicit conditions for the validity of (0.5).

## CHAPTER I

### A GENERAL IMPLICIT RESULT

#### 1. - Preliminaries.

Let  $\Sigma_1, \Sigma_2$  be locally given by  $u_1(x, \xi) = 0, u_2(x, \xi) = 0$ , as in the Introduction; in the following we shall suppose  $u_1, u_2$  positively homogeneous of degree 1, 0, respectively, and such that

$$(1.1) \quad \{u_1, u_2\}(\varrho) > 0.$$

Observe that the differential forms  $du_1, du_2, \sum_{t=1}^n \xi_t dx_t$  are linearly independent at  $\varrho$ ; this is a direct consequence of (1.1), or (0.1). Let  $p(x, \xi)$  be in  $A_k^{m,M}(\Sigma_1, \Sigma_2, \varrho)$ ; using Taylor's formula we have from (0.3)

$$(1.2) \quad p_m(x, \xi) = \sum_{\alpha/k + \beta = M} \sigma_{\alpha\beta}(x, \xi) u_2^\alpha(x, \xi) u_1^\beta(x, \xi).$$

In the sum  $\alpha \geq 0, \beta \geq 0$  are integers and the functions  $\sigma_{\alpha\beta}$  are smooth positively homogeneous of degree  $m - \beta$  (here, as in the following, we argue on a sufficiently small conic neighborhood of  $\rho$ ). Consider the  $M$  roots  $r_1, \dots, r_M$  of the equation

$$(1.3) \quad \sum_{\alpha/k + \beta = M} \alpha_{\alpha\beta}(\rho) r_\beta = 0 .$$

In view of (0.4) we have  $\text{Im } r_v \neq 0$  for every  $v, 1 \leq v \leq M$ ; suppose for example  $\text{Im } r_1 \geq \dots \geq \text{Im } r_M$  and denote  $M^+(M^-)$  the number of the  $r_v$ 's such that  $\text{Im } r_v > 0$  ( $\text{Im } r_v < 0$ ). It is easy to check that the integers  $M^+, M^-$  are independent of the initial choice of the local coordinates  $u_1, u_2$ , if  $k$  is odd; when  $k$  is even  $M^+, M^-$  do depend on  $u_1, u_2$ , but the integers  $\min\{M^+, M^-\}, \max\{M^+, M^-\}$  do not.

In general, using Taylor's formula and (0.3) we can write for every  $j, 0 \leq j \leq M k / (k + 1)$ :

$$(1.4) \quad p_{m-j}(x, \xi) = \sum_{\alpha/k + \beta = M - j(k+1)/k} \sigma_{\alpha\beta}(x, \xi) u_2^\alpha(x, \xi) u_1^\beta(x, \xi) ,$$

where  $\sigma_{\alpha\beta}$  are positively homogeneous of degree  $m + (\alpha - \beta - kM) / (k + 1)$ .

**DEFINITION 1.1** (Symbols with multiple roots). *Let  $k$  be odd; assume  $kM^- \geq M^+ > 0$  ( $kM^+ \geq M^- > 0$ ). We define  ${}^+A_k^{m, M^+, M^-}(\Sigma_1, \Sigma_2, \rho)$  (respectively  ${}^-A_k^{m, M^+, M^-}(\Sigma_1, \Sigma_2, \rho)$ ) to be the class of symbols  $p(x, \xi) \in A_k^{m, M}(\Sigma_1, \Sigma_2, \rho)$ ,  $M = M^+ + M^-$ , such that for some smooth positively homogeneous function of degree 1,  $r_0(x, \xi)$ :*

(I) *for every  $j, 0 \leq j < M^+$  ( $0 \leq j < M^-$ ),  $r_0(x, \xi)$  is a root of multiplicity  $M^+ - j$  ( $M^- - j$ ) of the equation*

$$(1.5) \quad \sum_{\alpha/k + \beta = M - j(k+1)/k} \sigma_{\alpha\beta}(x, \xi) r^\beta = 0 ,$$

for all  $(x, \xi)$  in a conic neighborhood of  $\rho$ .

(II) *Noting as before  $r_1, \dots, r_M$  the roots of (1.3), we have  $r_1 = \dots = r_{M^+} = r_0(\rho)$ ,  $\text{Im } r_0(\rho) > 0$  and  $\text{Im } r_v < 0$  for  $M^+ < v \leq M$  (respectively:  $r_{M^++1} = \dots = r_M = r_0(\rho)$ ,  $\text{Im } r_0(\rho) < 0$  and  $\text{Im } r_v > 0$  for  $0 \leq v \leq M^+$ ).*

Conditions (I), (II) do not depend on the choice of  $u_1, u_2$ ; (I) means that in (1.4) we may factorize for  $0 \leq j < M^\pm$

$$(1.6) \quad p_{m-j}(x, \xi) = \left( \sum_{\alpha/k + \beta = M^\mp - j/k} \tau_{\alpha\beta}(x, \xi) u_2^\alpha(x, \xi) u_1^\beta(x, \xi) \right) \cdot (u_1(x, \xi) - r_0(x, \xi) u_2^k(x, \xi))^{M^\pm - j} ,$$

where  $\tau_{\alpha\beta}$  are smooth positively homogeneous of degree  $m - \beta - M^\pm$ . A detailed study of operators with symbols with multiple roots will be given in Section 7; observe that if  $k$  is even,  $M \geq 2$  and  $M^+ = 1$  ( $M^- = 1$ ) every symbol in  $A_k^{m,M}(\Sigma_1, \Sigma_2, \varrho)$  can be regarded as an element of  ${}^+A_k^{m,1,M^-}(\Sigma_1, \Sigma_2, \varrho)$  (respectively  ${}^-A_k^{m,M^+,1}(\Sigma_1, \Sigma_2, \varrho)$ ).

It will be convenient now to specify further the choice of  $u_1, u_2$ , according to

**PROPOSITION 1.2.** *There exist smooth positively homogeneous functions  $u_1(x, \xi), u_2(x, \xi)$ , of degree 1, 0, respectively, such that  $\Sigma_1, \Sigma_2$  are given by  $u_1(x, \xi) = 0, u_2(x, \xi) = 0$  in a conic neighborhood of  $\varrho$  and*

$$(1.1)' \quad \{u_1, u_2\}(x, \xi) = 1,$$

for every  $(x, \xi)$  in the same neighborhood.

To prove Proposition 1.2 just fix  $u_1$  and consider the Cauchy problem  $H_{u_1}u_2 = \{u_1, u_2\} = 1, u_2 = 0$  on  $\Sigma_2$ ; a solution  $u_2$  exists since  $H_{u_1}$  is transverse to the initial manifold  $\Sigma_2$ .

**PROPOSITION 1.3 (Canonical form).** *Let  $u_1, u_2$  be as in Proposition 1.2 and let  $U_1, U_2$  be classical pseudo differential operators with principal symbols  $u_1(x, \xi), u_2(x, \xi)$ . Every operator  $P = p(x, D), p(x, \xi) \in A_k^{m,M}(\Sigma_1, \Sigma_2, \varrho)$ , can be written in the form*

$$(1.7) \quad P = \sum_{0 \leq j \leq Mk/(k+1)} \sum_{\alpha/k + \beta = M - j(k+1)/k} U_2^\alpha C_{\alpha\beta} U_1^\beta + W,$$

where  $C_{\alpha\beta}, W$  are suitable classical pseudo differential operators of degree  $m + (\alpha - \beta - kM)/(k + 1), m - Mk/(k + 1) - \varepsilon$  ( $\varepsilon > 0$ ) respectively. We shall note  $c_{\alpha\beta}$  the homogeneous principal symbol of  $C_{\alpha\beta}$ .

**PROPOSITION 1.4 (Canonical form for multiple roots).** *Take  $P = p(x, D)$  with symbol  $p(x, \xi) \in \pm A_k^{m,M^+,M^-}(\Sigma_1, \Sigma_2, \varrho)$ . Let  $U_1, U_2$  be defined as in Proposition 1.3 and introduce  $Z = U_1 - R_0 U_2^k$ , where  $R_0$  has principal symbol  $r_0(x, \xi), r_0$  as in Definition 1.1.  $P$  can be written in the form*

$$(1.8) \quad P = \sum_{0 \leq j < M^\pm} \left( \sum_{\alpha/k + \beta = M^\mp - j/k} U_2^\alpha A_{\alpha\beta} Z^\beta \right) Z^{M^\pm - j} + \sum_{M^\pm \leq j \leq Mk/(k+1)} \sum_{\alpha/k + \beta = M - j(k+1)/k} U_2^\alpha A'_{\alpha\beta} Z^\beta + T,$$

where  $A_{\alpha\beta}, A'_{\alpha\beta}, T$  are suitable classical pseudo differential operators of degree  $m - \beta - M^\pm, m + (\alpha - \beta - kM)/(k + 1), m - Mk/(k + 1) - \varepsilon$  ( $\varepsilon > 0$ ), respec-

tively. We shall note  $a_{\alpha\beta}(x, \xi)$ ,  $a'_{\alpha\beta}(x, \xi)$  the homogeneous principal symbols of  $A_{\alpha\beta}$ ,  $A'_{\alpha\beta}$ .

PROOFS. – To prove Proposition 1.3, observe first that the composition  $w(x, D) = w_1(x, D)w_2(x, D)$ ,  $w_1(x, \xi) \in A_k^{m_1, M_1}(\Sigma_1, \Sigma_2, \varrho)$ ,  $w_2(x, \xi) \in A_k^{m_2, M_2}(\Sigma_1, \Sigma_2, \varrho)$ , has symbol  $w(x, \xi) \in A_k^{m_1+m_2, M_1+M_2}(\Sigma_1, \Sigma_2, \varrho)$ , as we get easily from the inequalities (0.3) and the standard asymptotic expansion for products. In particular, if we define  $c_{\alpha\beta} = \sigma_{\alpha\beta}$ ,  $\alpha/k + \beta = M$ ,  $\sigma_{\alpha\beta}$  as in (1.2), and if we consider operators  $C_{\alpha\beta}$  with principal symbols  $c_{\alpha\beta}$ , setting

$$q(x, D) = P - \sum_{\alpha/k + \beta = M} U_2^\alpha C_{\alpha\beta} U_1^\beta, \quad q(x, \xi) \sim \sum_{j=1}^\infty q_{m-j}(x, \xi),$$

we have that  $q_{m-j}(x, \xi)$  satisfies (0.3) for  $1 \leq j \leq Mk/(k+1)$ . Therefore using Taylor's formula we can write

$$(1.9) \quad q_{m-1}(x, \xi) = \sum_{\alpha/k + \beta = M - (k+1)/k} c_{\alpha\beta}(x, \xi) u_2^\alpha(x, \xi) u_1^\beta(x, \xi).$$

Then we argue in a similar way on the difference

$$q(x, D) - \sum_{\alpha/k + \beta = M - (k+1)/k} U_2^\alpha C_{\alpha\beta} U_1^\beta,$$

where  $C_{\alpha\beta}$  are operators with principal symbols  $c_{\alpha\beta}$ ,  $c_{\alpha\beta}$  as in (1.9). Iterating the procedure, we arrive finally at (1.7).

The proof of Proposition 1.4 is analogous. First we define  $a_{\alpha\beta}$ ,  $\alpha/k + \beta = M^\mp$ , by imposing the identity

$$(1.10) \quad \sum_{\alpha/k + \beta = M^\mp} a_{\alpha\beta}(x, \xi) u_2^\alpha(x, \xi) (u_1(x, \xi) - r_0(x, \xi) u_2^k(x, \xi))^\beta = \sum_{\alpha/k + \beta = M^\mp} \tau_{\alpha\beta}(x, \xi) u_2^\alpha(x, \xi) u_1^\beta(x, \xi),$$

where  $\tau_{\alpha\beta}$ ,  $\alpha/k + \beta = M^\mp$ , are the functions in (1.6). Let  $A_{\alpha\beta}$  be pseudo differential operators with principal symbols  $a_{\alpha\beta}$ . The operator

$$P_0 = \left( \sum_{\alpha/k + \beta = M^\mp} U_2^\alpha A_{\alpha\beta} Z^\beta \right) Z^{M^\mp}$$

has symbol in  $\pm A_k^{m, M^+, M^-}(\Sigma_1, \Sigma_2, \varrho)$ , as we obtain again from the standard asymptotic expansion for symbols of products. Then consider the difference

$s(x, D) = P - P_0$ ,  $s(x, \xi) \sim \sum_{j=1}^\infty s_{m-j}(x, \xi)$ ; we have that all the functions  $s_{m-j}$ ,

$1 \leq j < M^+$ , can be factorized as in (1.6). Arguing similarly on  $s_{m-1}(x, \xi)$  and iterating the procedure we obtain (1.8).

Observe that the assumption (1.1) (which will be essential in the following) is not necessary for decompositions of the type (1.7), (1.8). Note also that the values at  $\varrho$  of the principal symbols  $c_{\alpha\beta}$ ,  $a_{\alpha\beta}$ ,  $a'_{\alpha\beta}$  are uniquely determined, once  $u_1, u_2$  are fixed.

Let  $P = p(x, D)$ ,  $p(x, \xi) \in \pm A_k^{m, M^+, M^-}(\Sigma_1, \Sigma_2, \varrho)$ , be written as in (1.8); using the notations of Proposition 1.4 we define the polynomial

$$(1.11) \quad Q_p^\pm(\lambda) = \sum_{0 \leq j \leq M^\pm} (-i)^{M^\pm - j} a_{kM^\mp - j, 0}(\varrho) \lambda(\lambda - 1) \dots \dots (\lambda - M^\pm + j + 1) + a'_{kM^\mp - M^\pm, 0}(\varrho).$$

The  $M^\pm$  roots of  $Q_p^\pm(\lambda) = 0$  do not depend on the choice of  $u_1, u_2$  in (1.1)', as we shall check after Proposition 2.3. These invariants will be used in Section 7.

**2. - Application of Fourier integral operators.**

Let  $P = p(x, D)$ ,  $p(x, \xi) \in A_k^{m, M}(\Sigma_1, \Sigma_2, \varrho)$ , be written in the canonical form (1.7); using the notations of Proposition 1.3 we define the ordinary differential operator

$$(2.1) \quad \mathfrak{L}_p = \sum_{0 \leq j \leq Mk/(k+1)} \sum_{\alpha/k + \beta = M - j(k+1)/k} c_{\alpha\beta}(\varrho) t^\alpha D_t^\beta.$$

The expression of  $\mathfrak{L}_p$  depends on the initial choice of  $u_1, u_2$ . Actually, starting from another couple  $u'_1, u'_2, u''_1, u''_2$  as in Proposition 1.2, and writing the corresponding canonical form of  $P$ , we obtain an operator  $\mathfrak{L}'_p$  which becomes  $\mathfrak{L}_p$  after a transformation of the form

$$(2.2) \quad t' = \gamma t, \quad 0 \neq \gamma \in \mathbf{R}.$$

The easy check is left to the reader. We can now state a general implicit result.

**THEOREM 2.1.** - *The following conditions are equivalent:*

- (i)  $\text{Ker } \mathfrak{L}_p \cap \mathfrak{S}(\mathbf{R}) = \{0\}$ .
- (ii)  $P = p(x, D)$  satisfies (0.5).

(iii) For any  $\varphi \in S^0(\Omega \times \mathbf{R}^n)$  with  $\varphi(\varrho) \neq 0$  and having sufficiently narrow support, there exists a proper linear continuous operator  $E: H_{\text{loc}}^s(\Omega) \rightarrow H_{\text{loc}}^{s+m-Mk/(k+1)}(\Omega)$ ,  $s \in \mathbf{R}$ , such that  $EP - \varphi(x, D)$  is smoothing, and  $WF'(E) \subset \text{diag}(\Omega \times \mathbf{R}^n)$ .

Condition (i) is obviously invariant for the transformations (2.2). Observe that (iii) implies the hypoellipticity of  $P$  in some conic neighborhood  $\Gamma$  of  $\varrho$ :

$$(2.3) \quad \Gamma \cap WF(Pf) = \Gamma \cap WF(f), \quad \text{for every } f \in \mathcal{D}'(\Omega).$$

PROOF OF THEOREM 2.2. Fix a canonical homogeneous transformation  $\chi: y = y(x, \xi)$ ,  $\eta = \eta(x, \xi)$ , such that in a conic neighborhood of  $\varrho$

$$(2.4) \quad y_n(x, \xi) = u_2(x, \xi), \quad \eta_n(x, \xi) = u_1(x, \xi)$$

(starting from (2.4) and using (1.1)', one can construct recursively the other coordinates  $y_{n-1}, \eta_{n-1}, \dots, y_1, \eta_1$  by means of a standard argument). Let  $F$  be any elliptic Fourier integral operator associated with  $\chi$ ; if  $A$  is classical with principal symbol  $a(x, \xi)$ , then the principal symbol of  $FAF^{-1}$  is given by  $a \circ \chi^{-1}$ . In particular the principal symbols of  $FU_1F^{-1}$ ,  $FU_2F^{-1}$  are  $\eta_n, y_n$ , respectively. We may specify the choice of  $F$  in the following way.

LEMMA 2.2. Let  $U_1, U_2$  be classical pseudo differential operators with principal symbols  $u_1, u_2$ , as in Proposition 1.3. Then there exists an elliptic Fourier integral operator  $F$  associated to the canonical transformation  $\chi$  such that  $FU_1F^{-1} - D_{y_n}$ ,  $FU_2F^{-1} - y_n$  are smoothing in a conic neighborhood of  $\varrho$ .

For the proof let us refer to Boutet de Monvel [1], Lemma (10.8), where the symbol of  $F$  is constructed explicitly by successive approximations. For such  $F$  we have from (1.7)

$$(2.5) \quad P' = FPF^{-1} = \sum_{0 \leq j \leq Mk/(k+1)} \sum_{\alpha/k + \beta = M - j(k+1)/k} y_n^\alpha C'_{\alpha\beta} D_{y_n}^\beta + W',$$

where  $C'_{\alpha\beta}$  are classical pseudo differential operators with principal symbol  $e_{\alpha\beta} \circ \chi^{-1}$ ;  $W'$  is of order  $m - Mk/(k+1) - \varepsilon$ , for some  $\varepsilon > 0$ . Note  $\Sigma'_1 = \{\eta_n = 0\}$ ,  $\Sigma'_2 = \{y_n = 0\}$ ,  $\Sigma' = \{y_n = \eta_n = 0\}$ ,  $\varrho' = \chi(\varrho) \in \Sigma'$ . The symbol  $p'(y, \eta)$  of  $P'$  is in  $A_k^{m, M}(\Sigma'_1, \Sigma'_2, \varrho')$  and (2.5) is a canonical form of  $P'$ ; since  $e'_{\alpha\beta}(\varrho') = e_{\alpha\beta}(\varrho)$ , it is  $\mathcal{L}_{p'} = \mathcal{L}_p$ . Now we may apply the results of [16]. In particular first we check on (2.5) that  $\mathcal{L}_{p'}$  and the « test operator » of [16], Definition 2.3, coincide; then we use Theorems 2.1, 3.1 of [16], which state the equivalence of (i), (ii), (iii) in the flat case. Since  $P$  satisfies (ii), (iii) if and only if  $P'$  does, Theorem 2.1 is proved.

Theorem 2.1 applies in particular to symbols with multiple roots; it will be convenient to use in this case the following equivalent expression of  $\mathfrak{L}_p$ .

PROPOSITION 2.3. *Let  $p(x, \xi)$  be in  $\pm A_k^{m, M^+, M^-}(\Sigma_1, \Sigma_2, \varrho)$ . Write  $P = p(x, D)$  in the canonical form (1.8) and set with the notations of Proposition 1.4*

$$(2.6) \quad \mathfrak{L}_p^\pm = \sum_{0 \leq j < M^\pm} \left( \sum_{\alpha/k + \beta = M^\mp - j/k} a_{\alpha\beta}(\varrho) t^\alpha D_t^\beta \right) D_t^{M^\pm - j} + \\ + \sum_{M^\pm \leq j \leq Mk/(k+1)} \sum_{\alpha/k + \beta = M - j(k+1)/k} a'_{\alpha\beta}(\varrho) t^\alpha D_t^\beta.$$

If  $\mathfrak{L}_p$  is defined as in (2.1), with the same choice of  $u_1, u_2$  in Proposition 1.3 and 1.4, then we have

$$(2.7) \quad \mathfrak{L}_p = \exp[ir_0(\varrho) t^{k+1}/(k+1)] \mathfrak{L}_p^\pm \exp[-ir_0(\varrho) t^{k+1}/(k+1)].$$

PROOF. — We shall limit ourselves to a sketch, leaving to the reader a detailed (and stricter) proof. Let us write out explicitly the right-hand side of (2.7):

$$(2.8) \quad \sum_{0 \leq j \leq M^\pm} \left( \sum_{\alpha/k + \beta = M^\mp - j/k} a_{\alpha\beta}(\varrho) t^\alpha \mathfrak{Z}^\beta \right) \mathfrak{Z}^{M^\pm - j} + \\ + \sum_{M^\pm \leq j \leq Mk/(k+1)} \sum_{\alpha/k + \beta = M - j(k+1)/k} a'_{\alpha\beta}(\varrho) t^\alpha \mathfrak{Z}^\beta, \quad \mathfrak{Z} = D_t - r_0(\varrho) t^k.$$

Both  $\mathfrak{L}_p$  in (2.1) and the operator in (2.8) are obtained from (1.7), (1.8), respectively, by replacing formally  $U_1$  with  $D_t$ ,  $U_2$  with  $t$  and  $C_{\alpha\beta}, A_{\alpha\beta}, A'_{\alpha\beta}$  with the value at  $\varrho$  of their principal symbols. Now, starting from (1.7), we may arrive at the canonical form (1.7) through a finite number of commutations, of three types:

$$(2.9) \quad U_1 U_2 = U_2 U_1 + [U_1, U_2],$$

$$(2.10) \quad A U_2 = U_2 A + [A, U_2],$$

$$(2.11) \quad U_1 A = A U_1 + [U_1, A],$$

where  $A$  is a classical pseudo differential operator with principal symbol  $a(x, \xi)$  of suitable order. The ordinary differential operators which we obtain after every commutation through an obvious arrangement of the foregoing formal proceeding coincide with the operator in (2.8). In fact, recalling

that the principal symbol of  $[U_1, U_2]$  is given by  $-i\{u_1, u_2\}$  and replacing in (2.9)  $U_1, U_2, [U_1, U_2]$  with  $D_t, t, -i\{u_1, u_2\}(\varrho)$ , we get the trivial identity  $D_t t = tD_t - i$ , since  $\{u_1, u_2\}(\varrho) = 1$  in view of (1.1)'. On the other hand it is easy to see that  $[A, U_2], [U_1, A]$  in (2.10), (2.11) do not give any contribution in the expressions of the ordinary differential operators, so that (2.10), (2.11) correspond to the obvious identities  $a(\varrho)t = ta(\varrho), D_t a(\varrho) = a(\varrho)D_t$ . In this way, step by step, we arrive at (2.7).

It follows from (2.7) that also the definition of  $\mathfrak{L}_p^\pm$  is invariant modulo transformations of the type (2.2). Observe that  $Q_p^\pm(\lambda)$  in (1.10) is the indicial polynomial of  $\mathfrak{L}_p^\pm$ :

$$(2.12) \quad \mathfrak{L}_p^\pm t^\lambda = Q_p^\pm(\lambda) t^{\lambda + kM^+ - M^\pm} + O(t^{\lambda + kM^+ - M^\pm - 1}).$$

Since  $Q^\pm(\lambda)$  changes for a multiplicative factor after the transformation (2.2), it is therefore proved that the roots of  $Q_p^\pm(\lambda) = 0$  do not depend on the choice of  $u_1, u_2$ .

## CHAPTER II

### ORDINARY DIFFERENTIAL OPERATORS WITH POLYNOMIAL COEFFICIENTS

#### 3. - Solutions in $\mathcal{S}(\mathbb{R})$ .

Here  $\tau, r_1, \dots, r_M$  are fixed complex numbers,  $\tau \neq 0, \text{Im } r_v > 0$  for  $1 < v < M^+ < M, \text{Im } r_v < 0$  for  $M^+ < v < M = M^+ + M^-$ ;  $c_{\alpha\beta}, \alpha/k + \beta = M$ , are defined by the identity  $\sum_{\alpha/k + \beta = M} c_{\alpha\beta} r^\beta = \tau(r - r_1) \dots (r - r_M)$ . We consider the ordinary differential operator

$$(3.1) \quad \mathfrak{L} = \mathfrak{L}_k^{\tau, r_1, \dots, r_M}(c) = \sum_{\alpha/k + \beta \leq M} c_{\alpha\beta} t^\alpha D_t^\beta,$$

where  $\alpha \geq 0, \beta \geq 0$  are integers and  $c = (c_{\alpha\beta}), \alpha/k + \beta < M$ , is regarded in this section as parameter in  $\mathbb{C}^K, K = kM(M - 1)/2$ . Actually, concerning the applications we purpose, it will be not restrictive to assume in (3.1)  $c_{\alpha\beta} = 0$  whenever  $j = M - \beta - (M + \alpha - \beta)/(k + 1)$  is not an integer,

$0 < j \leq Mk/(k + 1)$ ; i.e. we may limit ourselves to operators of the form

$$(3.2) \quad \sum_{0 \leq j \leq Mk/(k+1)} \sum_{\alpha/k + \beta = M - j(k+1)/k} c_{\alpha\beta} t^\alpha D_t^\beta.$$

We want to investigate the existence of non-trivial solutions  $y \in \mathcal{S}(\mathbf{R})$  of the equation  $\mathcal{L}y = 0$ . Our starting-point is the following theorem, whose proof is a straightforward generalization of the proof of Proposition 2.4.2 in [9] (for the case  $M = 2$  see Sibuya [18], Chapter 2).

**THEOREM 3.1.** *For every given value of  $\tau, r_1, \dots, r_M, k$ , there exists two  $M$ -tuples of functions,  $y_1^+(t, c), \dots, y_M^+(t, c), y_1^-(t, c), \dots, y_M^-(t, c)$ , which are entire analytic in  $\mathbf{C} \times \mathbf{C}^k$  with respect to the complex variable  $t$  and the parameter  $c = (c_{\alpha\beta})$ ,  $\alpha/k + \beta < M$ , such that for every fixed  $c \in \mathbf{C}^k$ :*

- I)  $y_v^+, y_v^-, v = 1, \dots, M$ , are two fundamental systems of solutions for the equation  $\mathcal{L}_{\tau, r_1, \dots, r_M}^k(c)y = 0$ .
- II) For any  $y = \sum_{v=1}^M \mu_v^+ y_v^+ = \sum_{v=1}^M \mu_v^- y_v^-$ ,  $\mu_v^+, \mu_v^- \in \mathbf{C}$ , we have  $y|_{\mathbf{R}_+} \in \mathcal{S}(\mathbf{R}_+)$  if and only if  $\mu_v^+ = 0$  for  $M^+ < v \leq M$ ; moreover, when  $k$  is odd  $y|_{\mathbf{R}_-} \in \mathcal{S}(\mathbf{R}_-)$  if and only if  $\mu_v^- = 0$  for  $M^+ < v \leq M$ , when  $k$  is even  $y|_{\mathbf{R}_-} \in \mathcal{S}(\mathbf{R}_-)$  if and only if  $\mu_v^- = 0$  for  $1 \leq v \leq M^+$ .

Therefore, for any fixed  $c \in \mathbf{C}^k$ :

$$(3.3) \quad \dim[\text{Ker } \mathcal{L} \cap \mathcal{S}(\mathbf{R}_+)] = M^+,$$

$$(3.4) \quad \dim[\text{Ker } \mathcal{L} \cap \mathcal{S}(\mathbf{R}_-)] = \begin{cases} M^+ & \text{for } k \text{ odd,} \\ M^- & \text{for } k \text{ even.} \end{cases}$$

From (3.3), (3.4) and easy linear algebra one gets

$$(3.5) \quad \text{for } k \text{ odd: } \quad M^+ - M^- \leq \dim[\text{Ker } \mathcal{L} \cap \mathcal{S}(\mathbf{R})] \leq M^+$$

$$(3.6) \quad \text{for } k \text{ even: } \quad \dim[\text{Ker } \mathcal{L} \cap \mathcal{S}(\mathbf{R})] \leq \min\{M^+, M^-\}.$$

In particular we have:

**COROLLARY 3.2.** *When  $k$  is odd: if  $M^+ = 0$  then  $\text{Ker } \mathcal{L} \cap \mathcal{S}(\mathbf{R}) = \{0\}$ ; if  $M^+ > M^-$  then there exists a non-trivial solution  $y \in \mathcal{S}(\mathbf{R})$  of the equation  $\mathcal{L}y = 0$ .*

*When  $k$  is even: if  $M^+ = 0$ , or  $M^- = 0$ , then  $\text{Ker } \mathcal{L} \cap \mathcal{S}(\mathbf{R}) = \{0\}$ .*

When  $k$  is odd and  $M^- \geq M^+ > 0$ , or else  $k$  is even and  $M > M^+ > 0$ , the existence of non-trivial solutions in  $S(\mathbf{R})$  depends on the parameter  $c$ . Assume first  $k$  is odd,  $M^- \geq M^+ > 0$ , and consider the  $M \times 2M^+$  matrix

$$(3.7) \quad \begin{bmatrix} y_1^+ & \dots & y_{M^+}^+ & y_1^- & \dots & y_{M^+}^- \\ (y_1^+)' & \dots & (y_{M^+}^+)' & (y_1^-)' & \dots & (y_{M^+}^-)' \\ \vdots & & \vdots & \vdots & & \vdots \\ (y_1^+)^{(M-1)} & \dots & (y_{M^+}^+)^{(M-1)} & (y_1^-)^{(M-1)} & \dots & (y_{M^+}^-)^{(M-1)} \end{bmatrix}.$$

Choose  $M^- - M^+ + 1$  independent minors in (3.7) and let  $f_1(c), \dots, f_{M^- - M^+ + 1}(c)$  be their values at a fixed point  $t_0 \in \mathbf{C}$ . It is clear that  $\dim[\text{Ker } \mathcal{L} \cap S(\mathbf{R})] > 0$  if and only if  $f_1(c) = \dots = f_{M^- - M^+ + 1}(c) = 0$ .

More precisely, write  $E(\mathbf{C}^k)$  for the ring of the entire analytic functions in  $\mathbf{C}^k$  and denote  $\mathfrak{J}_{\tau, r_1, \dots, r_M}^k$  the ideal of  $E(\mathbf{C}^k)$  generated by  $f_1, \dots, f_{M^- - M^+ + 1}$  (see the terminology in Gunning and Rossi [7], for example). The definition of  $\mathfrak{J}_{\tau, r_1, \dots, r_M}^k$  does not depend on the choice of the  $M$ -tuples  $y_1^\pm, \dots, y_M^\pm$  in Theorem 3.1, the minors in (3.7) and the point  $t_0 \in \mathbf{C}$  where the determinants are evaluated; we have:

**COROLLARY 3.3.** *Let  $k$  be odd. Let  $\tau, r_1, \dots, r_M$  be fixed with  $M^- \geq M^+ > 0$ . Then there exists a non-trivial solution  $y \in S(\mathbf{R})$  of the equation  $\mathcal{L}_{\tau, r_1, \dots, r_M}^k y = 0$  if and only if  $c \in \text{loc } \mathfrak{J}_{\tau, r_1, \dots, r_M}^k$ .*

The same statement holds when  $k$  is even and  $M > M^+ > 0$ , if one defines in this case  $\mathfrak{J}_{\tau, r_1, \dots, r_M}^k = (f) = E(\mathbf{C}^k) \cdot f$ ,  $f$  being the value at any fixed  $t_0 \in \mathbf{C}$  of the Wronskian  $W(y_1^+, \dots, y_{M^+}^+, y_{M^+ + 1}^-, \dots, y_M^-)$ . The analytic subvariety  $\text{loc } \mathfrak{J}_{\tau, r_1, \dots, r_M}^k \subset \mathbf{C}^k$  is always non-trivial, i.e.  $\text{loc } \mathfrak{J}_{\tau, r_1, \dots, r_M}^k \neq \mathbf{C}^k$ ,  $\text{loc } \mathfrak{J}_{\tau, r_1, \dots, r_M}^k \neq \emptyset$ ; this will be easy consequence of the examples in the next section.

**4. - The case  $M^+ = 1$ .**

We want to study in detail certain operators of the type (3.2) for which  $M \geq 2$  and  $M^+ = 1$ , i.e.  $\text{Im } r_1 > 0, \text{Im } r_v < 0$  for  $v = 2, \dots, M$ .

If  $M^+ = 1$ , every non-trivial solution  $Y^+ \in S(\mathbf{R}_+)$  can be written, for some constant  $\mu \in \mathbf{C}$ ,  $Y^+ = \mu y_1^+ (y_1^+(t, c))$  as in Theorem 3.1; here we argue on a fixed value of the parameter  $c$ ; moreover, if in addition the operator is of the type (3.2), we have for  $Y^+$  an asymptotic expansion of the form

$$(4.1) \quad Y^+ \sim \exp[ir_1 t^{k+1}/(k+1)] t^\delta \sum_{n=0}^\infty \omega_n t^{-n}, \quad t \in \mathbf{R}_+;$$

where  $\vartheta, \omega_n$  are independent of  $t$  (cf. [9], Remark 2.4.1). It will be convenient to use the following particularization of Corollary 3.3.

**PROPOSITION 4.1.** *Let  $\mathfrak{L}$  be of the type (3.2), with  $M^+ = 1$ ; let  $k$  be odd. Let  $Y^+(t)$  be a non-trivial entire analytic function such that  $\mathfrak{L}Y^+ = 0, Y^+|_{\mathbf{R}_+} \in \mathfrak{S}(\mathbf{R}_+)$ . Then there exists a non-trivial solution  $y \in \mathfrak{S}(\mathbf{R})$  of the equation  $\mathfrak{L}y = 0$  if and only if*

$$(4.2)_I \quad (Y^+)^{(h)}(0) = 0 \quad \text{for every even } h, 0 < h < M - 1,$$

or else

$$(4.2)_{II} \quad (Y^+)^{(h)}(0) = 0 \quad \text{for every odd } h, 0 < h < M - 1.$$

**PROOF.** Observe first that if  $y(t)$  is solution of  $\mathfrak{L}y = 0$ , also  $y(-t)$  is solution. In particular,  $Y^-(t) = Y^+(-t) \in \mathfrak{S}(\mathbf{R}_-)$  is (non-trivial) solution. Then  $Y^\pm$  can be identified with  $y_1^\pm$  in Theorem 3.1, modulo multiplicative constants; the matrix (3.7) reads for  $t = 0$

$$(4.4) \quad \begin{bmatrix} Y^+(0) & Y^+(0) \\ (Y^+)'(0) & -(Y^+)'(0) \\ \vdots & \vdots \\ (Y^+)^{(M-1)}(0) & (-1)^{(M-1)}(Y^+)^{(M-1)}(0) \end{bmatrix}.$$

It is clear that all the  $2 \times 2$ -minors vanish if and only if one of the two conditions (4.2)<sub>I</sub>, (4.2)<sub>II</sub> is satisfied, and an application of the arguments of the preceding section gives the proof.

First we shall consider the operator

$$(4.5) \quad \mathcal{N} = \mathcal{N}_{\tau, r_1, r_0}^{k, M}(a) = \tau(D_t - r_1 t^k)(D_t - r_0 t^k)^{M-1} + \sum_{0 < j \leq M-1} a_j t^{k-j}(D_t - r_0 t^k)^{M-j-1},$$

where  $k$  is odd,  $k \geq M - 1 > 0, \tau \neq 0, \text{Im } r_1 > 0, \text{Im } r_0 < 0$  and  $a = (a_1, \dots, a_{M-1}) \in \mathbf{C}^{M-1}$ . Developing the terms in the sum in (4.5)  $\mathcal{N}$  can be rewritten in the form (3.2), with  $M^+ = 1$ ; therefore the hypotheses of Proposition 4.1 are satisfied. We shall express the related solutions  $Y^+ \in \mathfrak{S}(\mathbf{R}_+)$  by means of Meijer's  $G$ -functions and we shall deduce the following explicit result on the « eigenvalues » of  $\mathcal{N}$ .

**THEOREM 4.2.** *Let  $S_0, S_1, \dots, S_{M-1}$  be the solutions of the linear system*

$$(4.6) \quad S_0 = 1, \quad \sum_{p=0}^j \binom{M-j-1}{j-p} B_{j-p}^{(j-M+1)} S_p = i^j a_j \tau^{-1}(r_0 - r_1)^{-1},$$

$$j = 1, \dots, M-1$$

( $B_n^{(m)}$  are the generalized Bernoulli numbers; see Luke [12], 2.8(1), for example). Define for  $\lambda \in \mathbf{C}$

$$(4.7) \quad Q^-(\lambda) = \sum_{s=0}^{M-1} S_{M-s-1} \lambda^s.$$

There exists a non-trivial solution  $y \in \mathcal{S}(\mathbf{R})$  of  $\mathcal{N}_{\tau, r_1, r_0}^{k, M}(a)y = 0$  if and only if one of the following two conditions is satisfied:

- I) for every even integer  $h, 0 \leq h \leq M-1$ , there exists an integer  $N_h > 0$  such that  $Q^-(h - (k+1)N_h) = 0$ ;
- II) for every odd integer  $h, 0 < h \leq M-1$ , there exists an integer  $N_h > 0$  such that  $Q^-(h - (k+1)N_h) = 0$ .

Observe that  $Q^-(\lambda)$  in (4.7) can be identified, modulo a multiplicative constant, with the indicial polynomial of

$$(4.5)' \quad \mathcal{N}^- = \exp[-ir_0 t^{k+1}/(k+1)] \mathcal{N}^+ \exp[ir_0 t^{k+1}/(k+1)]$$

$$= \tau(D_t + (r_0 - r_1)t^k) D_t^{M-1} + \sum_{0 < j \leq M-1} a_j t^{k-j} D_t^{M-j-1}.$$

Precisely, we have

$$(4.6)' \quad \mathcal{N}^- t^\lambda = \tau(r_0 - r_1)(-i)^{M-1} Q^-(\lambda) t^{\lambda+k-M+1} + O(t^{\lambda+k-M}).$$

This will be a direct consequence of the subsequent identity (4.13).

**PROOF OF THEOREM 4.2.** Consider the ordinary differential operator in the plane of the complex variable  $z$

$$(4.8) \quad \mathcal{M} = (\delta + \varrho_0 - 1)(\delta + \varrho_1 - 1) \dots (\delta + \varrho_{M-1} - 1) - z(\delta + \alpha_1) \dots (\delta + \alpha_{M-1}),$$

where  $\delta = z d/dz$  and

$$(4.9) \quad \varrho_h = 1 - h/(k+1), \quad h = 0, 1, \dots, M-1.$$

Noting  $\lambda_1, \dots, \lambda_{M-1}$  the roots of  $Q^-(\lambda) = 0$  ( $Q^-$  as in (4.7)) we set in (4.8)

$$(4.10) \quad \alpha_j = -\lambda_j/(k+1), \quad j = 1, \dots, M-1.$$

Let  $U(z)$  be any solution of  $\mathcal{M}U = 0$  in a conic neighborhood in  $\mathbf{C}$  of the ray  $\{z = i(r_1 - r_0)x, x \in \mathbf{R}_+\}$ ; we claim that

$$(4.11) \quad y(t) = \exp[ir_0 t^{k+1}/(k+1)]U(i(r_1 - r_0)t^{k+1}/(k+1))$$

is a solution of  $\mathcal{N}y = 0$  in a conic neighborhood of  $\mathbf{R}_+$ . In fact, after the change of variable  $z = i(r_1 - r_0)t^{k+1}/(k+1)$  the operator  $\delta$  becomes  $(k+1)^{-1}t d/dt$ , so that  $\delta + \varrho_h - 1 = (k+1)^{-1}(t d/dt - h)$ ,  $\delta + \alpha_j = (k+1)^{-1}(t d/dt - \lambda_j)$ ; then use the following identities (see [12], 2.9(5), 2.9(11))

$$(4.12) \quad t d/dt(t d/dt - 1) \dots (t d/dt - M + 1) = t^M d^M/dt^M,$$

$$(4.13) \quad (t d/dt - \lambda_1) \dots (t d/dt - \lambda_{M-1}) = \sum_{m=0}^{M-1} \left[ \sum_{q=0}^{M-m-1} \binom{M-q-1}{M-q-m-1} B_{M-a-m-1}^{(-m)} S_a \right] t^m d^m/dt^m;$$

in (4.13)  $S_0, S_1, \dots, S_{M-1}$  ( $S_0 = 1$ ) are exactly the coefficients of the equation  $\sum_{s=0}^{M-1} S_{M-s-1} \lambda^s = 0$ , whose solutions are  $\lambda_1, \dots, \lambda_{M-1}$ . In view of (4.6) we have

$$(4.14) \quad \mathcal{M} = (k+1)^{-M} t^M \left( d^M/dt^M + i(r_0 - r_1) t^k d^{M-1}/dt^{M-1} + \sum_{0 < j \leq M-1} i^{j+1} \tau^{-1} \alpha_j t^{k-j} d^{M-j-1}/dt^{M-j-1} \right)$$

and, since

$$(4.15) \quad \exp[ir_0 t^{k+1}/(k+1)] d/dt = (d/dt - ir_0 t^k) \exp[ir_0 t^{k+1}/(k+1)],$$

the claim is easily proved.

The equation  $\mathcal{M}U = 0$  is a particular case of the generalized hypergeometric equation (see [12], 5.1(2,19)) and a solution for  $\text{Re } z < 0$  is given by

$$(4.16) \quad U(z) = \sum_{h=0}^{M-1} \frac{(-z)^{1-\varrho_h} \prod_{0 \leq j \leq M-1, j \neq h} \Gamma(\varrho_h - \varrho_j)}{\prod_{1 \leq j \leq M-1} \Gamma(\varrho_h - \alpha_j)} U_h(z),$$

where  $U_h$  is the generalized hypergeometric function

$$(4.17) \quad U_h(z) = \sum_{n=0}^{\infty} \left\{ \frac{\prod_{1 \leq j \leq M-1} (1 + \alpha_j - \varrho_h)_n}{\prod_{0 \leq j \leq M-1, j \neq h} (1 + \varrho_j - \varrho_h)_n} \right\} z^n/n!$$

(we use the notation  $(\alpha)_n = (\alpha + 1) \dots (\alpha + n - 1)$ ,  $(\alpha)_0 = 1$ ).

In (4.16)  $\Gamma$  is the Euler's gamma function and we understand  $1/\Gamma(z) = 0$  when  $z = 0, -1, -2, \dots$ ; moreover the principal branch of the multivalued function  $(-z)^{1-\varrho_n}$  is chosen, in such way that  $(1)^{1-\varrho_n} = 1$ . The solution  $U$  can be expressed by means of a Meijer's  $G$ -function (see [12], 5.1(19), 5.2(1)), which admits the following integral representation of the Mellin-Barnes type

$$(4.18) \quad U(z) = G_{M-1, M}^{M, 0}(-z |_{1-\varrho_0, \dots, 1-\varrho_{M-1}}^{1-\alpha_1, \dots, 1-\alpha_{M-1}}) = (2\pi i)^{-1} \int_L \frac{\prod_{j=0}^{M-1} \Gamma(1 - \varrho_j - \zeta)}{\prod_{j=1}^{M-1} \Gamma(1 - \alpha_j - \zeta)} (-z)^\zeta d\zeta$$

(see [12], 5.2(2, 3, 4) for the definition of the path  $L$  of integration). Using standard arguments of asymptotic analysis (cf. [12], 5.7(12, 13, 14, 15)), from (4.18) one obtains for  $U$  an expansion of the type:

$$(4.19) \quad U(z) \sim \exp[z] z^{\vartheta'} \sum_{n=0}^{\infty} \omega'_n z^{-n}, \quad \operatorname{Re} z < 0,$$

with suitable  $\vartheta', \omega'_n$ ; (4.19) can be differentiated term-by-term.

Now, applying (4.11) and recalling (4.9), we define

$$(4.20) \quad Y^+(t) = \exp[ir_0 t^{k+1}/(k+1)] \sum_{h=0}^{M-1} A_h t^h U_h(ir_1 - r_0) t^{k+1}/(k+1)$$

with

$$(4.21) \quad A_h = \frac{[i(r_0 - r_1)/(k+1)]^{h/(k+1)} \prod_{0 \leq j \leq M-1, j \neq h} \Gamma((j-h)/(k+1))}{\prod_{1 \leq j \leq M-1} \Gamma(1 - \alpha_j - h/(k+1))}.$$

We know that  $Y^+$  is a solution of  $\mathcal{N}y = 0$  in a conic neighborhood of  $\mathbf{R}_+$ ; actually, (4.20), (4.21), (4.17) define an entire analytic function, which is certainly solution in the whole of  $\mathbf{C}$ . Applying (4.18), (4.19) we obtain

$$(4.22) \quad Y^+ = \exp[ir_0 t^{k+1}/(k+1)] G_{M-1, M}^{M, 0}(i(r_0 - r_1) t^{k+1}/(k+1) |_{1-\varrho_0, \dots, 1-\varrho_{M-1}}^{1-\alpha_1, \dots, 1-\alpha_{M-1}}) \sim \exp[ir_1 t^{k+1}/(k+1)] t^\vartheta \sum_{n=0}^{\infty} \omega_n t^{-n}, \quad t \in \mathbf{R}_+$$

for suitable  $\vartheta, \omega_n$  (cf. (4.1)); since  $\operatorname{Re}(ir_1) < 0$ , it is evident that  $Y^+|_{\mathbf{R}_+} \in \mathcal{S}(\mathbf{R}_+)$ . Then we may apply Proposition 4.1. Recalling that  $k \geq M-1$  and observing that  $U_h(0) = 1$ , we have from (4.20), (4.21)

$$(4.23) \quad (Y^+)^{(h)}(0) = h! A_h, \quad h = 0, 1, \dots, M-1.$$

On the other hand  $A_h = 0$  if and only if for some  $j, j = 1, \dots, M - 1$ , it is  $(k + 1)\alpha_j = (k + 1)N - h, N > 0$  integer; therefore the conditions (4.2)<sub>I</sub>, (4.2)<sub>II</sub> are equivalent to I), II) in the statement of the theorem, in view of (4.10). The proof is complete.

For  $M = 2$ , replacing  $r_0$  with  $r_2$ , we have in (4.5) the operator

$$(4.24) \quad \mathcal{N}_{\tau, r_1, r_2}^{k, 2}(a_1) = \tau(D_t - r_1 t^k)(\overline{D_t - r_2 t^k}) + a_1 t^{k-1},$$

$\tau \neq 0, \text{Im } r_1 > 0, \text{Im } r_2 < 0, k$  odd; the polynomial  $Q^-(\lambda)$  in (4.7) comes down to

$$(4.25) \quad Q^-(\lambda) = \lambda + ia_1/[(r_2 - r_1)\tau]$$

and Theorem 4.2 says that a non-trivial solution  $y \in \mathcal{S}(\mathbf{R})$  of  $\mathcal{N}_{\tau, r_1, r_2}^{k, 2}(a_1)y = 0$  exists if and only if for some integer  $N > 0$

$$(4.26) \quad (k + 1)N - ia_1/[(r_2 - r_1)\tau] = 0 \quad \text{or } 1$$

(see Gilioli - Trèves [5], for example).

Finally, we shall consider some operators of the type (3.2) for which  $k$  is even and  $M^+ = 1$ . In particular, let  $k$  be even in the expression of  $\mathcal{N}_{\tau, r_1, r_2}^{k, 2}(a_1)$  in (4.24). Recall that the Tricomi's  $\Psi$ -function is given by

$$(4.27) \quad \Psi(v, w; z) = \frac{\Gamma(1 - w)}{\Gamma(v - w + 1)} \Phi(v, w; z) + \\ + \frac{\Gamma(w - 1)}{\Gamma(v)} z^{1-w} \Phi(v - w + 1, 2 - w; z), \quad w \notin \mathbf{N}, \text{Re } z > 0,$$

where the principal branch of  $z^{1-w}$  is chosen and

$$(4.28) \quad \Phi(v, w; z) = \sum_{n=0}^{\infty} \frac{(v)_n}{(w)_n} z^n/n!$$

(see Tricomi [22], 4.6(7); the  $\Psi$ -function could be regarded as a particular case of the  $\mathcal{G}$ -function). Define for  $t$  in a conic neighborhood of  $\mathbf{R}_+$

$$(4.29) \quad J^+(t) = \exp[ir_1 t^{k+1}/(k + 1)] \cdot \\ \cdot \Psi(ia_1/[\tau(k + 1)(r_2 - r_1)] + k/(k + 1), k/(k + 1); i(r_2 - r_1) t^{k+1}/(k + 1))$$

and for  $t$  in a conic neighborhood of  $\mathbf{R}_-$

$$(4.30) \quad J^-(t) = \exp[ir_2 t^{k+1}/(k + 1)] \cdot \\ \cdot \Psi(ia_1/[\tau(k + 1)(r_1 - r_2)], k/(k + 1); i(r_1 - r_2) t^{k+1}/(k + 1)).$$

$J^+$  and  $J^-$  extend to entire analytic functions; using the properties of  $\Psi$  in Tricomi [22] and arguing as in the proof of Theorem 4.2 we obtain  $\mathcal{N}_{\tau, r_1, r_2}^{k, 2}(a_1)J^\pm = 0$ ,  $J^\pm|_{\mathbf{R}^\pm} \in \mathcal{S}(\mathbf{R}_\pm)$ . Then, according to the remark after Corollary 3.3, we compute the value of the Wronskian  $W(J^+, J^-)(t)$ ; from (4.27), (4.28), (4.29), (4.30) we get easily

$$(4.31) \quad W(J^+, J^-)(0) = T \cos \left( \pi \left\{ \frac{k}{2(k+1)} + ia_1 / [\tau(k+1)(r_2 - r_1)] \right\} \right),$$

with

$$(4.32) \quad T = 2[i(r_2 - r_1)]^{1/(k+1)}(k+1)^{k/(k+1)} \frac{\sin \{k\pi/[2(k+1)]\}}{\sin \{\pi/(k+1)\}}.$$

In conclusion, when  $k$  is even the equation  $\mathcal{N}_{\tau, r_1, r_2}^{k, 2}(a_1)y = 0$  has a non-trivial solution  $y \in \mathcal{S}(\mathbf{R})$  if and only if  $W(J^+, J^-) = 0$ , that is:

$$(4.33) \quad (k+1)L - ia_1/[(r_2 - r_1)\tau] = -\frac{1}{2}$$

for some integer  $L \in \mathbf{Z}$ . Using again Meijer's  $G$ -functions, one could handle in the same way the more general equation  $\mathcal{N}_{\tau, r_1, r_0}^{k, M}(a)y = 0$ , in the case when  $k$  is even.

### 5. - Solutions of exponential type.

Let us denote  $E_{r_0}^k$  the vector space of all the functions of the exponential type

$$(5.1) \quad y(t) = \exp[ir_0 t^{k+1}/(k+1)]X(t),$$

where  $X$  is an arbitrary polynomial with complex coefficients. If  $k \geq 1$  is odd and  $\text{Im } r_0 > 0$ , we have  $E_{r_0}^k \subset \mathcal{S}(\mathbf{R})$  and the following obvious inclusion holds for every  $\mathfrak{L}$  in (3.1):

$$(5.2) \quad \text{Ker } \mathfrak{L} \cap E_{r_0}^k \subset \text{Ker } \mathfrak{L} \cap \mathcal{S}(\mathbf{R}).$$

As the asymptotic expansion (4.1) suggests, (5.2) becomes particularly significant when  $M^+ = 1$  and  $r_1 = r_0$  ( $k$  in (5.2) and  $k$  in (3.1) are the same odd integer), or else in general when the operator is of the form (2.8) with  $\text{Im } r_0 > 0$ . Actually, under suitable additional hypotheses, we shall see that for  $\mathfrak{L}$  of this type the assumption  $\mathfrak{L}y = 0$ ,  $y \in \mathcal{S}(\mathbf{R})$ , implies automatically  $y \in E_{r_0}^k$ .

Consider

$$(5.3) \quad \mathcal{R} = \sum_{0 \leq j < M^+} \left( \sum_{\alpha/k + \beta = M^- - j/k} a_{\alpha\beta} t^\alpha \mathfrak{z}^\beta \right) \mathfrak{z}^{M^+ - j} + \\ + \sum_{M^+ \leq j \leq M^+ k / (k+1)} \sum_{\alpha/k + \beta = M^- - j(k+1)/k} a'_{\alpha\beta} t^\alpha \mathfrak{z}^\beta, \quad \mathfrak{z} = D_t - r_0 t^k,$$

where  $\text{Im } r_0 > 0$  and  $\text{Im } r_v < 0$  for  $M^+ < v \leq M^+ + M^- = M$ , if we note  $r_{M^+ + 1}, \dots, r_M$  the roots of the equation  $\sum_{\alpha/k + \beta = M^-} a_{\alpha\beta} (r - r_0)^\beta = 0$ ;  $a_{\alpha\beta}, \alpha/k + \beta < M^-$ ,  $a'_{\alpha\beta}$  are complex parameters and we understand  $kM^- \geq M^+ > 0$ .

**PROPOSITION 5.1.** *Assume  $\mathcal{R}y = 0$ . Then  $y(t)$  is in  $E_{r_0}^k$  if and only if for every integer  $m$ ,  $0 \leq m < (k + 1)/2$ , the function*

$$(5.4) \quad \mathfrak{Y}_m(t) = y(t \exp[2m\pi i / (k + 1)]), \quad t \in \mathbf{C},$$

is in  $\mathcal{S}(\mathbf{R})$  (we mean: the restriction of  $\mathfrak{Y}_m$  to the real axis is in  $\mathcal{S}(\mathbf{R})$ ).

If in addition in (5.3)  $M$  is even and  $k + 1 = HM$ , for some integer  $H \geq 1$ , then in order to have  $y \in E_{r_0}^k$  it is sufficient to suppose  $\mathfrak{Y}_m \in \mathcal{S}(\mathbf{R})$  for every  $m = jH$ ,  $0 \leq j < M/2$ .

Observing that  $\mathfrak{Y}_0 = y$ , from Proposition 5.1 we get the following

**COROLLARY 5.2** (Helffer - Rodino [9]). *Assume in (5.3)  $k = 1$ , or else  $M = 2$ ; then*

$$(5.5) \quad \text{Ker } \mathcal{R} \cap E_{r_0}^k = \text{Ker } \mathcal{R} \cap \mathcal{S}(\mathbf{R}).$$

Since the determination of  $\dim[\text{Ker } \mathcal{R} \cap E_{r_0}^k]$  is a merely algebraic matter, (5.5) leads to an explicit expression of the related eigenvalues. In particular, if  $M = 2$ ,  $\mathcal{R}$  comes down to the operator in (4.24) and using Corollary 5.2 we recapture easily formula (4.26). For the case  $k = 1$ ,  $M^+ = M^-$ , see the next Theorem 5.5.

When  $k > 1$ ,  $M > 2$ , the simple assumption  $y \in \mathcal{S}(\mathbf{R})$  does not imply in general  $y \in E_{r_0}^k$ ; for example, if in (4.5) it is  $k \geq 3$ ,  $M \geq 3$ ,  $a_1 = 0$ , the solutions  $y \in \mathcal{S}(\mathbf{R})$  of  $\mathcal{N}y = 0$  are no longer of exponential type.

**PROOF OF PROPOSITION 5.1.** First let us show how the second part of the Proposition can be deduced from the first part. Integrating by series the equation  $\mathcal{R}y = 0$  we see easily that in the case  $k + 1 = HM$  a fundamental system of solutions is given by functions of the form  $\bar{y}_h(t) = t^h F_h(t^{k+1})$ ,  $h = 0, 1, \dots, M - 1$ , where  $F_h(z)$ ,  $z \in \mathbf{C}$ , is entire analytic. Let  $y(t)$  satisfy the hypotheses of the second part of the statement, that is:

$\Re y = 0$  and

$$(5.6) \quad y(t \exp[j2\pi i/M]) \in \mathcal{S}(\mathbf{R})$$

for  $0 \leq j < M/2$ ; note that, if  $M$  is even, we may actually assume in (5.6)  $j = 0, 1, \dots, M-1$ . We have for suitable constants  $\mu_h \in \mathbf{C}$

$$(5.7) \quad y(t) = \sum_{h=0}^{M-1} \mu_h t^h F_h(t^{k+1})$$

and then for  $j = 0, 1, \dots, M-1$

$$(5.8) \quad y(t \exp[j2\pi i/M]) = \sum_{h=0}^{M-1} \mu_h \exp[jh2\pi i/M] t^h F_h(t^{k+1}),$$

since  $\exp[j(k+1)2\pi i/M] = \exp[jH2\pi i] = 1$ . Observing that the determinant  $(\exp[jh2\pi i/M])_{j,h=0,1,\dots,M-1}$  is of the Cauchy-Vandermonde type and solving in (5.8) with respect to  $\mu_h t^h F_h(t^{k+1})$ , we obtain

$$(5.9) \quad \mu_h t^h F_h(t^{k+1}) = \sum_{j=0}^{M-1} \varepsilon_{h,j} y(t \exp[j2\pi i/M]),$$

$h = 0, 1, \dots, M-1$ , with suitable constants  $\varepsilon_{h,j}$ . From (5.6) and (5.9) we get

$$(5.10) \quad \mu_h t^h F_h(t^{k+1}) \in \mathcal{S}(\mathbf{R}),$$

and then  $F_h(t^{k+1}) \in \mathcal{S}(\mathbf{R})$  whenever  $\mu_h \neq 0$ . Therefore

$$(5.11) \quad \mathfrak{Y}_m(t) = \sum_{j=0}^{M-1} \mu_j \exp[hm2\pi i/(k+1)] t^j F_j(t^{k+1})$$

is in  $\mathcal{S}(\mathbf{R})$  for every  $m \in \mathbf{Z}$ , so that we may conclude  $y \in E_{r_0}^k$ , in view of the first part of the statement.

To prove the first part of the Proposition we shall use the following extension of the Phragmen-Lindelöf theorem (see for example Titchmarsh [20], pp. 177-180).

LEMMA 5.3. *Let  $f(t)$  be analytic for*

$$(5.12) \quad R \leq |t| < \infty, \quad \varphi_1 \leq \arg t < \varphi_2,$$

where  $R, \varphi_1, \varphi_2$  are real constants. Let

$$(5.13) \quad |f(t)| \leq C \exp[\nu|t|^\eta]$$

in the same region, for some real constants  $C, \nu$  and  $\eta$  such that  $\varphi_2 - \varphi_1 < \pi/\eta$ . If  $f$  is bounded as  $|t| \rightarrow \infty$  on the lines  $\arg t = \varphi_1$  and  $\arg t = \varphi_2$ , then  $f$  is bounded uniformly in the region (5.12).

We shall also need a generalization of the asymptotic expansion (4.1) to the operator  $\mathcal{R}$  in (5.3).

LEMMA 5.4. Let  $S$  be an open sector of the  $t$ -plane with vertex at the origin and a positive central angle not exceeding  $\pi/(k+1)$ . There exist  $M^+$  entire analytic functions,  $y_{S,1}(t), \dots, y_{S,M^+}(t)$ , such that

- (I)  $y_{S,1}, \dots, y_{S,M^+}$  are independent solutions of the equation  $\mathcal{R}y = 0$ .
- (II) In  $S$  we have for  $v = 1, \dots, M^+$  asymptotic expansions of the form:

$$(5.14) \quad y_{S,v}(t) \sim \exp[ir_0 t^{k+1}/(k+1)] \left( \sum_h t^{\vartheta_{vh}} (\log t)^{\zeta_h} \sum_{n=0}^{\infty} \omega_{vhn} t^{-n} \right);$$

the sum in  $h$  is finite;  $\vartheta_{vh}, \zeta_{vh}, \omega_{vhn}$  are suitable constants which do not depend on  $t$  and  $S$ .

(III) Let  $\varphi$  be a fixed real number such that  $\Theta = \{t = \exp[i\varphi]x, x \in \mathbf{R}_+\} \subset S$ . Let  $y(t)$  be any solution of  $\mathcal{R}y = 0$ , such that  $y(\exp[i\varphi]x) \in \mathcal{S}(\mathbf{R}_+)$  (we shall also write:  $y|_{\Theta} \in \mathcal{S}(\mathbf{R}_+)$ ). Then  $y = \sum_{v=1}^{M^+} \mu_v y_{S,v}$  for some constants  $\mu_v \in \mathbf{C}$ .

To prove Lemma 5.4, it will be sufficient to observe that

$$(5.15) \quad \begin{aligned} \mathcal{R}^+ t^\lambda &= \exp[-ir_0 t^{k+1}/(k+1)] \mathcal{R} \exp[ir_0 t^{k+1}/(k+1)] t^\lambda = \\ &= Q^+(\lambda) t^{\lambda+kM^- - M^+} + O(t^{\lambda+kM^- - M^+ - 1}), \end{aligned}$$

with

$$(5.16) \quad \begin{aligned} Q^+(\lambda) &= \sum_{0 \leq j < M^+} (-i)^{M^+ - j} a_{kM^- - j, 0} \lambda(\lambda - 1) \dots \\ &\dots (\lambda - M^+ + j + 1) + a'_{kM^- - M^+, 0}. \end{aligned}$$

Since the degree of the indicial polynomial  $Q^+(\lambda)$  is  $M^+$ , using Remarks 2.4.1; 2.2.6 in [9] we can construct easily  $y_{S,1}, \dots, y_{S,M^+}$  satisfying (I), (II), (III).

Let us end the proof of Proposition 5.1. We note for  $m = 0, 1, \dots, k$

$$(5.17) \quad \Theta_m = \{t \in \mathbf{C}, \arg t = 2m\pi/(k + 1)\}$$

and for a sufficiently small  $\varepsilon > 0$  ( $\varepsilon = 1/10$  is enough, for example)

$$(5.18) \quad S_m^+ = \{t \in \mathbf{C}, (2m - \varepsilon)\pi/(k + 1) < \arg t < (2m + 1 - 2\varepsilon)\pi/(k + 1)\},$$

$$(5.19) \quad S_m^- = \{t \in \mathbf{C}, (2m - 1 + 2\varepsilon)\pi/(k + 1) < \arg t < (2m + \varepsilon)\pi/(k + 1)\}.$$

Let us apply Lemma 5.4 to every sector  $S_m^\pm$ . The hypothesis  $\mathfrak{Y}_m \in \mathfrak{S}(\mathbf{R})$ ,  $0 \leq m < (k + 1)/2$ , is equivalent to the assumption  $y|_{\Theta_m} \in \mathfrak{S}(\mathbf{R}_+)$ , for all  $m$ ,  $m = 0, 1, \dots, k$ , and since  $\Theta_m \subset S_m^\pm$ , we have from (III):

$$(5.20) \quad y(t) = \sum_{v=1}^{M^+} \mu_{S_m^\pm, v} y_{S_m^\pm, v}(t), \quad \mu_{S_m^\pm, v} \in \mathbf{C},$$

where  $y_{S_m^\pm, 1}, \dots, y_{S_m^\pm, M^+}$  admit in  $S_m^\pm$  the asymptotic expansions (5.14). Then for  $N \geq 2 + \max \vartheta_{v_h}$  the function

$$(5.21) \quad f(t) = t^{-N} \exp[-ir_0 t^{k+1}/(k + 1)]y(t)$$

is bounded in  $(\cup S_m^\pm) \cap \{t \in \mathbf{C}, |t| \geq 1\}$ . Actually,  $f$  is bounded in the whole region  $\{t \in \mathbf{C}, |t| \geq 1\}$ . In fact we have from the theory of the asymptotic integration that for any  $\sigma > 0$

$$(5.22) \quad |y(t)| \leq C \exp[(\sigma + \max_v \operatorname{Re}(ir_v)]|t|^{k+1}/(k + 1)], \quad t \in \mathbf{C},$$

for a suitable constant  $C$  (see for example [9], Theorem 2.2.3 and Remark 2.4.1) so that we can apply Lemma 5.3 repeatedly to  $f$  in (5.21), with

$$\eta = k + 1, \quad \nu = [\sigma - \operatorname{Re}(ir_0) + \max_v \operatorname{Re}(ir_v)]/(k + 1)$$

in (5.13) and

$$\varphi_1 = (2m + 1 - 3\varepsilon)\pi/(k + 1), \quad \varphi_2 = (2m + 1 + 3\varepsilon)\pi/(k + 1),$$

$m = 0, 1, \dots, k$ . In conclusion: the function

$$(5.23) \quad X(t) = \exp[-ir_0 t^{k+1}/(k + 1)]y(t) = f(t)t^N$$

is analytic in  $\mathbf{C}$ , with a pole at  $\infty$ ; thus it is indeed a polynomial. Since in the opposite direction  $y \in E_{r_0}^k$  implies trivially  $\mathfrak{Y}_m \in \mathfrak{S}(\mathbf{R})$ ,  $m \in \mathbf{Z}$ , the proof of Proposition 5.1 is complete.

In the applications of Section 7 we shall refer to the following consequence of Corollary 5.2.

**THEOREM 5.5.** *Let  $M$  be even. Consider the operator*

$$(5.24) \quad \mathfrak{C} = \sum_{0 \leq j \leq M/2} \left( \sum_{\alpha + \beta = M/2 - j} a_{\alpha\beta} t^\alpha (D_t - r_0 t)^\beta \right) (D_t - r_0 t)^{M/2 - j},$$

where  $\text{Im } r_0 > 0$  and all the roots of the equation  $\sum_{\alpha + \beta = M/2} a_{\alpha\beta} (r - r_0)^\beta = 0$  have negative imaginary part. Define

$$(5.25) \quad Q^+(\lambda) = a_{0,0} + \sum_{0 \leq j < M/2} (-i)^{M/2 - j} a_{M/2 - j, 0} \lambda(\lambda - 1) \dots (\lambda - M/2 + j + 1).$$

We have  $\text{Ker } \mathfrak{C} \cap \mathfrak{S}(\mathbf{R}) = \{0\}$  if and only if  $Q^+(N) \neq 0$  for every integer  $N \geq 0$ .

**PROOF.**  $\mathfrak{C}$  in (5.24) is of the form (5.3) with  $k = 1$ ,  $M^+ = M^- = M/2$ . It follows from Corollary 5.2 that  $\text{Ker } \mathfrak{C} \cap E_{r_0}^1 = \text{Ker } \mathfrak{C} \cap \mathfrak{S}(\mathbf{R})$ . Consider

$$(5.26) \quad \mathfrak{C}^+ = \exp[-ir_0 t^2/2] \mathfrak{C} \exp[ir_0 t^2/2] = \sum_{0 \leq j \leq M/2} \left( \sum_{\alpha + \beta = M/2 - j} a_{\alpha\beta} t^\alpha D_t^\beta \right) D_t^{M/2 - j}.$$

It is clear that  $\text{Ker } \mathfrak{C} \cap E_{r_0}^1 \neq \{0\}$  if and only if  $\mathfrak{C}^+ X = 0$ , for some non-trivial polynomial  $X(t) = \sum_{0 \leq n \leq N} v_n t^n$ ,  $v_n \in \mathbf{C}$ ,  $v_N = 1$ . Now we have in general

$$(5.27) \quad \mathfrak{C}^+ X = Q^+(N) t^N + \sum_{0 \leq n < N} \left( Q^+(n) v_n + \sum_{n < p \leq N} \sigma_{np} v_p \right) t^n,$$

where the constants  $\sigma_{np}$  depend on  $a_{\alpha\beta}$ . To obtain  $\mathfrak{C}^+ X = 0$  it is therefore necessary that  $Q^+(N) = 0$ , for some integer  $N \geq 0$ . This condition is also sufficient for the existence of a non-trivial polynomial solution. In fact, if  $Q^+(\lambda) = 0$  has one integer root  $N \geq 0$ , the coefficients of  $X$  can be determined inductively by

$$(5.28) \quad v_n = \left( - \sum_{n < p \leq N} \sigma_{np} v_p \right) / Q^+(n), \quad 0 \leq n < N.$$

If there are  $h \geq 2$  integers  $N_1, \dots, N_h$  which satisfy  $Q^+(\lambda) = 0$ , we just consider  $N = \min \{N_1, \dots, N_h\}$ , so that in (5.28) it is always  $Q^+(n) \neq 0$ . Theorem 5.5 is proved.

REMARK 5.6. Note that the existence of an integer root of the indicial equation is always necessary to have  $\text{Ker } \mathcal{R} \cap E_{r_0}^k \neq \{0\}$ , for every  $\mathcal{R}$  of the type (5.3). In fact  $\mathcal{R}y = 0$  has a non-trivial solution in the class  $E_{r_0}^k$  if and only if  $\mathcal{R}^+X = 0$  for some non-trivial polynomial  $X(t) = \sum_{0 \leq n \leq N} v_n t^n$ ,  $v_n \in \mathbf{C}$ ,  $v_N = 1$  (the operator  $\mathcal{R}^+$  is defined as in (5.15)). On the other hand from (5.15) we have

$$(5.29) \quad \mathcal{R}^+X(t) = Q^+(N)t^{N+kM^- - M^+} + O(t^{N+kM^- - M^+ - 1}),$$

where the indicial polynomial  $Q^+(\lambda)$  is defined in general by (5.16); therefore  $\mathcal{R}^+X = 0$  implies  $Q^+(N) = 0$ .

### CHAPTER III

### CONCLUSIONS

#### 6. - An explicit result.

Let  $M^+, M^-$  be defined as in Section 1; combining Theorem 2.1 and Corollary 3.2 we obtain

THEOREM 6.1. *Let  $p(x, \xi)$  be in  $A_k^{m, M}(\Sigma_1, \Sigma_2, \varrho)$  and note  $P = p(x, D)$ . If  $k$  is odd and  $M^+ = 0$ , or else is even and  $\min \{M^+, M^-\} = 0$ , then  $P$  satisfies (0.5). If  $k$  is odd and  $M^+ > M^-$ , then  $P$  does not satisfy (0.5).*

In effect, we think it is possible with small modifications of the proofs in [16] to obtain a non-hypoellipticity result when  $k$  is odd and  $M^+ > M^-$ . For  $M = 1$  we get from Theorem 6.1 a particularization of the general result of [4], [10], [21] on subelliptic estimates:

COROLLARY 6.2. *Let  $\Sigma$  be a closed conic submanifold of codimension 2 of  $\Omega \times \mathbf{R}^n$  and let  $p(x, \xi) \sim \sum_{j=0}^{\infty} p_{m-j}(x, \xi)$  be a classical symbol in  $\Omega \times \mathbf{R}^n$ , with  $p_m = 0$  exactly on  $\Sigma$ . Assume that for every  $\varrho \in \Sigma$  there exist a conic neigh-*

neighborhood  $\Gamma$  of  $\rho$  and a complex number  $z$  such that  $zp_m = a + ib$ , where  $d_\xi a \neq 0$  in  $\Gamma$  and  $H_a^j b = 0$  on  $\Sigma \cap \Gamma$  if  $j < k$  but  $H_a^k b \neq 0$ , for a fixed  $k$  independent of  $\rho$ . Then  $p(x, \xi) \in A_k^{m,1}(\Sigma_1, \Sigma_2, \rho)$  at any  $\rho \in \Sigma$  for a suitable choice of  $\Sigma_1, \Sigma_2$ . Moreover, if  $k$  is even the operator  $P = p(x, D)$  satisfies in every  $K \subset\subset \Omega$  the estimates

$$(6.1) \quad \|f\|_{s+m-k/(k+1)} \leq C_{s,K} \{ \|Pf\|_s + \|f\|_{s+m-k/(k+1)-\varepsilon} \}, \quad f \in C_0^\infty(K),$$

for some  $\varepsilon > 0$ . If  $k$  is odd  $P$  satisfies (6.1) if and only if it is always  $H_a^k b > 0$ .

PROOF. In a neighborhood of a fixed  $\rho \in \Sigma$  we may assume without loss of generality that  $p_m = a + ib$  is homogeneous of degree 1, with  $H_a^j b = 0$  for  $j < k$  and  $H_a^k b \neq 0$  on  $\Sigma$ . Noting  $c = H_a^{k-1}$ , we see that  $\{a = c = 0\} \subset \Sigma$  and  $\{a, c\} = H_a^k b \neq 0$ . This implies that the differential of  $a$  and  $c$  are linearly independent. Letting  $\Sigma_1 = \{a = 0\}$ ,  $\Sigma_2 = \{c = 0\}$ , we have locally  $\Sigma = \Sigma_1 \cap \Sigma_2$ , since one of these manifolds is contained in the other and they both have the same codimension. It is also evident that  $\Sigma$  is non-involutive.

Consider the Cauchy problem  $H_a c' = 1, c' = 0$  on  $\Sigma_2$ . Since  $H_a$  is transverse to  $\Sigma_2$ , a solution  $c'$  exists, and it is homogeneous of degree 0. Using  $a$  and  $c'$  as coordinates and then taking a Taylor expansion of  $b$ , we may deduce that  $b = hc'^k + h'a$ , for some functions  $h \neq 0$  and  $h'$ . Dividing by the elliptic factor  $1 + ih'$ , we may replace  $p_m$  by  $a + igc'^k$  where  $Reg \neq 0$  near  $\rho$ . It is therefore clear that  $p(x, \xi) \in A_k^{m,1}(\Sigma_1, \Sigma_2, \rho)$ .

Now, when  $k$  is even we have always  $\min \{M^+, M^-\} = 0$ , so that (0.5) is satisfied with  $M = 1$ , in view of Theorem 6.1. When  $k$  is odd we have  $M^+ = 0, M^- = 1$  if  $Reg(\rho) > 0, M^+ = 1, M^- = 0$  if  $Reg(\rho) < 0$ ; since  $\text{sign } Reg(\rho) = \text{sign } h(\rho) = \text{sign } H_a^k b$ , it follows from Theorem 6.1 that in this case (0.5) is satisfied if and only if  $H_a^k b > 0$ . Then a standard partition of unity argument gives the conclusion.

### 7. - Symbols with multiple roots.

When  $k$  is odd and  $M^- \geq M^+ > 0$ , or else  $k$  is even and  $\min \{M^+, M^-\} > 0$ , necessary and sufficient conditions for the validity of (0.5) can be deduced from a combination of Theorem 2.1, Corollary 3.3 and the remark after Corollary 3.3. Precise statements are left to the reader.

Here we limit ourselves to consider some operators with symbol in the classes  $\pm A_k^{m, M^+, M^-}(\Sigma_1, \Sigma_2, \rho)$  of Definition 1.1. Actually, using the

results of Sections 4, 5, one obtains for such operators necessary and sufficient conditions of pure algebraic type, whereas in the general case Corollary 3.3 leads to the problematic computation of the zeros of certain entire analytic functions. Consider first  $P = p(x, D)$ , with  $p(x, \xi) \in \mathcal{A}_k^{m,1,M-1}(\Sigma_1, \Sigma_2, \varrho)$ . Let us write  $P$  in the canonical form for multiple roots, according to Proposition 1.4; with lightly different notations, it is:

$$(7.1) \quad P = (U_2^k A_0 + A_{00} Z) Z^{M-1} + \sum_{0 < j \leq M-1} U_2^{k-j} A_j Z^{M-j-1} + A_M$$

where

$$(7.2) \quad Z = U_1 - R_0 U_2^k .$$

$U_1, U_2$  have principal symbol  $u_1(x, \xi), u_2(x, \xi), u_1, u_2$  as in Proposition 1.2;  $R_0$  is of degree 1 with principal symbol  $r_0(x, \xi), \text{Im } r_0(\varrho) < 0$ . The degrees of  $A_0, A_{00}, A_j, 0 < j \leq M-1$ , are  $m - M + 1, m - M, m - M + 1$ , with principal symbols  $a_0(x, \xi), a_{00}(x, \xi), a_j(x, \xi)$ , respectively; the degree of  $A_M$  is  $m - M$ . We have  $a_{00}(\varrho) \neq 0$  and, noting  $r_1 = (r_0 a_{00} - a_0)/a_{00}, \text{Im } r_1(\varrho) > 0$ . Let us recall that in Definition 1.1  $k$  is odd and  $k \geq M - 1 > 0$ . According to (1.11), we associate to  $p(x, \xi)$  the polynomial

$$(7.3) \quad Q_p^-(\lambda) = a_{M-1}(\varrho) + \sum_{0 \leq j < M-1} (-i)^{M-j-1} a_j(\varrho) \lambda(\lambda - 1) \dots (\lambda - M + j + 2) .$$

As it was observed at the end of Section 2, the roots of  $Q_p^-(\lambda) = 0$  do not depend on the representation (7.1).

**THEOREM 7.1.** *Let  $p(x, \xi)$  be in  $\mathcal{A}_k^{m,1,M-1}(\Sigma_1, \Sigma_2, \varrho)$  and let  $Q_p^-(\lambda)$  be defined as in (7.3). Then  $P = p(x, D)$  satisfies (0.5) if and only if there exist an odd integer  $h_1$  and an even integer  $h_2, 0 \leq h_1, h_2 \leq M - 1$ , such that*

$$Q_p^-(h_1 - (k + 1)N) \neq 0, \quad Q_p^-(h_2 - (k + 1)N) \neq 0$$

for every integer  $N > 0$ .

Theorem 7.1 is a direct consequence of Theorem 2.1, Proposition 2.3, Theorem 4.2 and of the remark after Theorem 4.2. Note that Theorems 6.1, 7.1 allow a complete discussion of the validity of (0.5) for operators with symbol in  $\mathcal{A}_k^{m,2}(\Sigma_1, \Sigma_2, \varrho)$ , when  $k$  is odd; when  $k$  is even explicit results for  $M = 2$  can be deduced from Theorem 6.1 and formula (4.33) (cf. [3], [14], [23]).

Consider now  $P = p(x, D)$ , with  $p(x, \xi) \in {}^+A_1^{m, M/2, M/2}(\Sigma_1, \Sigma_2, \rho)$ , where we assume  $M$  is even. Write  $P$  in the canonical form for multiple roots:

$$(7.4) \quad P = \sum_{0 \leq j \leq M/2} \left( \sum_{\alpha + \beta = M/2 - j} U_2^\alpha A_{\alpha\beta} Z^\beta \right) Z^{M/2 - j} + T,$$

where  $Z$  is defined as in (7.2), but now  $\text{Im } r_0(\rho) > 0$ ;  $A_{\alpha\beta}$  are classical pseudo differential operators of degree  $m - \beta - M/2$ , with principal symbols  $a_{\alpha\beta}$ ;  $T$  is of degree  $m - M/2 - 1$ . All the roots of the equation  $\sum_{\alpha + \beta = M/2} a_{\alpha\beta}(\rho)(r - r_0(\rho))^\beta = 0$  have negative imaginary part. The indicial polynomial is given in this case by

$$(7.5) \quad Q_p^+(\lambda) = a_{0,0}(\rho) + \sum_{0 \leq j < M/2} (-i)^{M/2 - j} a_{M/2 - j, 0}(\rho) \lambda(\lambda - 1) \dots \dots (\lambda - M/2 + j + 1).$$

**THEOREM 7.2.** *Let  $p(x, \xi)$  be in  ${}^+A_1^{m, M^+, M^-}(\Sigma_1, \Sigma_2, \rho)$  with  $M^+ = M^- = M/2$ , and let  $Q_p^+(\lambda)$  be defined as in (7.5). Then  $P = p(x, D)$  satisfies (0.5) if and only if  $Q_p^+(N) \neq 0$  for every integer  $N \geq 0$ .*

Theorem 7.2 is a direct consequence of Theorem 2.1, Proposition 2.3 and Theorem 5.5. Using Corollary 5.2, we may obtain a more general result (the condition for the validity of (0.5) is less explicit, but still algebraic):

**PROPOSITION 7.3** (cf. [8], [9], [13], [17]). *Let  $p(x, \xi)$  be in  ${}^+A_k^{m, M^+, M^-}(\Sigma_1, \Sigma_2, \rho)$ ,  $M^- \geq M^+ > 0$ . Let  $\mathfrak{L}_p^+$  be defined as in (2.6). Assume  $P = p(x, D)$  satisfies (0.5); then:*

$$(7.6) \quad \mathfrak{L}_p^+ X \neq 0 \quad \text{for every polynomial with complex coefficients } X \neq 0.$$

When  $k = 1$  (7.6) is also sufficient for the validity of (0.5).

In fact, defining  $\mathfrak{L}_p$  as in (2.1), we have from Proposition 2.3 that  $\mathfrak{L}_p y = 0$  has a non-trivial solution in the class  $E_{r_0}^k$  of Section 5 if and only if  $\mathfrak{L}_p^+ X = 0$  for some polynomial  $X \neq 0$ . Condition (7.6) is obviously invariant for the transformations (2.2).

Keeping in mind Remark 5.6 we have in particular from Proposition 7.3 that an operator  $P = p(x, D)$ ,  $p(x, \xi) \in {}^+A_1^{m, M^+, M^-}(\Sigma_1, \Sigma_2, \rho)$ ,  $M^- \geq M^+ > 0$ , satisfies (0.5) if the corresponding indicial equation has no integer root  $N \geq 0$ ; however such condition is no longer necessary for the validity of (7.6) and (0.5) in the case  $M^- > M^+$ .

## REFERENCES

- [1] L. BOUTET DE MONVEL, *Hypoelliptic operators with double characteristics and related pseudo differential operators*, Comm. Pure Appl. Math., **27** (1974), pp. 585-639.
- [2] L. BOUTET DE MONVEL - A. GRIGIS - B. HELFFER, *Paramétrixes d'opérateurs pseudo-différentiels à caractéristiques multiples*, Astérisque, **34-35** (1976), pp. 93-121.
- [3] L. BOUTET DE MONVEL - F. TRÈVES, *On a class of pseudo differential operators with double characteristics*, Invent. Math., **24** (1974), pp.1-34.
- [4] YU. B. EGOROV, *Subelliptic operators*, Uspekhi Mat. Nauk, **30** (1975), pp. 57-114; Russian Math. Surveys, **30** (1975), pp. 59-118.
- [5] A. GILIOLI - F. TRÈVES, *An example in the solvability theory of linear PDE's* Amer. J. Math., **96** (1974), pp. 367-385.
- [6] V. V. GRUŠIN, *On a class of hypoelliptic operators*, Mat. Sb., **83** (1970), pp. 456-473; Math. USSR-Sb., **12** (1970), pp. 458-476.
- [7] R. C. GUNNING - H. ROSSI, *Analytic functions of several complex variables*, Prentice-Hall, Inc. (1965), Englewood Cliffs.
- [8] B. HELFFER - L. RODINO, *Elliptic pseudo differential operators degenerate on a symplectic submanifold*, Bull. Amer. Math. Soc., **82** (1976), pp. 619-622.
- [9] B. HELFFER - L. RODINO, *Opérateurs différentiels ordinaires intervenant dans l'étude de l'hypoellipticité*, Boll. Un. Mat. Ital., **14-B** (1977), pp. 491-522.
- [10] L. HÖRMANDER, *Pseudo differential operators and nonelliptic boundary value problem*, Ann. of Math., **83** (1966), pp. 129-209.
- [11] L. HÖRMANDER, *Fourier integral operators, I*, Acta Math., **121** (1971), pp. 79-183.
- [12] Y. L. LUKE, *The special functions and their approximations*, Academic Press (1969), New York and London.
- [13] M. MASCARELLO RODINO - L. RODINO, *Two examples of pseudo differential operators with multiple characteristics*, Boll. Un. Mat. Ital., **13-A** (1976), pp. 391-398.
- [14] A. MENIKOFF, *Some examples of hypoelliptic partial differential equations*, Math. Ann., **221** (1976), pp. 167-181.
- [15] J. NOURRIGAT, *Paramétrixes pour une classe d'opérateurs hypoelliptiques*, Comm. Partial Diff. Equations, **3** (1978), pp. 407-441.
- [16] C. PARENTI - L. RODINO, *Paramétrices for a class of pseudo differential operators, I, II*, Ann. Mat. Pura Appl., **125** (1980), pp.221-278.
- [17] L. RODINO, *Nonsolvability of a hygher order degenerate elliptic pseudo differential equation*, Amer. J. Math., **102** (1980), pp. 1-12.
- [18] Y. SIBUYA, *Global theory of a second order linear ordinary differential equation with a polynomial coefficient*, North-Holland (1975), Amsterdam.
- [19] J. SJÖSTRAND, *Paramétrices for pseudo differential operators with multiple characteristics*, Ark. Mat., **12** (1974), pp. 85-130.
- [20] E. C. TITCHMARSH, *The theory of functions*, Oxford Press (1939), Oxford.
- [21] F. TRÈVES, *A new method of proof of the subelliptic estimates*, Comm. Pure Appl. Math., **24** (1971), pp. 71-115.
- [22] F. G. TRICOMI, *Funzioni speciali*, Ed. Tirrenia (1965), Torino.

- [23] K. YAMAMOTO, *Parametrixes for pseudo differential equations with double characteristics*, Hokkaido Math. J., **5** (1976), pp. 280-301.
- [24] B. HELFFER - J. NOURRIGAT, *Construction de paramétrixes pour une nouvelle classe d'opérateurs pseudodifférentiels*, J. Differential Equations, **32** (1979), pp. 41-64.
- [25] J. NOURRIGAT, *Hypoellipticité et paramétrixes pour une classe d'opérateurs différentiels à caractéristiques multiples*, Thèse d'état (1979).

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