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Generation of analytic semigroups by elliptic operators of second order in Hölder spaces

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0. – Introduction.

Let Ω be a bounded open subset of R^n with boundary $\partial\Omega$ of class C^2. If $x = (x, \dots, x_n) \in R^n$, $\|x\|$ will denote its euclidean norm. By the same symbol we shall denote also the norm of the points $z = (z_1, \dots, z_n) \in \boldsymbol{C}^n$. If u is a complex valued function defined in Ω we set $Du = (D_1 u, \dots, D_n u)$ where $D_i = \partial/\partial x_i$ and, as usual,

$$\|u\|_{0,\Omega} = \left\{ \int_\Omega |u|^2 \, dx \right\}^{\frac{1}{2}} \tag{0.1}$$

$$|u|_{1,\Omega} = \left\{ \int_\Omega \|Du\|^2 \, dx \right\}^{\frac{1}{2}} \tag{0.2}$$

$$\|u\|_{1,\Omega} = \left\{ \|u\|^2_{0,\Omega} + |u|^2_{1,\Omega} \right\}^{\frac{1}{2}}. \tag{0.3}$$

$H^1(\Omega)$ and $H^1_0(\Omega)$ are the usual Sobolev spaces, i.e. the completion in the norms (0.3) and (0.2) respectively of the space $C^\infty(\bar{\Omega})$ or $C^\infty_0(\Omega)$.

$C^{0,\alpha}(\bar{\Omega})$, with $0 < \alpha < 1$, is the space of the complex valued functions $u\colon \bar{\Omega} \to \boldsymbol{C}$ such that

$$[u]_{\alpha,\bar{\Omega}} = \sup_{x,y \in \bar{\Omega}} \frac{|u(x) - u(y)|}{\|x - y\|^\alpha} < +\infty \tag{0.4}$$

with the norm

$$\|u\|_{C^{0,\alpha}(\bar{\Omega})} = \|u\|_{0,\Omega} + [u]_{\alpha,\bar{\Omega}}. \tag{0.5}$$

It is well known that the norm (0.5) is equivalent to the norm $\sup_\Omega |u| + [u]_{\alpha,\bar{\Omega}}$.

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We define

$$Eu = -\sum_{ij=1}^{n} D_i(a_{ij}D_ju) + \sum_{i=1}^{n} b_iD_iu .$$

The operator E will have coefficients a_{ij}, b_i subject to the following assumptions:

(0.I) a_{ij}, $ij = 1, \ldots, n$, are real valued and continuous functions in $\bar{\Omega}$ and satisfy the ellipticity condition: there is a constant $\nu > 0$ such that

$$\sum_{ij} a_{ij}(x)\xi_i\xi_j \geqslant \nu\|\xi\|^2 \tag{0.6}$$

for all $x \in \bar{\Omega}$ and $\xi \in R^n$.

(0.II) $b_i \in L^\infty(\Omega)$, i.e. are measurable and bounded complex valued functions on Ω.

We set

$$\lambda = \frac{1}{2\nu}\sup_{\Omega}\sum_i |b_i(x)|^2 \tag{0.7}$$

and for all $u, \varphi \in H^1(\Omega)$

$$a(u, \varphi) = \int_{\Omega} \sum_{ij} a_{ij}D_juD_i\bar{\varphi} + \sum_i b_iD_iu\cdot\bar{\varphi}\, dx . \tag{0.8}$$

We have that

$$\left|\int_{\Omega}\sum_i b_iD_iu\cdot\bar{u}\, dx\right| \leqslant \frac{\nu}{2}|u|^2_{1,\Omega} + \lambda\|u\|^2_{0,\Omega} \tag{0.9}$$

and then, for all $u \in H^1(\Omega)$,

$$\mathscr{R}e\, a(u, u) \geqslant \frac{\nu}{2}|u|^2_{1,\Omega} - \lambda\|u\|^2_{0,\Omega} . \tag{0.10}$$

Let μ be a complex parameter and $f \in L^2(\Omega)$. We consider a solution u of the Dirichlet problem

$$\begin{cases} u \in H^1_0(\Omega) \\ \mu\int_{\Omega} u\bar{\varphi}\, dx + a(u, \varphi) = \int_{\Omega} f\bar{\varphi}\, dx , \quad \forall\varphi \in H^1_0(\Omega). \end{cases} \tag{0.11}$$

If $\mathscr{R}e\, \mu \geqslant \lambda$, such a solution exists and is unique because of (0.10).

The aim of the present paper is to prove that if

(0.III) $f \in C^{0,\alpha}(\bar{\Omega})$ and $f = 0$ on $\partial\Omega$,

then there exists $\lambda_0 \geqslant \lambda$ such that, if $\mathcal{R}e\,\mu > \lambda_0$, the following estimate holds

$$\|u\|_{C^{0,\alpha}(\bar{\Omega})} \leqslant \frac{K}{|\mu| - \lambda_0} \|f\|_{C^{0,\alpha}(\bar{\Omega})} \tag{0.12}$$

where the constant K does not depend on μ and λ_0.

The method consist of proving the known result that the solution of (0.11) is α-Hölder continuous on $\bar{\Omega}$ by making use of the techniques of [5] (see also [7]).

Without condition $f = 0$ on $\partial\Omega$, estimate (0.12) is well known with the L^p-norms [2], [3] and C^0-norms [8], [9], [10] but not yet for Hölder norms. Indeed estimate (0.12) is not true without the condition that f vanishes on $\partial\Omega$ [11], [12].

As it is known, inequality (0.12) is linked with the problem of knowing whether the elliptic operator E generates an analytic semigroup on the closure in the Hölder norm of the operator's domain. Such a space can be characterized as the subspace of the functions belonging to $C_0^{0,\alpha}(\bar{\Omega})$ such that

$$\lim_{\delta \to 0} \sup_{\substack{x,y \in \bar{\Omega} \\ \|x-y\| < \delta}} \frac{|f(x) - f(y)|}{\|x - y\|^\alpha} = 0 .$$

I believe that the result of the present paper could be extended further, by means of the same technique, also to more general operators and more general boundary conditions.

Thanks are due to my friend G. Da Prato who suggested me the question.

1. - Preliminaries.

If A is a measurable subset of R^n, with positive measure, and $u\colon A \to \boldsymbol{C}$, we denote by u_A the average of u on A

$$u_A = \frac{1}{\text{meas } A} \int_A u(x)\, dx .$$

Let Ω be an open bounded subset of R^n with diameter d_Ω. We set

$$B(x_0, \sigma) = \{x \in R^n\colon \|x - x_0\| < \sigma\} ,$$
$$\Omega(x_0, \sigma) = \Omega \cap B(x_0, \sigma) .$$

Recall that $u \in \mathscr{L}^{2,\lambda}(\Omega)$, $0 \leqslant \lambda \leqslant n + 2$, if $u \in L^2(\Omega)$ and

$$[u]^2_{\mathscr{L}^{2,\lambda}(\Omega)} = \sup \sigma^{-\lambda} \int_{\Omega(x_0,\sigma)} \|u - u_{\Omega(x_0,\sigma)}\|^2 \, dx < +\infty , \tag{1.1}$$

where the supremum is taken over all $\Omega(x_0, \sigma)$ with $x_0 \in \Omega$ and $0 < \sigma \leqslant d_\Omega$ (see [5], [7]).

We have the following result [4]:

LEMMA 1.I. *If Ω has the cone property and $\alpha \in (0, 1)$, then*

$$u \in \mathscr{L}^{2,n+2\alpha}(\Omega) \Rightarrow u \in C^{0,\alpha}(\bar{\Omega}) \tag{1.2}$$

and

$$[u]_{\alpha,\bar{\Omega}} \leqslant c[u]_{\mathscr{L}^{2,n+2\alpha}(\Omega)} . \tag{1.3}$$

Let V be a closed subspace of $H^1(\Omega)$ with

$$H^1_0(\Omega) \subset V \subset H^1(\Omega)$$

and with induced norm. Let $f \in L^2(\Omega)$ and $\mathscr{R}e\, \mu > \lambda$ (1). It is well known that if u is the solution of the problem

$$\begin{aligned} & u \in V \\ & \mu \int_\Omega u\bar{\varphi} \, dx + a(u, \varphi) = \int_\Omega f\bar{\varphi} \, dx , \quad \forall \varphi \in V \end{aligned} \tag{1.4}$$

then the following estimate holds

$$\|u\|_{0,\Omega} \leqslant \frac{K}{|\mu| - \lambda} \|f\|_{0,\Omega} \tag{1.5}$$

where the constant K does not depend on μ and λ. We give a proof for the reader's convenience and because we will need it for the following lemma.

In (1.4) we take $\varphi = u$ and recall (0.10). We obtain that

$$|u|^2_{1,\Omega} \leqslant \frac{2}{\nu} \left| \int_\Omega f\bar{u} \, dx \right| . \tag{1.6}$$

(1) λ is defined as in (0.7).

On the other hand

$$(1.7)\qquad |\mu|\cdot\|u\|^2_{0,\Omega}\leqslant\Big|\int_\Omega f\bar{u}\,dx\Big| + |a(u,u)|\leqslant\Big|\int_\Omega f\bar{u}\,dx\Big| + c|u|^2_{1,\Omega} + \lambda\|u\|^2_{0,\Omega}\,.$$

From (1.6) and (1.7) it is clear that

$$(|\mu|-\lambda)\|u\|^2_{0,\Omega}\leqslant c\Big|\int_\Omega f\bar{u}\,dx\Big|$$

and then (1.5) easily follows.

In a similar way we can prove also the following lemma:

LEMMA 1.II. *If u is the solution of problem* (1.4) *with $\mathcal{R}e\,\mu>\lambda$ and $V=H^1(\Omega)$, then*

$$(1.8)\qquad \|u-u_\Omega\|_{0,\Omega}\leqslant\frac{K}{|\mu|-\lambda}\|f-f_\Omega\|_{0,\Omega}$$

with K independent of μ and λ.

PROOF. In (1.4) we take $\varphi=u-u_\Omega$ and we obtain that

$$\mu\int_\Omega|u-u_\Omega|^2\,dx + a(u-u_\Omega,\,u-u_\Omega)=\int_\Omega(f-f_\Omega)(\bar{u}-\bar{u}_\Omega)\,dx\,.$$

From this point, the proof is completely analogous to that of estimate (1.5).

We will prove now two lemmas concerning Caccioppoli type inequalities.

LEMMA 1.III. *Let $u\in H^1(\Omega)$ be a solution of the equation*

$$(1.9)\qquad \mu\int_\Omega u\bar{\varphi}\,dx + a(u,\varphi)=0\,,\qquad \forall\varphi\in H^1_0(\Omega)$$

with $\mathcal{R}e\,\mu>\lambda$. Let $B(\varrho)$, $B(\sigma)$ be concentric open balls contained in Ω with $0<\varrho<\sigma$. Then we have the following estimates

$$(1.10)\qquad \int_{B(\varrho)}\|Du\|^2\,dx\leqslant\frac{c}{(\sigma-\varrho)^2}\int_{B(\sigma)}|u|^2\,dx\,,$$

$$(1.11)\qquad \int_{B(\varrho)}\|Du\|^2\,dx\leqslant c\Big(\frac{\sigma}{\varrho}\Big)^n(\sigma-\varrho)^{-2}\int_{B(\sigma)}|u-u_{B(\sigma)}|^2\,dx\,,$$

where the constants c do not depend on μ, ϱ, σ, λ.

PROOF. Let θ be a real valued function, of class $C^\infty(R^n)$, with

$$0\leqslant\theta\leqslant 1\,,\quad \theta=1 \quad \text{on } B(\varrho)\,,\quad \theta=0 \quad \text{in } R^n\setminus B(\sigma) \tag{1.12}$$
$$|D_i\theta|\leqslant c(\sigma-\varrho)^{-1}\,,\qquad i=1,\dots,n\,.$$

Choosing $\varphi=\theta^2 u$ in (1.9), we obtain that

$$\int\limits_{B(\sigma)}\theta^2\sum_{ij}a_{ij}D_juD_i\bar u\,dx+\mu\int\limits_{B(\sigma)}\theta^2|u|^2\,dx=-2\int\limits_{B(\sigma)}\theta\sum_{ij}D_i\theta a_{ij}D_ju\bar u\,dx-$$
$$-\int\limits_{B(\sigma)}\theta^2\sum_i b_iD_iu\bar u\,dx\,.$$

Then, recalling (0.9), we get for all $\varepsilon>0$

$$\frac{\nu}{2}\int\limits_{B(\sigma)}\theta^2\|Du\|^2\,dx+(\mathcal{R}e\,\mu-\lambda)\int\limits_{B(\sigma)}\theta^2|u|^2\,dx\leqslant\varepsilon\int\limits_{B(\sigma)}\theta^2\|Du\|^2\,dx+\frac{c}{(\sigma-\varrho)^2}\int\limits_{B(\sigma)}|u|^2\,dx\,.$$

Since $\mathcal{R}e\,\mu>\lambda$, choosing ε small enough, we obtain (1.10).

In a similar way we can prove (1.11): we set

$$\mathcal{M}(A)=\int\limits_A\theta^2\,dx\,,\qquad u_{\theta,A}=\frac{1}{\mathcal{M}(A)}\int\limits_A\theta^2u\,dx\,. \tag{1.13}$$

Taking $\varphi=\theta^2(u-u_{\theta,B(\sigma)})$ in (1.9) and noting that

$$\int\limits_{B(\sigma)}\theta^2u(\bar u-\bar u_{\theta,B(\sigma)})\,dx=\int\limits_{B(\sigma)}\theta^2|u-u_{\theta,B(\sigma)}|^2\,dx$$

as before for the proof of (1.10), we get the inequality

$$\int\limits_{B(\varrho)}\|Du\|^2\,dx\leqslant c(\sigma-\varrho)^{-2}\int\limits_{B(\sigma)}|u-u_{\theta,B(\sigma)}|^2\,dx\,. \tag{1.14}$$

From (1.14), we easily deduce (1.11) because

$$\int\limits_{B(\sigma)}|u-u_{\theta,B(\sigma)}|^2\,dx\leqslant\frac{1}{\mathcal{M}(B(\sigma))}\int\limits_{B(\sigma)}dx\int\limits_{B(\sigma)}|u(x)-u(y)|^2\,dy\leqslant$$
$$\leqslant c\,\frac{\text{meas}\,B(\sigma)}{\mathcal{M}(B(\sigma))}\int\limits_{B(\sigma)}|u-u_{B(\sigma)}|^2\,dx\leqslant c\,\frac{\text{meas}\,B(\sigma)}{\text{meas}\,B(\varrho)}\int\limits_{B(\sigma)}|u-u_{B(\sigma)}|^2\,dx\,.$$

We shall denote

$$B^+(\sigma) = \{x \in B(0,\sigma)\colon x_n > 0\}\,, \qquad \Gamma(\sigma) = \{x \in B(0,\sigma)\colon x_n = 0\}\,. \tag{1.15}$$

Now a_{ij} and b_i, $ij = 1, \dots, n$, are functions defined on $B^+(1)$ which satisfy, in $B^+(1)$, the assumptions (0.I), (0.II), and

$$\lambda = \frac{1}{2\nu} \sup_{B^+(1)} \sum_i |b_i(x)|^2\,. \tag{1.16}$$

Lemma 1.IV. *If $u \in H^1(B^+(1))$ is a solution of the equation*

$$\mu \int_{B^+(1)} u\bar{\varphi}\, dx + \int_{B^+(1)} \sum_{ij} a_{ij} D_j u D_i \bar{\varphi} + \sum_i b_i D_i u \bar{\varphi}\, dx = 0\,, \quad \forall \varphi \in H^1_0(B^+(1)) \tag{1.17}$$

which vanishes on $\Gamma(1)$ and $\mathcal{R}e\, \mu > \lambda$, then for all ϱ, σ, with $0 < \varrho < \sigma \leqslant 1$, the following inequality holds

$$\int_{B^+(\varrho)} \|Du\|^2\, dx \leqslant c(\sigma - \varrho)^{-2} \int_{B^+(\sigma)} |u|^2\, dx \tag{1.18}$$

with c independent of $\mu, \varrho, \sigma, \lambda$.

Proof. Let θ be a function as described in (1.12). Since u vanishes on the flat part $\Gamma(1)$, in (1.17) we can assume $\varphi = \theta^2 u$ and, by proceeding exactly as in the proof of (1.10), we obtain (1.18).

We conclude this section recalling two lemmas proved in [5] (see also [7], CAP. I, n. 1).

Lemma 1.V. *Let φ and ω be nonnegative functions defined in $(0, d]$ and let A and α be positive constants. If*

$$\lim_{\sigma \to 0} \omega(\sigma) = 0 \tag{1.19}$$

$$\varphi(t\sigma) \leqslant \{At^\alpha + \omega(\sigma)\}\varphi(\sigma)\,, \qquad \forall t \in (0,1) \text{ and } \forall \sigma \in (0, d] \tag{1.20}$$

then for all $\varepsilon > 0$ there is $\sigma_\varepsilon \in (0, d]$ such that $\forall \sigma \in (0, \sigma_\varepsilon)$ and $\forall t \in (0, 1)$

$$\varphi(t\sigma) \leqslant (1 + A) t^{\alpha - \varepsilon} \varphi(\sigma)\,. \tag{1.21}$$

Lemma 1.VI. *Let φ be a nonnegative function defined in $(0, d]$. Let A, α, β*

be positive constants with $\beta < \alpha$ *and* $B \geqslant 0$. *If for all* $t \in (0, 1)$ *and* $\sigma \in (0, d]$

$$\varphi(t\sigma) \leqslant At^{\alpha}\varphi(\sigma) + \sigma^{\beta}B \tag{1.22}$$

then $\forall\varepsilon \in (0, \alpha - \beta)$, $\forall t \in (0, 1)$, $\forall\sigma \in (0, d]$

$$\varphi(t\sigma) \leqslant At^{\alpha-\varepsilon}\varphi(\sigma) + KB(t\sigma)^{\beta} \tag{1.23}$$

where K *depends only on* A, α, β, ε (2).

2. – Estimates for solutions of equations with constant coefficients.

In this section we shall prove, for solutions of homogeneous equations with constant coefficients, some estimates which are a main step in our method.

Let b_{ij} be, $ij = 1, \ldots, n$, real constants satisfying the ellipticity condition (0.6) and μ a complex parameter with $\mathcal{R}e\,\mu \geqslant 0$. $B(\sigma)$ and $B^+(\sigma)$ are the sphere or the hemisphere defined as above.

We will prove the two theorems below:

THEOREM 2.I. *If* $u \in H^1(B(\sigma))$ *is a solution of the equation*

$$\mu\int_{B(\sigma)} u\bar{\varphi}\,dx + \int_{B(\sigma)} \sum_{ij} b_{ij}D_j u D_i\bar{\varphi}\,dx = 0\,, \qquad \forall\varphi \in H^1_0(B(\sigma)) \tag{2.1}$$

then, for all $t \in (0, 1)$

$$\int_{B(t\sigma)} \|Du\|^2\,dx \leqslant ct^n \int_{B(\sigma)} \|Du\|^2\,dx \tag{2.2}$$

where c *does not depend on* μ, σ, t.

THEOREM 2.II. *If* $u \in H^1(B^+(1))$ *is a solution of the equation*

$$\mu\int_{B^+(1)} u\bar{\varphi}\,dx + \int_{B^+(1)} \sum_{ij} b_{ij}D_j u D_i\bar{\varphi}\,dx = 0\,, \qquad \forall\varphi \in H^1_0(B^+(1)) \tag{2.3}$$

(2) $K = \dfrac{(1 + A)^{2\alpha/\varepsilon}}{(1 + A)^{(\alpha-\beta)/\varepsilon} - A}$.

which vanishes on $\Gamma(1)$, *then for all* $\sigma \leqslant 1$ *and* $t \in (0,1)$

$$\int_{B^+(t\sigma)} \|Du\|^2\, dx \leqslant ct^n \int_{B^+(\sigma)} \|Du\|^2\, dx \tag{2.4}$$

where c *does not depend on* μ, σ, t.

Estimates (2.2) and (2.4) are well known in the case of $\mu = 0$ [5]. Since the most important thing is that the constants c, which appear in (2.2) and (2.4), should not depend on μ, it will be necessary to repeat shortly the proof.

PROOF OF THEOREM 2.I. Since $\partial B(\sigma)$ is smooth, without loss of generality we can suppose that $u \in C^\infty(\overline{B(\sigma)})$. Then estimate (1.10), with c independent of μ, holds for all derivatives of u. It is enough to repeat the proof of [5] coroll. 7.I and 7.II:

From (1.10) we deduce that for all integer k

$$\int_{B(\sigma/2)} \sum_{|\alpha| \leqslant k} \sum_{i=1}^{n} |D^\alpha D_i u|^2\, dx \leqslant c(\sigma) \int_{B(\sigma)} \|Du\|^2\, dx \tag{2.5}$$

where c depends only on σ.

Then, for Sobolev imbedding theorem, if $0 < t \leqslant \frac{1}{2}$

$$\int_{B(t\sigma)} \|Du\|^2 dx \leqslant c(\sigma)\, t^n \sup_{\overline{B(\sigma/2)}} \|Du\|^2 \leqslant c(\sigma)\, t^n \int_{B(\sigma)} \|Du\|^2\, dx\,. \tag{2.6}$$

Finally, inequality (2.6) is trivial for $\frac{1}{2} \leqslant t < 1$. Hence we conclude that for all $t \in (0,1)$

$$\int_{B(t\sigma)} \|Du\|^2\, dx \leqslant c(\sigma)\, t^n \int_{B(\sigma)} \|Du\|^2\, dx\,. \tag{2.7}$$

To specify the dependence on σ, we dilate the spatial coordinates as usual. The function $U(y) = u(\sigma y)$ is a C^∞ solution of the equation

$$\mu\sigma^{-2}\, U(y) - \sum_{ij} b_{ij} D_{ij} U(y) = 0\,, \qquad \text{in } B(1)\,.$$

Then, since the constant c in (2.7) does not depend on the coefficient of U, for all $t \in (0,1)$

$$\int_{B(t)} \|DU\|^2\, dy \leqslant ct^n \int_{B(1)} \|DU\|^2\, dy \tag{2.8}$$

with c independent of μ and σ.

(2.2) easily follows from (2.8).

PROOF OF THEOREM 2.II. The proof will be divided in two steps. First let us suppose that

$$b_{ij} = \delta_{ij} b_{ij}$$

δ_{ij} being Kronecker's symbol.

Equation (2.3) becomes

$$\mu \int_{B^+(1)} u\bar{\varphi}\, dx + \int_{B^+(1)} \sum_i b_{ii} D_i u D_i \bar{\varphi}\, dx = 0 , \quad \forall \varphi \in H_0^1(B^+(1)) . \tag{2.9}$$

Since u vanishes on $\Gamma(1)$, the proof is obtained by transforming our problem into an interior problem by a type of reflection, and then by using the result of theorem 2.I. Let

$$U(x) = \begin{cases} u(x) & \text{for } x_n > 0 \\ -u(x', -x_n) & \text{for } x_n < 0 \end{cases}$$

where $x' = (x_1, \dots, x_{n-1})$. Then $U \in H^1(B(1))$ is a solution of the equation

$$\mu \int_{B(1)} U\bar{\varphi}\, dx + \int_{B(1)} \sum_i b_{ii} D_i U D_i \bar{\varphi}\, dx = 0 , \qquad \forall \varphi \in H_0^1(B(1)) \tag{2.10}$$

and, by theorem 2.I, $\forall \sigma \leqslant 1$ and $\forall t \in (0, 1)$

$$\int_{B(t\sigma)} \|DU\|^2\, dx \leqslant ct^n \int_{B(\sigma)} \|DU\|^2\, dx$$

hence

$$\int_{B^+(t\sigma)} \|Du\|^2\, dx \leqslant ct^n \int_{B^+(\sigma)} \|Du\|^2\, dx . \tag{2.11}$$

In the general case note that, if u is a solution of equation (2.3) and vanishes on $\Gamma(1)$, then $u \in C^\infty(B^+(1) \cup \Gamma(1))$ and

$$\mu u - \sum_{ij} b_{ij} D_{ij} u = 0 \qquad \text{on } B^+(1) . \tag{2.12}$$

Without loss of generality we can suppose that $b_{ij} = b_{ji}$ and then the differential equation (2.12), by a suitable linear transformation $\mathscr{T}$ on the spatial coordinates, can be written as

$$\mu U - \sum_i D_{ii} U = 0 ,$$

where $U(x) = u(\mathcal{T}x)$. $B^+(1)$ is carried by $\mathcal{T}$ onto an open set with a flat part of its boundary, corresponding to $\Gamma(1)$, which belongs to the plane $x_n = 0$.

We are thus back to the preceding case and leave the details to the reader.

3. – Estimates for solutions of homogeneous equation with continuous coefficients.

In this section, we shall be concerned with homogeneous equations, defined in Ω or in a hemisphere $B^+(1)$, with continuous coefficients. The hypotheses (0.I) and (0.II) are always assumed. The results just obtained for constant coefficients equations will be applied here to estimate norms of the type

$$\int_{B(\sigma)} |u - u_{B(\sigma)}|^2 \, dx \,, \qquad \int_{B^+(\sigma)} |u|^2 \, dx \,.$$

THEOREM 3.I. *Let $u \in H^1(\Omega)$ be a solution of the equation*

$$\mu \int_{\Omega} u\bar{\varphi} \, dx + a(u, \varphi) = 0 \,, \qquad \forall \varphi \in H_0^1(\Omega) \tag{3.1}$$

with $\mathcal{R}e\, \mu > \lambda$ ([3]). Then for all $\varepsilon > 0$ there is σ_ε, independent on μ, such that for all open balls $B(\sigma) \subset\subset \Omega$, with $\sigma < \sigma_\varepsilon$, and for every $t \in (0.1)$

$$\int_{B(t\sigma)} |u - u_{B(t\sigma)}|^2 \, dx \leqslant c t^{n+2-\varepsilon} \int_{B(\sigma)} |u - u_{B(\sigma)}|^2 \, dx \,, \tag{3.2}$$

where c does not depend on μ, t, σ, λ.

PROOF. Let us denote by $\omega(\sigma)$ the modulus of continuity of the coefficients $a_{ij}(x)$ and fix $B(\sigma) = B(x_0, \sigma) \subset \Omega$. In $B(\sigma)$ u can be split as $v + w$ where w is the solution of the Dirichlet problem

$$\left\{ \begin{aligned} & w \in H_0^1(B(\sigma)) \\ & \mu \int_{B(\;)} w\bar{\varphi} \, dx + \int_{B(\sigma)} \sum_{ij} a_{ij}(x_0) D_j w D_i \bar{\varphi} \, dx = \\ & \quad = \int_{B(\sigma)} \sum_{ij} (a_{ij}(x_0) - a_{ij}) D_j u D_i \bar{\varphi} \, dx - \int_{B(\sigma)} \sum_i b_i D_i u \bar{\varphi} \, dx \,, \quad \forall \varphi \in H_0^1(B(\sigma)) \end{aligned} \right. \tag{3.3}$$

([3]) λ is defined as in (0.7).

whereas $v \in H^1(B(\sigma))$ is a solution of the equation

$$\mu\int_{B(\sigma)} v\bar{\varphi}\,dx + \int_{B(\sigma)} \sum_{ij} a_{ij}(x_0) D_j v D_i \bar{\varphi}\,dx = 0\,, \qquad \forall \varphi \in H^1_0(B(\sigma))\,. \tag{3.4}$$

Choosing $\varphi = w$ in (3.3), for all $\varepsilon > 0$ we get the following estimate for w

$$\mathscr{R}e\,\mu\int_{B(\sigma)} |w|^2\,dx + \nu\int_{B(\sigma)} \|Dw\|^2\,dx \leqslant c(\varepsilon)(\omega^2(\sigma) + \sigma^2)\int_{B(\sigma)} \|Du\|^2\,dx + \varepsilon\int_{B(\sigma)} \|Dw\|^2\,dx\,.$$

Because $\mathscr{R}e\,\mu > 0$, choosing ε small enough, we obtain

$$\int_{B(\sigma)} \|Dw\|^2\,dx \leqslant o(\sigma)\int_{B(\sigma)} \|Du\|^2\,dx \tag{3.5}$$

where $o(\sigma)$ goes to 0 when $\sigma \to 0$.

Using theorem 2.I, we can estimate v in the following way, for all $t \in (0,1)$,

$$\int_{B(t\sigma)} \|Dv\|^2\,dx \leqslant ct^n \int_{B(\sigma)} \|Dv\|^2\,dx\,. \tag{3.6}$$

As $u = v + w$, from (3.5) and (3.6) we get, by a standard argument, that for every $t \in (0,1)$

$$\int_{B(t\sigma)} \|Du\|^2\,dx \leqslant \{ct^n + o(\sigma)\}\int_{B(\sigma)} \|Du\|^2\,dx\,. \tag{3.7}$$

Then, by lemma 1.V, for every $\varepsilon > 0$ there is σ_ε such that if $\sigma < \sigma_\varepsilon$ and $t \in (0,1)$

$$\int_{B(t\sigma)} \|Du\|^2\,dx \leqslant (1 + c)\,t^{n-\varepsilon}\int_{B(\sigma)} \|Du\|^2\,dx\,. \tag{3.8}$$

On the other hand by Poincaré inequality

$$\int_{B(t\sigma)} |u - u_{B(t\sigma)}|^2\,dx \leqslant c(n)\,t^2\sigma^2\int_{B(t\sigma)} \|Du\|^2\,dx\,. \tag{3.9}$$

Because $\mathscr{R}e\,\mu > \lambda$, from (1.11) where we assume $\varrho = \sigma/2$, if $t \in (0,1)$ we get that

$$\int_{B(\sigma/2)} \|Du\|^2\,dx \leqslant c\sigma^{-2}\int_{B(\sigma)} |u - u_{B(\sigma)}|^2\,dx\,. \tag{3.10}$$

From (3.8) (3.9) (3.10), (3.2) easily follows if $0 < t < \frac{1}{2}$. Finally (3.2) is trivial for $\frac{1}{2} \leqslant t < 1$.

THEOREM 3.II. *If $u \in H^1(B^+(1))$ is a solution of the equation*

$$\mu \int_{B^+(1)} u\bar{\varphi}\, dx + \int_{B^+(1)} \sum_{ij} a_{ij} D_j u D_i \bar{\varphi} + \sum_i b_i D_i u \bar{\varphi}\, dx = 0\,, \quad \forall \varphi \in H_0^1(B^+(1)) \tag{3.11}$$

which vanishes on $\Gamma(1)$ and $\mathcal{Re}\,\mu > \lambda$ ([4]*), then for all $\varepsilon > 0$ there is $\sigma_\varepsilon \leqslant 1$, independent of μ, such that if $\sigma < \sigma_\varepsilon$ and $t \in (0, 1)$*

$$\int_{B^+(t\sigma)} |u|^2\, dx \leqslant c t^{n+2-\varepsilon} \int_{B^+(\sigma)} |u|^2\, dx \tag{3.12}$$

where c does not depend on μ, t, σ, λ.

The proof of this theorem is completely analogous to that of theorem 3.I; this time we must use theorem 2.II and estimate (1.18) instead of theorem 2.I and (1.11). Thus the proof is left to the reader. Note only that because of the fact that u vanishes on $\Gamma(1)$ we have, instead of (3.9), the Poincaré inequality

$$\int_{B^+(t\sigma)} |u|^2\, dx \leqslant c(n)\, t^2 \sigma^2 \int_{B^+(t\sigma)} |D_n u|^2\, dx\,.$$

4. – Local interior and boundary estimates for the Hölder norm of u.

Theorems 3.I and 3.II enable us to obtain local interior or boundary estimates for the Hölder norm of a solution of the equation

$$\mu \int_\Omega u\bar{\varphi}\, dx + a(u, \varphi) = \int_\Omega f\bar{\varphi}\, dx\,, \qquad \forall \varphi \in H_0^1(\Omega) \tag{4.1}$$

or

$$\mu \int_{B^+(1)} u\bar{\varphi}\, dx + \int_{B^+(1)} \sum_{ij} a_{ij} D_j u D_i \bar{\varphi} + \sum_i b_i D_i u \bar{\varphi}\, dx = \int_{B^+(1)} f\bar{\varphi}\, dx\,, \quad \forall \varphi \in H_0^1(B^+(1)). \tag{4.2}$$

In both cases we suppose that assumption (0.I) and (0.II) are fulfilled.

[4] λ is defined as in (1.15).

THEOREM 4.I. *If $u \in H^1(\Omega)$ is a solution of equation* (4.1), *where $f \in C^{0,\alpha}(\bar{\Omega})$ and $\mathcal{R}e\,\mu > \lambda$* ([5]), *then for every ball $B(\sigma) \subset\subset \Omega$ we have the estimate*

$$\int_{B(\sigma)} |u - u_{B(\sigma)}|^2\,dx \leqslant c\sigma^{n+2\alpha}\left\{\|u\|^2_{0,\Omega} + \frac{1}{(|\mu| - \lambda)^2}[f]^2_{\alpha,\bar{\Omega}}\right\} \tag{4.3}$$

where c does not depend on μ, λ, σ.

PROOF. Fix a positive $\varepsilon < 2(1-\alpha)$ and $B(\sigma) \subset\subset \Omega$ with $\sigma < \sigma_\varepsilon$. In $B(\sigma$ we write $u = v + w$ where w is the solution of the Neumann problem

$$\begin{aligned} & w \in H^1(B(\sigma)) \\ & \mu\int_{B(\sigma)} w\bar{\varphi}\,dx + a(w,\varphi) = \int_{B(\sigma)} f\bar{\varphi}\,dx\,, \qquad \forall\varphi \in H^1(B(\sigma)) \end{aligned} \tag{4.4}$$

whereas $v \in H^1(B(\sigma))$ is a solution of the equation

$$\mu\int_{B(\sigma)} v\bar{\varphi}\,dx + a(v,\varphi) = 0\,, \qquad \forall\varphi \in H^1_0(B(\sigma))\,. \tag{4.5}$$

There exists a unique w because of (0.10) and $\mathcal{R}e\,\mu > \lambda$.

Using lemma 1.II we get the following estimate on w:

$$\int_{B(\sigma)} |w - w_{B(\sigma)}|^2\,dx \leqslant \frac{K}{(|\mu| - \lambda)^2}\int_{B(\sigma)} |f - f_{B(\sigma)}|^2\,dx \leqslant \frac{c\sigma^{n+2\alpha}}{(|\mu| - \lambda)^2}[f]^2_{\alpha,\bar{\Omega}} \tag{4.6}$$

with c independent on μ, λ, σ.

Since $\sigma < \sigma_\varepsilon$, by theorem 3.I we deduce the following estimate on v:

$$\int_{B(t\sigma)} |v - v_{B(t\sigma)}|^2\,dx \leqslant ct^{n+2-\varepsilon}\int_{B(\sigma)} |v - v_{B(\sigma)}|^2\,dx \tag{4.7}$$

for all $t \in (0,1)$, where c does not depend on μ, t, σ, λ.

As $u = v + w$, from (4.6) and (4.7) we get that for every $t \in (0,1)$

$$\int_{B(t\sigma)} |u - u_{B(t\sigma)}|^2\,dx \leqslant ct^{n+2-\varepsilon}\int_{B(\sigma)} |u - u_{B(\sigma)}|^2\,dx + \frac{c\sigma^{n+2\alpha}}{(|\mu| - \lambda)^2}[f]^2_{\alpha,\bar{\Omega}}\,. \tag{4.8}$$

([5]) λ is defined as in (0.7).

Because $n + 2 - \varepsilon > n + 2\alpha$, from, (4.8) by using lemma 1.VI, (4.3) easily follows for $\sigma < \sigma_\varepsilon$ (see [5] and [7]). Finally (4.3) is trivial for $\sigma \geqslant \sigma_\varepsilon$.

REMARK 4.I. Because of lemma 1.I, from (4.8) it easily follows that, under the hypotheses of the previous theorm, for every subset $\Omega_0 \subset\subset \Omega$, $[u]_{\alpha,\bar{\Omega}_0}$ is finite and

$$\|u\|_{0,\Omega_0} + [u]_{\alpha,\bar{\Omega}_0} \leqslant c\left\{\|u\|_{0,\Omega} + \frac{1}{|\mu| - \lambda}[f]_{\alpha,\bar{\Omega}}\right\} \tag{4.9}$$

where c does not depend on μ and λ (see [5]).

THEOREM 4.II. *Let $u \in H^1(B^+(1))$ be a solution of equation (4.2) which vanishes on $\Gamma(1)$. Suppose that $f \in C^{0,\alpha}(\overline{B^+(1)})$ and $f = 0$ on $\Gamma(1)$. Suppose that $\mathcal{R}e\,\mu > \lambda$* (6). *Then for every $\sigma < 1$ the following estimate holds*

$$\int_{B^+(\sigma)} |u - u_{B^+(\sigma)}|^2\, dx \leqslant c\sigma^{n+2\alpha}\left\{\|u\|^2_{0,B^+(1)} + \frac{1}{(|\mu| - \lambda)^2}[f]^2_{\alpha,\overline{B^+(1)}}\right\} \tag{4.10}$$

with c independent of μ, λ, σ.

PROOF. The proof is analogous to that of theorem 4.I. Fix a positive $\varepsilon < 2(1 - \alpha)$ and $\sigma < \sigma_\varepsilon \wedge 1$. In $B^+(\sigma)$ we write $u = v + w$ where w is the solution of the Dirichlet problem

$$\begin{cases} w \in H^1_0(B^+(\sigma)) \\ \mu\displaystyle\int_{B^+(\sigma)} w\bar{\varphi}\, dx + \int_{B^+(\sigma)} \sum_{ij} a_{ij} D_j w D_i \bar{\varphi} + \sum_i b_i D_i u \bar{\varphi}\, dx = \int_{B^+(\sigma)} f\bar{\varphi}\, d\tau\,, \\ \qquad\qquad \forall \varphi \in H^1_0(B^+(\sigma))\,, \end{cases} \tag{4.11}$$

whereas $v \in H^1(B^+(\sigma))$ is a solution of the equation

$$\mu\int_{B^+(\sigma)} v\bar{\varphi}\, dx + \int_{B^+(\sigma)} \sum_{ij} a_{ij} D_j v D_i \bar{\varphi} + \sum_i b_i D_i v \bar{\varphi}\, d\tau = 0\,, \qquad \forall \varphi \in H^1_0(B^+(\sigma)) \tag{4.12}$$

which vanishes on $\Gamma(\sigma)$.

By (1.5) the following estimate on w holds

$$\int_{B^+(\sigma)} |w|^2\, dx \leqslant \frac{c}{(|\mu| - \lambda)^2}\int_{B^+(\sigma)} |f|^2\, dx\,. \tag{4.13}$$

(6) λ is defined as in (1.15).

As f vanishes on $\Gamma(\sigma)$

$$\int_{B^+(\sigma)} |w|^2\, dx \leqslant \frac{c\sigma^{n+2\alpha}}{(|\mu|-\lambda)^2}[f]^2_{\alpha,\overline{B^+(1)}}\,. \tag{4.14}$$

Since $\sigma < \sigma_\varepsilon$, by theorem 3.II we deduce the following estimate on v:

$$\int_{B^+(t\sigma)} |v|^2\, dx \leqslant ct^{n+2-\varepsilon}\int_{B^+(\sigma)} |v|^2\, dx \tag{4.15}$$

for all $t \in (0, 1)$. The constants c which appear in (4.14) and (4.15) do not depend on μ and λ.

As $u = v + w$, from (4.14) and (4.15) we obtain, for every $t \in (0, 1)$,

$$\int_{B^+(t\sigma)} |u|^2\, dx \leqslant ct^{n+2-\varepsilon}\int_{B^+(\sigma)} |u|^2\, dx + c\,\frac{\sigma^{n+2\alpha}}{(|\mu|-\lambda)^2}[f]^2_{\alpha,\overline{B^+(1)}}\,. \tag{4.16}$$

Since $n + 2 - \varepsilon > n + 2\alpha$, from (4.16), by using lemma 1.VI and the Poincaré inequality, we easily obtain (4.10) for $\sigma < \sigma_\varepsilon$ (see [5]). On the other hand (4.10) is trivial for $\sigma_\varepsilon \leqslant \sigma < 1$.

REMARK 4.II. Because of lemma 1.I, from (4.3) and (4.10) it easily follows that, under the hypotheses of the previous theorem, for every $\sigma < 1$ the $[u]_{\alpha,\overline{B^+(\sigma)}}$ is finite and

$$\|u\|_{0,B^+(\sigma)} + [u]_{\alpha,\overline{B^+(\sigma)}} \leqslant c\left\{\|u\|_{0,B^+(1)} + \frac{1}{|\mu|-\lambda}[f]_{\alpha,\overline{B^+(1)}}\right\} \tag{4.17}$$

where the constant c does not depend on μ and λ (see [5]).

5. – Conclusion of the proof of estimate (0.12).

Now we can conclude the proof of estimate (0.12) which was the aim of the present paper.

The conclusion of the proof is, as usual, by a covering argument. By the definition of boundary of class C^2, about every $x_0 \in \partial\Omega$ there is an open neighborhood U which can be mapped, by a mapping of class C^2 together with its inverse, onto the sphere $B(1) = B(0, 1)$ and in particular $U \cap \Omega$ is carried in $B^+(1)$ and $U \cap \partial\Omega$ in the flat part $\Gamma(1)$. Such a mapping preserves the desired properties of u and the properties (0.I), (0.II) of the coefficients a_{ij} and b_i in the trasformed differential equation.

Since $\partial\Omega$ is a compact set, only a finite number of such neighborhoods are needed to cover it, say $U_1, \ldots, U_m$.

For each U_i we can suppose σ close enough to 1 so that, if $U_i(\sigma)$ is the inverse image of $B(\sigma)$, $U_1(\sigma), \ldots, U_m(\sigma)$ still cover $\partial\Omega$.

Then there exists an open subset $\Omega_0 \subset\subset \Omega$ such that $\Omega_0, U_1(\sigma), \ldots, U_m(\sigma)$ cover $\bar{\Omega}$.

REMARK 4.I can be applied to subset Ω_0, then if $\mathscr{R}e\,\mu > \lambda$ we have estimate (4.9).

REMARK 4.II can be applied to each of the mapped neighborhoods $U_i(\sigma)$, and therefore there exist $\lambda_i > 0$, $i = 1, \ldots, m$ such that if $\mathscr{R}e\,\mu > \lambda_i$

$$\|u\|_{C^{0,\alpha}(\overline{U_i(\sigma)\cap\Omega})} \leqslant c_i \left\{ \|u\|_{0,\Omega} + \frac{1}{|\mu| - \lambda_i} \|f\|_{C^{0,\alpha}(\bar{\Omega})} \right\}. \tag{5.1}$$

Set

$$\lambda_0 = \max\{\lambda_1, \lambda_2, \ldots, \lambda_m, \lambda\}.$$

From estimates (4.9) and (5.1) we derive that, if $\mathscr{R}e\,\mu > \lambda_0$

$$\|u\|_{C^{0,\alpha}(\bar{\Omega})} \leqslant c \left\{ \|u\|_{0,\Omega} + \frac{1}{|\mu| - \lambda_0} \|f\|_{C^{0,\alpha}(\bar{\Omega})} \right\} \tag{5.2}$$

where c does not depend on μ and λ_0.

To obtain (0.12) we only need to apply (1.5) with λ_0 instead of λ.

REFERENCES

[1] S. AGMON, *Lectures on elliptic boundary value problem*, Van Nostrand, Princeton, N. J., (1965).

[2] S. AGMON, *On the eingenfunctions and on the eingenvalues of general elliptic boundary value problems*, Comm. Pure Appl. Math., **15** (1962), pp. 119-147.

[3] F. E. BROWDER, *On the spectral theory of elliptic differential operators I*, Math. Ann., **142** (1960/61), pp. 22-130.

[4] S. CAMPANATO, *Proprietà di hölderianità di alcune classi di funzioni*, Ann. Scuola Norm. Sup. Pisa, **17** (1963), pp. 175-188.

[5] S. CAMPANATO, *Equazioni ellittiche del II° ordine e spazi* $\mathscr{L}^{2,\lambda}$, Ann. Mat. Pura Appl., **68** (1965), pp. 321-382.

[6] S. CAMPANATO, *Equazioni ellittiche non variazionali a coefficienti continui*, Ann. Mat. Pura Appl., **86** (1970), pp. 125-154.

[7] S. CAMPANATO, *Sistemi ellittici in forma divergenza. Regolarità all'interno*, Quaderni Scuola Norm. Sup. Pisa, 1980.

[8] K. Masuda, *Manuscript for seminar at Kyoto Univ.*, (1970).
[9] H. B. Stewart, *Generation of analytic semigroups by strongly elliptic operators*, Trans. Amer. Math. Soc., **199** (1974), pp. 141-162.
[10] H. B. Stewart, *Generation of analytic semigroups by strongly elliptic operators under general boundary conditions*, Trans. Amer. Math. Soc., **259** (1980), pp. 299-310.
[11] W. von Wahl, *Gebrochene potenzen eines elliptischen operators and parabolische differentialgleichungen in räumen hölderstetiger functionen*, Nachr. Akad. Wiss. Göttingen Math. Phys. Kl., **11** (1972), pp. 231-258.
[12] W. von Wahl, *Einige bemerkungen zu meiner arbeit « Gebrochene potenzen eines elliptischen operators und parabolische differentialgleichungen in räumen hölderstetiger funktionen »*, Manuscripta Math., **11** (1974), pp. 199-201.

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