S. CAMpanato
P. CANnarSA

Differentiability and partial Hölder continuity of the solutions of non-linear elliptic systems of order $2m$ with quadratic growth


<http://www.numdam.org/item?id=ASNSP_1981_4_8_2_285_0>
Differentiability and Partial Hölder Continuity of the Solutions of Non-Linear Elliptic Systems of Order $2m$ with Quadratic Growth.

S. CAMPANATO - P. CANNARS A

1. Introduction.

Let $\Omega$ be a bounded open set of $\mathbb{R}^n$ with points $x = (x_1, ..., x_n)$; here $m$ and $N$ are integers $>1$, $(\cdot, \cdot)_N$ and $\|\cdot\|_N$ are the scalar product and the norm in $\mathbb{R}^N$. We shall drop the subscript $N$ when there is no danger of confusion.

Let $\alpha = (\alpha_1, ..., \alpha_n)$ be a multi-index and $|\alpha| = \alpha_1 + ... + \alpha_n$. We denote by $\mathfrak{R}$ the cartesian product $\prod_{|\alpha| \leq m} \mathbb{R}^N$ and by $p = \{p^\alpha\}_{|\alpha| \leq m}$, $p^\alpha \in \mathbb{R}^N$, a typical point in $\mathfrak{R}$.

If $p \in \mathfrak{R}$ we set

$$
(p')^2 = \sum_{|\alpha| \leq m} \|p^\alpha\|^2_N, \quad (p')^2 = \sum_{|\alpha| = m} \|p^\alpha\|^2_N, \quad (p')^2 = \sum_{|\alpha| < m} \|p^\alpha\|^2_N.
$$

We define, as usual,

$$D_i = \frac{\partial}{\partial x_i}, \quad D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} ... D_n^{\alpha_n}$$

and, if $u: \Omega \to \mathbb{R}^n$, then

$$Du = \{D^\alpha u\}_{|\alpha| \leq m}, \quad D'u = \{D^\alpha u\}_{|\alpha| = m}, \quad D'u = \{D^\alpha u\}_{|\alpha| < m}.$$

$C^{h,\lambda}(\bar{\Omega}, \mathbb{R}^n)$, $h$ integer $>0$ and $0 < \lambda < 1$, is the space of those vectors $u: \bar{\Omega} \to \mathbb{R}^n$ which satisfy a Hölder condition of exponent $\lambda$, together with

Pervenuto alla Redazione il 28 Maggio 1980.
all their derivatives $D^s u$, $|x| < h$; if $u \in C^h(\overline{\Omega}, R^N)$, then

$$\|u\|_{C^h(\overline{\Omega}, R^N)} = \sup_{\Omega} \sum_{|\alpha| \leq h} \|D^\alpha u\| + \sum_{|\alpha| = h} [D^\alpha u]_{1, h}$$

where

$$[u]_{1, h} = \sup_{x, y \in \Omega} \frac{\|u(x) - u(y)\|}{\|x - y\|^h}.$$

$H^s, p(\Omega, R^N)$ and $H^s, p(\Omega, R^N)$, $s$ integer $> 0$ and $p > 1$, are the usual Sobolev spaces and, if $1 < p < + \infty$, then

$$|u|_{s, p, \Omega} = \left\{ \int_\Omega \sum_{|\alpha| = s} \|D^\alpha u\|^p dx \right\}^{\frac{1}{p}}$$

$$\|u\|_{s, p, \Omega} = \left\{ \sum_{s=0}^{\frac{1}{p}} |u|_{s, p, \Omega} \right\}^{\frac{1}{p}}$$

$H^s, p(\Omega, R^N) = L^p(\Omega, R^N)$ and

$$|u|_{0, p, \Omega} = \left\{ \int_\Omega \|u\|^p dx \right\}^{\frac{1}{p}}.$$

If $p = 2$, then we shall simply write $H^s$, $H^s_0$, $|\cdot|_{s, \Omega}$, $\|\cdot\|_{s, \Omega}$.

Let $a^s(x, p)$, $|x| < m$, be vectors of $R^N$, defined in $\Omega \times \mathbb{R}$, measurable in $x$ and continuous in $p$; assume that $\forall (x, p) \in \Omega \times \mathbb{R}$ with $\|p\| < K$

$$\|a^s(x, p)\| < M(K) \left\{ f^s(x) + \|p\| \right\}$$

$$\|a^s(x, p)\| < M(K) \left\{ f^s(x) + \|p\|^2 \right\}$$

where

$$f^s \in L^2(\Omega)$$

$$f^s \in L^1(\Omega)$$

Let us consider the differential system of order $2m$

$$\sum_{|\alpha| = m} (-1)^{|\alpha|} D^\alpha a^s(x, Du) = 0$$

which is assumed to be strongly elliptic, i.e. the vector functions $p' \rightarrow a^s(x, p)$, $|x| = m$, are differentiable and there exists $\nu(K) > 0$ such that

$$\sum_{\beta, \gamma = 1}^N \sum_{|\alpha| = |\beta| = m} \frac{\partial a^s(x, p)}{\partial p^\beta} \xi^\beta_{h, K} \xi^\gamma_{h, K} > \nu(K) \sum_{|\alpha| = m} \|\xi^\alpha\|^2 \overline{\Omega}.$$
for every set \( \{ \xi^m \} \) of vectors in \( \mathbb{R}^n \) and for every \((x, p) \in \Omega \times \mathbb{R} \) with \( \| p^r \| < K \).

A solution of system (1.8) is a vector \( u \in H^m \cap H^{m-1, \infty}(\Omega, \mathbb{R}^N) \) such that

\[
\int_{\Omega} \sum_{|\alpha| \leq m} (a^\alpha(x, Du)|D^\alpha \varphi|) \, dx = 0
\]

\( \forall \varphi \in H^m_0 \cap H^{m-1, \infty}(\Omega, \mathbb{R}^N) \).

In this paper we shall investigate the problem of the local differentiability of the solutions of system (1.10): if \( 0 < \lambda < 1 \) and \( u \in H^m \cap C^{m-1, \lambda}(\Omega, \mathbb{R}^N) \) is a solution of system (1.10), then under what conditions on the vectors \( a^\alpha(x, p) \) can we show that

\[
u \in H^{m+1}_{\text{loc}}(\Omega, \mathbb{R}^N) \]

We take solutions of class \( C^{m-1, \lambda}(\Omega, \mathbb{R}^N) \) because it is already known that, if \( u \in H^m \cap H^{m-1, \infty}(\Omega, \mathbb{R}^N) \), problem (1.11) is, in general, answered negatively even if the vectors \( a^\alpha(x, p) \) are very smooth.

If \( m = 1 \) (second order systems), it is well known that the answer to problem (1.11) is positive under the following conditions: denoting by \( p \) the vector \((u, p^1, ..., p^n)\), where \( u \) and \( p^i \) are vectors of \( \mathbb{R}^N \), then

\[
a^i(x, p), \quad i = 1, ..., n, \text{ are of class } C^1 \text{ in } \bar{\Omega} \times \mathbb{R}^{n+1} \]

and \( \forall (x, p) \in \Omega \times \mathbb{R}^{n+1} \) with \( \| u \| < K \)

\[
\| a^i(x, p) \| < M(K) \left( 1 + \sum_{i=1}^n \| p^i \| \right), \quad i = 1, ..., n, \tag{1.13}
\]

\[
\| a^\alpha(x, p) \| < M(K) \left( 1 + \sum_{i=1}^n \| p^i \| \right), \quad i = 1, ..., n, \tag{1.14}
\]

\[
\left\| \frac{\partial a^i(x, p)}{\partial p_k} \right\| < M(K), \quad i = 1, ..., n, \tag{1.15}
\]

\[
\left\| \frac{\partial a^i(x, p)}{\partial u_k} \right\| + \left\| \frac{\partial a^i(x, p)}{\partial x_r} \right\| < M(K) \left( 1 + \sum_{i=1}^n \| p^i \| \right), \quad i = 1, ..., n, \tag{1.16}
\]

\[
\left\| \frac{\partial a^\alpha(x, p)}{\partial p_k} \right\| < M(K) \left( 1 + \sum_{i=1}^n \| p^i \| \right), \tag{1.17}
\]

\[
\left\| \frac{\partial a^\alpha(x, p)}{\partial u_k} \right\| + \left\| \frac{\partial a^\alpha(x, p)}{\partial x_r} \right\| < M(K) \left( 1 + \sum_{i=1}^n \| p^i \| \right), \tag{1.18}
\]
If $m = N = 1$ see for instance [8]; if $m = 1$, $N > 1$ see for example [9] and also [5], chapter V, n. 3.

In order to get the differentiability result (1.11) for second order systems ($m = 1$), the following inequality is essential: using the notation $B(x^0, \sigma) = \{x : \|x - x^0\| < \sigma\}$, if $u \in H^1 \cap C^{0,\theta}(\Omega, \mathbb{R}^N)$ is a solution of the strongly elliptic system

$$
\sum_{i=1}^{n} D_i a^i(x, D u) = a^0(x, D u)
$$

under the hypotheses (1.13) (1.14) with $K = \sup_{x} \|u(x)\|$, then there exists $a_0 > 0$ such that $\forall B(x^0, \sigma) \subset \Omega$ with $\sigma < a_0$ and $\forall \varphi \in H^1_0 \cap L^\infty(B(x^0, \sigma), \mathbb{R}^N)$

\begin{equation}
(1.19) \int_{B(x^0, \sigma)} \|\varphi\|^2 \sum_{i} \|D_i u\|^2 dx < o(\sigma) \int_{B(x^0, \sigma)} \sum_{i} \|D_i \varphi\|^2 dx
\end{equation}

where $o(\sigma)$ tends to zero with $\sigma$.

For strongly elliptic systems of order $2m > 2$, problem (1.11) remained unsolved because of the difficulty of proving a proper extension of inequality (1.19) (see for instance [10]).

In section 3 of this work we deal with problem (1.11) following a different method.

In addition to the strong ellipticity (1.9), we shall assume that the vectors $a^s(x, p)$ satisfy the following hypotheses:

\begin{equation}
(1.20) \text{if } |x| < m, \text{ then the vectors } a^s(x, p) \text{ are measurable in } x \forall p \in \mathbb{R}, \text{ continuous in } p \forall x \in \Omega, \text{ and for every } (x, p) \in \Omega \times \mathbb{R} \text{ with } \|p^s\| < K 
\end{equation}

$$
\|a^s(x, p)\| < M(K) \{f^s(x) + \|p\|^2\}
$$

where $f^s \in L^2(\Omega)$;

\begin{equation}
(1.21) \text{if } |x| = m, \text{ then the vectors } a^s(x, p) \text{ are of class } C^1 \text{ in } \overline{\Omega} \times \mathbb{R} \text{ and for every } (x, p) \in \Omega \times \mathbb{R} \text{ with } \|p^s\| < K 
\end{equation}

$$
\|a^s\| + \sum_{i=1}^{n} \left\|\frac{\partial a^s}{\partial x_i}\right\| + \sum_{k=1}^{N} \left\|\sum_{|\beta| \leq m} \frac{\partial a^s}{\partial p^\beta_k}\right\| < M(K) \{1 + \|p\|\}
$$

We remark that hypothesis (1.21) is a formal extension of the hypotheses (1.13) (1.15) (1.16), whereas hypothesis (1.20) is less restrictive than the assumptions we made on $a^s$ if $m = 1$.

As usual we denote by $H^\theta(\Omega, \mathbb{R}^N)$, $0 < \theta < 1$, the space consisting of
those vectors $u \in L^2(\Omega, \mathbb{R}^N)$ such that

$$
(1.22) \quad |u|^2_{0, \Omega} = \int_{\Omega} \int_{\partial \Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2\theta}} \, dy < +\infty
$$

and by $H^{m+\theta}(\Omega, \mathbb{R}^N)$ the subspace of $H^m(\Omega, \mathbb{R}^N)$ consisting of those vectors $u$ for which

$$
D^s u \in H^0(\Omega, \mathbb{R}^N), \quad |x| = m.
$$

In section 3 we show that, if the conditions (1.9) (1.20) (1.21) are fulfilled and $u \in H^{m} \cap C^{m-1,\lambda}(\Omega, \mathbb{R}^N)$ is a solution of system (1.10), then

$$
(1.23) \quad u \in H^{m+\theta}_{loc}(\Omega, \mathbb{R}^N), \quad \forall 0 < \theta < \lambda/2.
$$

As $u \in C^{m-1,\lambda}(\Omega, \mathbb{R}^N)$, from (1.23) and theorem 2.1 we get

$$
(1.24) \quad u \in H^{m+\theta}_{loc}(\Omega, \mathbb{R}^N), \quad \forall p < \frac{2(1 + \theta)}{n - 2\theta\lambda}.
$$

Now, by an induction process (theorem 3.IV) we show that

$$
(1.25) \quad u \in H^{m+\theta}_{loc}(\Omega, \mathbb{R}^N), \quad \forall 0 < \theta < 1.
$$

Then (1.24) implies

$$
(1.26) \quad u \in H^{m+\theta}_{loc}(\Omega, \mathbb{R}^N).
$$

Once we have this information we can easily prove the following local differentiability theorem, where

$$
Q^0(x, \sigma) = \{x: |x_i - x^0_i| < \sigma, \ i = 1, \ldots, n\}.
$$

**THEOREM 1.I.** If $u \in H^{m} \cap C^{m-1,\lambda}(\Omega, \mathbb{R}^N)$ is a solution of system (1.10) under the hypotheses (1.9) (1.20) (1.21), then

$$
(1.27) \quad u \in H^{m+1}_{loc}(\Omega, \mathbb{R}^N)
$$

and $\forall Q(x^0, 2\sigma) \subset \subset \Omega$

$$
(1.28) \quad \|u\|^2_{m+1, Q(x^0, \sigma)} < c\left\{1 + F^2 + \int_{Q(x^0, 3\sigma)} \|D^s u\|^4 \, dx\right\}
$$

where

$$
F = \sum_{|\alpha| < m} \|f^\alpha\|_{0, \Omega}
$$

and $c$ depends on $\sigma$ and on the norm $\|u\|_{C^{m-1,\lambda}(\overline{\Omega}, \mathbb{R})}$.
As we show in section 3, (1.28) can be replaced by the following equivalent inequality

\[(1.29) \quad |u|^2_{m+1, Q(x_0, r)} < c \left\{ 1 + F^2 + |u|^2_{m, Q(x_0, 2r)} \right\}. \]

Moreover, we may assume that the function \( f^a \), which appear in (1.20), satisfy the more general hypothesis

\[ F = \sum_{|\alpha| < m} D^{\alpha} f^a \in H^{-m+1}(\Omega). \] (1)

In this case \( F^a \) has to be replaced by \( \| F \|^2 \) in (1.28) and by \( \| F \|^2 + \epsilon, \forall \epsilon > 0 \), in (1.29).

In section 4 we prove that the differentiability result of theorem 1.1 and theorem 3.1 of [5] (chap. IV) allow us to obtain the partial Hölder continuity of the derivatives \( D^x u \), \( |x| = m \).

Assume that

\[ \frac{\partial a^0(x, p)}{\partial p_k}, \quad |x| = |\beta| = m \text{ and } 1 < k < N \]

are uniformly continuous in \( \bar{Q} \times \mathbb{R} \) and that in condition (1.20) we have

\[ f^a \in L^p(\Omega) \quad \text{with } p > n \]

or, in general,

\[ F = \sum_{|\alpha| < m} D^{\alpha} f^a \in H^{-m+1,p}(\Omega) \quad \text{with } p > n. \]

Then

**Theorem 1.11.** If \( u \in H^m \cap C^{m-1,1}(\overline{Q}, \mathbb{R}^N) \) is a solution of system (1.10) under the hypotheses (1.9) (1.20) (1.21), then there exists a set \( \Omega_0 \subset \Omega \), closed in \( \Omega \), such that

\[(1.30) \quad \mathcal{M}_{n-q}(\Omega_0) = 0 \quad \text{for a certain } q > 2 \]

\[(1.31) \quad D^x u \in C^{0,\mu}(\Omega \setminus \Omega_0, \mathbb{R}^N), \quad |x| = m, \ \forall 0 < \mu < 1 - n/p \]

where \( \mathcal{M}_{n-q} \) is the \((n-q)\)-dimensional Hausdorff measure.

It is now easy to prove higher regularity results for the solutions \( u \in H^m \cap C^{m-1,1}(\overline{Q}, \mathbb{R}^N) \) of system (1.10), using the theory of linear systems.

\[(1) \quad H^{-m+1}(\Omega) \text{ is the dual of } H^m_{0-1}(\Omega). \]
2. – Preliminary results.

In this section we mention a few results that will be used in the sequel of the work.

Here \( Q(\sigma) = Q(x^0, \sigma) = \{ x : |x_i - x_i^0| < \sigma, i = 1, \ldots, n \} \).

If \( u : Q(\sigma) \to \mathbb{R}^N, \ t \in (0, 1), \ x \in Q(\sigma) \) and \(|h| < (1 - t)\sigma\), then we define

\[
\tau_{i,A} u(x) = u(x + he^i) - u(x), \quad i = 1, \ldots, n
\]

where \( \{e^i\}_{i=1}^n \) is the standard base of \( \mathbb{R}^n \).

**Lemma 2.1.** If \( u \in L^p(Q(\sigma), \mathbb{R}^N) \), \( 1 < p < +\infty \), and there exists \( M > 0 \) such that

\[
\|\tau_{i,A} u\|_{0,p,Q(\sigma)} < |h| M, \quad \forall |h| < (1 - t)\sigma, \quad i = 1, \ldots, n
\]

then \( u \in H^{1,p}(Q(\sigma), \mathbb{R}^N) \) and

\[
\|D_i u\|_{0,p,Q(\sigma)} < M, \quad i = 1, \ldots, n.
\]

**Lemma 2.2.** If \( u \in H^{1,p}(Q(\sigma), \mathbb{R}^N) \), \( 1 < p < +\infty \), then \( \forall t \in (0, 1) \) and \(|h| < (1 - t)\sigma\)

\[
\|\tau_{i,A} u\|_{0,p,Q(\sigma)} < |h| \|D_i u\|_{0,p,Q(\sigma)}, \quad i = 1, \ldots, n.
\]

The previous lemmas are well known in the mathematical literature (see for instance [5], chap. 1).

**Lemma 2.3.** If \( u \in L^2(Q(3\sigma), \mathbb{R}^N) \) and, for \( \theta \in (0, 1) \)

\[
\sum_{i=1}^n \int_{-2^\theta}^{2^\theta} \frac{dh}{|h|^{1+2\theta}} \int_{Q(\sigma)} \|\tau_{i,A} u(x)\|^2 d x < +\infty
\]

then \( u \in H^0(Q(\sigma), \mathbb{R}^N) \) and

\[
\|u\|_{0, Q(\sigma)} < c(n) \sum_{i=1}^n \int_{-2^\theta}^{2^\theta} \frac{dh}{|h|^{1+2\theta}} \int_{Q(\sigma)} \|\tau_{i,A} u(x)\|^2 d x.
\]

See for instance [2], lemma II.3.

If \( Q \) is a bounded open set and \( \sigma > 0 \), then \( Q_\sigma \) denotes the set of those points whose distance from \( Q \) is less than \( \sigma \).
Lemma 2.1V. If \( \Omega, \Omega_1, \ldots, \Omega_6 \) are bounded open sets of \( R^n \) and \( \Omega = \bigcup_{i=1}^{k} \Omega_i \), \( \sigma > 0 \), \( 0 < \theta < 1 \), then \( \forall u \in H^\theta(\Omega, R^n) \)

\[
\|u\|_{b, \sigma}^2 < c(k, \theta, \sigma) \left\{ \|u\|_{b, \sigma}^2 + \sum_{i=1}^{k} \int_{\Omega_i \cap \Omega} \int_{\Omega_i} \frac{\|u(x) - u(y)\|^2}{\|x - y\|^{n+2\theta}} \, dy \right\}.
\]

See for instance [2], Lemma 1.3.

We now state a theorem which is interesting in itself and extends theorem 3.1.1 of [3], that deals with the case \( \theta = 1 \). We set

\[
u_x = (\text{meas } E)^{-1} \int_{E_u} u(x) \, dx.
\]

Theorem 2.1. If \( Q \) is a cube of \( R^n \) and \( u \in H^{1+\theta}(Q, R^n) \), \( 0 < \theta < 1 \) and \( 0 < \lambda < 1 \), then \( \forall t > 0 \) and for \( i = 1, \ldots, n \)

\[
\text{meas} \{ x \in Q : \|D_i u(x) - (D_i u)_Q \| > t \} < c'(n, \theta) \sum_{i} \frac{|D_i u|_{\| \cdot \|_{Q}}^{\theta/(1+\theta)} \cdot [u]_{L, Q}^{\theta/(1+\theta)}}{t^2}
\]

where \( q = 2(1 + \theta) n/(n - 2\theta \lambda) \).

In particular

\[
D_i u \in L^s(Q, R^n), \quad \forall 1 < s < q
\]

and

\[
\int_{Q} \|D_i u - (D_i u)_Q\|^s \, dx < c(n, q, \theta, s) (\text{meas } Q)^{1-s/q} \sum_{i} |D_i u|_{\| \cdot \|_{Q}}^{\theta/(1+\theta)} [u]_{L, Q}^{\theta/(1+\theta)}.
\]

The proof of this theorem is given in the appendix and follows the proof of theorem 3.1.1 of [3]. With formal modifications the previous theorem can be proved also for vectors \( u \in H^{1+\theta,p} \cap C^{0,\lambda}(\overline{Q}, R^n) \), \( p > 1 \).

3. - The local differentiability result.

In this section we prove the differentiability theorem (theorem 1.1). Here \( \Omega \) is a bounded open set of \( R^n \),

\[
Q(a^n, \sigma) = \{ x : |x_i - x^n_i| < \sigma, \ i = 1, \ldots, n \}
\]

and \( d(x^n) = \text{dist}(x^n, \partial \Omega) \).
\[ u \in H^m \cap C^{m-1,\lambda}(\overline{\Omega}, \mathbb{R}^N) \] is a solution of the following system

\[ \int_{\overline{\Omega}} \sum_{\alpha \leq m} (a^\alpha(x, Du)|D^\alpha \varphi) \, dx = 0 \]

(3.1)

\[ \forall \varphi \in H^m_0 \cap H^{m-\alpha}(\Omega, \mathbb{R}^N). \]

Assume that the hypotheses (1.9) (1.20) (1.21) are fulfilled and let us define

\[ K = \sup_D \|D'u\|, \quad U = \|u\|_{C^{m-\lambda}(\partial \Omega, \mathbb{R}^N)}, \quad F = \sum_{\alpha \leq m} \|f^\alpha\|_{0, \Omega}. \]

If \( i \) is an integer, \( 1 \leq i \leq n \), and \( h \in \mathbb{R} \), then we set

\[ \tau_{i,h} u(x) = u(x + he^i) - u(x). \]

The proof of theorem 1.1 is based on theorems 3.1, 3.11, 3.111 and 3.1IV that we are now going to demonstrate.

**Theorem 3.1.** If \( u \in H^m \cap C^{m-1,\lambda}(\Omega, \mathbb{R}^N) \) is a solution of system (3.1) under the hypotheses (1.9) (1.20) (1.21), then for every \( Q(x^0, \sigma) \subset \Omega \), for every \( \varphi \in C_0^\infty(\Omega) \) with \( \varphi > 0 \) and \( \varphi = 0 \) in \( \Omega \setminus Q(x^0, \sigma) \), for \( i = 1, \ldots, n \) and \( |h| < d(x^0) - \sigma \), the following inequality holds:

\[ \int_{\Omega} \psi^m |\tau_{i,h} D'u|^2 \, dx \leq c(n, U, \mathcal{P}) h^2 (1 + |u|^2_{m, Q(x^0, |h|)}) + \]

\[ + c(K, \nu) \int_{\Omega} \psi^m \|D'u\|^2 \cdot |\tau_{i,h} D'u|^2 \, dx - \sum_{\alpha \leq m} \int_{\Omega} (a^\alpha(x, Du)|\tau_{i,h} D^\alpha (\psi^{2m} \tau_{i,h} u)) \, dx \]

where \( \mathcal{P} = \sup_D \|D\varphi\| \).

**Proof.** Let \( Q(\sigma) = Q(x^0, \sigma) \subset \Omega \) and \( \varphi \in C_0^\infty(\Omega) \) with \( \varphi > 0 \), \( \varphi = 0 \) in \( \Omega \setminus Q(\sigma) \). Having fixed \( i \) integer, \( 1 \leq i \leq n \), and \( h \) such that \( |h| < d(x^0) - \sigma \), let us assume in (3.1)

\[ \varphi = \tau_{i,-h}(\psi^{2m} \tau_{i,h} u). \]

Then we get

\[ \int_{\Omega} \sum_{\alpha \leq m} (\tau_{i,h} a^\alpha(x, Du)|D^\alpha (\psi^{2m} \tau_{i,h} u)) \, dx = \]

\[ = -\int_{\Omega} \sum_{\alpha \leq m} (a^\alpha(x, Du)|\tau_{i,-h} D^\alpha (\psi^{2m} \tau_{i,h} u)) \, dx. \]
For the sake of simplicity let us set, if \( b(x, p) \) is a vector of \( \mathbb{R}^n \),

\[
\tilde{b} = \frac{1}{t} b(x + t e^t, Du + t \tau_{t,h} Du) dt.
\]

Then

\[
\tau_{t,h} a^\alpha(x, Du) = \sum_{|\beta| \leq m} \sum_{k=1}^N \tau_{t,h} D^\beta u_k \frac{\partial a^\alpha}{\partial P_k}.
\]

Now, if \( |x| = m \),

\[
D^\alpha (p^{2m} \tau_{t,h} u) = p^{2m} \tau_{t,h} D^\alpha u + p^m \sum_{\gamma < \alpha} C_{\alpha, \gamma} (\tau_{t,h} D^\gamma u)
\]

with

\[
|C_{\alpha, \gamma} (\psi)| < c(m, n) \Psi.
\]

Therefore, from (3.5) we obtain

\[
\int_{\Omega} \sum_{|\alpha| = m} \sum_{|\beta| < m} \sum_{k=1}^N \tau_{t,h} D^\beta u_k \frac{\partial a^\alpha}{\partial P_k} \tau_{t,h} D^\alpha u dx = 
\]

\[
= - \sum_{|\alpha| = m} \sum_{|\beta| < m} \sum_{k=1}^N \int_{\Omega} \left( \tau_{t,h} D^\beta u_k \frac{\partial a^\alpha}{\partial P_k} \right) \psi^m C_{\alpha, \gamma} \tau_{t,h} D^\gamma u dx +
\]

\[
- \sum_{|\alpha| = m} \sum_{|\beta| < m} \sum_{k=1}^N \int_{\Omega} \left( \tau_{t,h} D^\beta u_k \frac{\partial a^\alpha}{\partial P_k} \right) D^\alpha (p^{2m} \tau_{t,h} u) dx +
\]

\[
- \sum_{|\alpha| = m} \sum_{|\beta| < m} \sum_{k=1}^N \int_{\Omega} \left( \tau_{t,h} D^\beta u_k \frac{\partial a^\alpha}{\partial P_k} \right) D^\alpha (p^{2m} \tau_{t,h} u) dx = A + B + C + D.
\]

By the hypothesis of strong ellipticity (1.9) we get

\[
\int_{\Omega} \sum_{|\alpha| = m} \sum_{|\beta| < m} \sum_{k=1}^N \left( \tau_{t,h} D^\beta u_k \frac{\partial a^\alpha}{\partial P_k} \right) \tau_{t,h} D^\alpha u dx \geq \gamma \int_{\Omega} \| D^\alpha (\tau_{t,h} u) \|^2 dx.
\]

On the other hand, from hypothesis (1.21) and (3.6) it follows that, if \( |x| = m \),

\[
\sum_{|\alpha| = m} \sum_{|\beta| < m} \| \frac{\partial a^\alpha}{\partial P_k} \| < M(K),
\]

\[
\sum_{|\alpha| = m} \sum_{|\beta| < m} \| \frac{\partial a^\alpha}{\partial P_k} \| < M(K) \{ 1 + \| D' u \| + \| \tau_{t,h} D' u \| \}.
\]
Then
\[ |A| < c(K, \mathcal{P}) \int_{\Omega} |\nabla^m \tau_{t,h} D' u| \cdot |\tau_{t,h} D'' u| \, dx \]
and \( \forall \varepsilon > 0 \)
\[ |A| < \varepsilon \int_{\Omega} \psi^{2n} ||\tau_{t,h} D' u||^2 \, dx + c(K, \mathcal{P}, \varepsilon) ||\tau_{t,h} u||_{m-1, \Phi}^2 \cdot \psi^{2n} \cdot ||\tau_{t,h} D' u||^2 \, dx \]
And so by lemma 2.11
\[
(3.14) \quad |A| < \varepsilon \int_{\Omega} \psi^{2n} ||\tau_{t,h} D' u||^2 \, dx + \psi^{2n} c(K, \mathcal{P}, \varepsilon) \{1 + ||u||_{m, \Phi}^2, ||u||_{m, \Omega}^2 \}. 
\]
Let us now estimate \( B \):
\[ |B| < c(K) \int_{\Omega} (1 + ||D' u|| + ||\tau_{t,h} D' u||) ||\tau_{t,h} D'' u|| \cdot \psi^{2n} ||\tau_{t,h} D' u|| + c(\mathcal{P}) \psi^{2n} ||\tau_{t,h} D'' u|| \, dx. \]
By the fact that \( u \in C^{m-1, \lambda} (\Omega, \mathbb{R}^N) \)
\[ \sup_{Q(\varepsilon)} \|\tau_{t,h} D'' u\| = o(h) \]
where \( o(h) \) depends on \( U \) and tends to zero with \( h \). Then \( \forall \varepsilon > 0 \)
\[ |B| < \varepsilon c(K, \mathcal{P}, o(h)) \int_{\Omega} \psi^{2n} ||\tau_{t,h} D' u||^2 \, dx + 
+ c(K, \varepsilon) \int_{\Omega} \psi^{2n} ||D' u||^2 \cdot ||\tau_{t,h} D' u||^2 \, dx 
+ c(K, \mathcal{P}, \varepsilon) ||\tau_{t,h} u||_{m-1, \Phi}^2. \]
Therefore by lemma 2.11 we conclude that
\[ (3.15) \quad |B| < \varepsilon c(K, \mathcal{P}, o(h)) \int_{\Omega} \psi^{2n} ||\tau_{t,h} D' u||^2 \, dx + 
+ c(K, \varepsilon) \int_{\Omega} \psi^{2n} ||D' u||^2 \cdot ||\tau_{t,h} D' u||^2 \, dx + h^2 c(K, \mathcal{P}, \varepsilon) \{1 + ||u||_{m, \Phi}^2, ||u||_{m, \Omega}^2 \}. \]
Similarly
\[ |C| < k c(K) \int_{\Omega} (1 + ||D' u|| + ||\tau_{t,h} D' u||) (\psi^{2n} ||\tau_{t,h} D' u|| + c(\mathcal{P}) \psi^{2n} ||\tau_{t,h} D'' u||) \, dx \]
and \( \forall \varepsilon > 0 \)
\[ (3.17) \quad |C| < \varepsilon c(K, \mathcal{P}, ||h|| + h^3) \int_{\Omega} \psi^{2n} ||\tau_{t,h} D' u||^2 \, dx + 
+ h^2 c(K, \mathcal{P}, \varepsilon, \sigma) \{1 + ||u||_{m, \Phi}^2, ||u||_{m, \Omega}^2 \}. \]
From (3.10) (3.11) (3.14) (3.16) (3.17) we obtain inequality (3.3) if we take $\varepsilon$ and $|h|$ small enough:

$$\varepsilon < \varepsilon_0(\nu) \quad \text{and} \quad |h| < h_0(\nu, U) < d(x^0) - \sigma.$$ 

However, if $h_0 < |h| < d(x^0) - \sigma$, then (3.3) is trivial because

$$\int_{\Omega} \psi^{2mu} \|\tau_{t,h} D' u\|^2 \, dx < 2 \frac{h^2}{\bar{h}^2} \|u\|_{m, \Omega}^2.$$ 

**Lemma 3.1.** If $u \in H^{m,p} \cap C^{m-1,1}(\Omega, \mathbb{R}^N)$, $2 < p < 4$, then for every cube $Q(2\sigma) = Q(x^0, 2\sigma) \subset \Omega$, for every $\nu \in C^2(\Omega)$ with $\nu > 0$ and $\varphi = 0$ in $\Omega \setminus Q(\frac{3}{2}\sigma)$, for $i = 1, \ldots, n$, for $|h| < \sigma/4$ and $\forall \varepsilon > 0$ the following inequality holds:

$$\langle \sum_{|\alpha| < m} \left| \int_{\Omega} (a^\alpha(x, Du) \tau_{t,h} u \nabla (\psi^{2mu} \tau_{t,h} u)) \, dx \right| <$$

$$< \varepsilon \int_{\Omega} \psi^{2mu} \|\tau_{t,h} D' u\|^2 \, dx + c(e, \sigma, \nu, U) |h|^{p-2 + \lambda(2 - p/2)} \left(1 + F^{p/2} + |u|_{m, \Omega}^p \right)$$

where $\nu = \sup \|D\nu\|$.

**Proof.** By hypothesis (1.20) and Hölder’s inequality we get $\forall \varepsilon > 0$

$$\langle \sum_{|\alpha| < m} \left| \int_{\Omega} (a^\alpha(x, Du) \tau_{t,h} u \nabla (\psi^{2mu} \tau_{t,h} u)) \, dx \right| <$$

$$< c(K) \sum_{|\alpha| < m} \left| \int_{\Omega} (|f^\alpha| + |D' u|^p) \|\tau_{t,h} D^\alpha (\psi^{2mu} \tau_{t,h} u)\| \, dx \right| <$$

$$< \varepsilon |h|^{-2} \int_{Q(\frac{7}{16}\sigma)} \|\tau_{t,h} D^\alpha (\psi^{2mu} \tau_{t,h} u)\|^2 \, dx +$$

$$+ c(e, K) |h|^{p-2} \sum_{|\alpha| < m} \int_{Q(\frac{7}{16}\sigma)} (|f^\alpha| + \|D' u\|^{p/2} \|\tau_{t,h} D^\alpha (\psi^{2mu} \tau_{t,h} u)\|^{p/2}) \, dx.$$

Then, from lemma 2.11, (3.8) and the fact that $u \in C^{m-1,1}(\Omega, \mathbb{R}^N)$ we conclude that

$$\langle \sum_{|\alpha| < m} \left| \int_{\Omega} (a^\alpha(x, Du) \tau_{t,h} u \nabla (\psi^{2mu} \tau_{t,h} u)) \, dx \right| <$$

$$< c(\sigma) \varepsilon \|\psi^{2mu} \tau_{t,h} u\|^2_{m, \Omega} + c(e, U) |h|^{p-2 + \lambda(2 - p/2)} \left(F^{p/2} + |u|_{m, \Omega}^p \right) \left(1 + F^{p/2} + |h|_{m, \Omega}^p \right) <$$

$$< c(\sigma) \varepsilon \int_{\Omega} \psi^{2mu} \|\tau_{t,h} D' u\|^2 \, dx + c(e, \nu, U) |h|^{p-2 + \lambda(2 - p/2)} \left(1 + F^{p/2} + |h|_{m, \Omega}^p \right).$$
Using theorem 3.1 we can easily prove the following fractional differentiability result.

**THEOREM 3.1**. If \( u \in H^{m} \cap C^{m-1,1}(\Omega, \mathbb{R}^{N}) \) is a solution of system (3.1) under the hypotheses (1.9) (1.20) (1.21), then

\[
(3.20) \quad u \in H^{m+\theta}_{\text{loc}}(\Omega, \mathbb{R}^{N}), \quad 0 < \theta < \lambda/2
\]

and for every cube \( Q(3\sigma) = Q(x^{0}, 3\sigma) \subset \Omega \) \(^{2}\)

\[
(3.21) \quad |D' u|^{2}_{(Q(\sigma)} \leq C(\sigma, \lambda, U) \{1 + F + |u|^{2}_{m, Q(3\sigma)} \}.
\]

**PROOF.** Choose \( \varphi \in C_{0}^{\infty}(\Omega) \) with

\[
0 < \varphi < 1, \quad \varphi = 1 \text{ in } Q(\sigma), \quad \varphi = 0 \text{ in } \Omega \setminus Q(2\sigma).
\]

From inequality (3.3), hypothesis (1.20) and the fact that \( u \in C^{m-1,1}(\Omega, \mathbb{R}^{N}) \) we conclude that \( \forall \varphi \) \( < \sigma \)

\[
(3.22) \quad \sum_{i=1}^{N} \int_{Q(\sigma)} |\tau_{i,\lambda} D' u|^{2} \, dx < c(U, \sigma)(\lambda|\tau_{i,\lambda} D' u|^{2} + |u|^{2}_{m, Q(3\sigma)}) +
\]

\[
+ c(K) \sum_{|a| < m} \int_{Q(3\sigma)} (|f_{a}| + \|D' u\|^{2}) \|D^{a} u\| \, dx <
\]

\[
< c(U, \sigma)|h|^{2}(1 + F + |u|_{m, Q(3\sigma)}). \quad (\ast)
\]

Inequality (3.22) is trivial if \( \sigma < |\lambda| < 2\sigma \) and therefore, if \( 0 < \theta < \lambda/2 \), from (3.22) we easily get

\[
(3.23) \quad \sum_{i=1}^{N} \int_{-\sigma}^{\sigma} \frac{d\sigma}{|\lambda|^{1+2\sigma}} \int_{Q(\sigma)} |\tau_{i,\lambda} D' u|^{2} \, dx < c(U, \sigma, \theta, \lambda) \{1 + F + |u|^{2}_{m, Q(3\sigma)} \}.
\]

(3.20) and (3.21) follow from (3.23) using lemma 2.111.

A more general result is the following:

**THEOREM 3.III.** If \( u \in H^{m+\theta} \cap C^{m-1,1}(\Omega, \mathbb{R}^{N}), \ 0 < \theta < 1, \) is a solution

\(^{2}\) \( |D' u|^{2}_{Q} = \sum_{|a| = m} |D^{a} u|^{2}_{Q} \)

\(^{2}\) We note that this theorem is valid even if \( f \in L^{1}(\Omega), |x| < m.\)
of system (3.1) under the hypotheses (1.9) (1.20) (1.21), then

\begin{equation}
(3.24) \quad u \in H^{m+\theta}_{\text{loc}}(\Omega, R^n), \quad \forall \theta_1 < \frac{\lambda}{2} + \theta \left(1 - \frac{\lambda}{2}\right)
\end{equation}

and for every cube \(Q(3\sigma) = Q(x^0, 3\sigma) \subset \subset \Omega\)

\begin{equation}
(3.25) \quad |D' u|_{[\theta_1, Q(\sigma)]}^2 < c(U, \sigma) \left\{1 + F^{1+\theta} + |u|_{m, Q(\sigma)}^2 + |D' u|_{[\theta_1, Q(\sigma)]}^2\right\}.
\end{equation}

PROOF. Because of theorem 3.II we may assume \(0 < \theta < 1\). As \(\theta > 0\), by theorem 2.1 we get

\begin{equation}
(3.26) \quad u \in H^{m,p}_{\text{loc}}(\Omega, R^n), \quad \forall 2 < \frac{2(1 + \theta) n}{n - 2\theta \lambda}
\end{equation}

and for every cube \(Q \subset \subset \Omega\)

\begin{equation}
(3.27) \quad \sum_{|\alpha| = m} \int_Q \|D^\alpha u - (D^\alpha u)_{\sigma}\|^q \, dx < C(U) |D' u|_{L^q_{\sigma}}^{p/(1 + \theta)}.
\end{equation}

Now, choose \(\psi \in C_0^\infty(\Omega)\) with

\(0 < \psi < 1\), \(\psi = 1\) in \(Q(\sigma)\), \(\psi = 0\) in \(\Omega \setminus Q(2\sigma)\).

Let \(|h| < \sigma/2\). From inequality (3.3) and lemma 3.1, in which we assume \(\varepsilon\) small enough and \(p = 2(1 + \theta)\), we conclude that

\begin{equation}
(3.28) \quad \sum_{i=1}^n \int_{Q(\sigma)} \|\tau_{iA} D' u\|^q \, dx < c(U, \sigma)|h|^{\theta + \theta(1-\theta)}\left\{1 + F^{1+\theta} + |u|_{m, Q(\sigma)}^{2(1+\theta)} + \right.\n\end{equation}

\[+ \sum_{i=1}^n \int_{Q(2\sigma)} \|D' u\|^2 \cdot \|\tau_{iA} D^\alpha u\|^2 \, dx.\]

Having fixed \(p\) such that

\(2(1 + \theta) < \frac{2(1 + \theta) n}{n - 2\theta \lambda}\)

by Hölder’s inequality we get

\(\int_{Q(\sigma)} \|D' u\|^2 \cdot \|\tau_{iA} D^\alpha u\|^2 \, dx < c(K) h^{\varepsilon} |u|_{m, Q(\sigma)}^2 + \int_{Q(2\sigma)} \|D' u\|^2 \cdot \sum_{|\alpha| = m-1} \|\tau_{iA} D^\alpha u\|^2 \, dx < \)
Now, by lemma 2.11

\[ \left( \int_{Q} |\partial_{x} u(x, y)|^{2} \, dx \right)^{\frac{2}{p}} < C^{|\frac{m}{2}|} |u|^{\frac{2}{2}}_{m,p,Q(2\sigma)} \]

and by the fact that \( u \in C^{m-1,1}(\Omega, \mathbb{R}^{n}) \)

\[ \left( \int_{Q} |\partial_{x} u(x, y)|^{2p(1-\theta)/(p-2(1+\theta))} \, dx \right)^{1-2(1+\theta)/p} < C|u|^{2(1+\theta)/p}_{2} \]

Therefore

\[ (3.29) \quad \int_{Q} |\partial_{x} u(x, y)|^{2} \, dx < c(U, \sigma)|h|^{2\theta+2(1+\theta)} \{ 1 + |u|^{2(1+\theta)}_{m,p,Q(2\sigma)} \} \]

From (3.28) and (3.29) we deduce that \( \forall |h| < \sigma/2 \)

\[ (3.30) \quad \sum_{i=1}^{n} \int_{Q} |\partial_{x} u(x, y)|^{2} \, dx < c(U, \sigma)|h|^{2\theta+4(1+\theta)} \{ 1 + F^{1+\theta} + |u|^{2(1+\theta)}_{m,p,Q(2\sigma)} \} \]

Furthermore, from (3.27)

\[ |u|^{2(1+\theta)}_{m,p,Q(2\sigma)} < c(U) \left( |D u|^{2}_{2, Q(3\sigma)} + |D u|^{2}_{2, Q(3\sigma)} \right)^{2(1+\theta)} < c(U, \sigma) \left( |D u|^{2}_{2, Q(3\sigma)} + |u|^{2(1+\theta)}_{m,p,Q(2\sigma)} \right) \]

Now, by lemma 1 of the appendix

\[ |u|^{2(1+\theta)}_{m,p,Q(2\sigma)} < c(K) \{ 1 + |D u|^{2}_{2, Q(3\sigma)} \} \]

Hence we conclude that \( \forall |h| < \sigma/2 \)

\[ (3.31) \quad \sum_{i=1}^{n} \int_{Q} |\partial_{x} u(x, y)|^{2} \, dx < c(U, \sigma)|h|^{2\theta+4(1+\theta)} \{ 1 + F^{1+\theta} + |u|^{2(1+\theta)}_{m,p,Q(2\sigma)} + |D u|^{2}_{2, Q(3\sigma)} \} \]
The last inequality is trivial if $\sigma/2 < |h| < 2\sigma$ and so the proof finishes as in theorem 3.II.

By theorems 3.II and 3.III we prove, using an iteration argument, the following

**THEOREM 3.IV.** If $u \in H^m \cap C^{m-1,\lambda}(\Omega, R^n)$ is a solution of system (3.1) under the hypotheses (1.9) (1.20) (1.21), then

\begin{equation}
\text{(3.32)} \quad u \in H^{m+\theta}_\text{loc}(\Omega, R^n), \quad \forall 0 < \theta < 1
\end{equation}

and $\forall Q(\sigma) \subset Q(\sigma_0) \subset \Omega$ (*)

\begin{equation}
\text{(3.33)} \quad |D' u|^2_{\theta, Q(\sigma)} \leq c(\sigma, \sigma_0 - \sigma, U) \left\{ 1 + F^{1+\theta} + |u|^2_{m, Q(\sigma_0)} \right\}.
\end{equation}

In particular

\begin{equation}
\text{(3.34)} \quad u \in H^{m,\lambda}_{\text{loc}}(\Omega, R^n).
\end{equation}

**Proof.** Let $Q(3\rho) = Q(x^\sigma, 3\rho) \subset \Omega$ and choose $\sigma_0$ such that $0 < \sigma_0 < \lambda/2$.

From theorem 3.II we conclude that

\begin{equation}
\text{u} \in H^{m+\theta_1}_\text{loc} \cap C^{m-1,\lambda}(\overline{Q(\rho)}), R^n)
\end{equation}

and

\begin{equation}
\text{(3.35)} \quad |D' u|^2_{\theta_1, Q(\rho)} \leq c(\rho, U) \left\{ 1 + F + |u|^2_{m, Q(3\rho_0)} \right\}.
\end{equation}

Then, by theorem 3.III

\begin{equation}
\text{u} \in H^{m+\theta_i}_\text{loc} \cap C^{m-1,\lambda}(\overline{Q(3^{-i} \rho)}), R^n)
\end{equation}

with

\begin{equation}
\theta_i = \theta_0 + \theta_0 (1 - \theta_0) < \theta_0 + \frac{\lambda}{2} (1 - \theta_0)
\end{equation}

and

\begin{equation}
\text{(3.36)} \quad |D' u|^2_{\theta_i, Q(3^{-i} \rho)} \leq c(\rho, U) \left\{ 1 + F^{1+\theta_i} + |u|^2_{m, Q(3\rho_0)} \right\}.
\end{equation}

By induction we obtain that for every integer $i$

\begin{equation}
\text{u} \in H^{m+\theta_i}_\text{loc} \cap C^{m-1,\lambda}(\overline{Q(3^{-i} \rho)}), R^n)
\end{equation}

\(*\) $Q(\sigma) = Q(x^\sigma, \sigma), Q(\sigma_0) = Q(x^\sigma, \sigma_0).$
with
\[ \theta_i = \theta_0 \sum_{r=0}^{i} (1 - \theta_r)^r \]
and
\[ |D'u|^2_{p, Q(3^{-j} \sigma)} < c(\sigma, U) \{ 1 + F^{1 + \theta_0} + |u|^2_{m, q(Q(2\sigma))} \}. \]

Now, from lemma 2.IV it follows that \( \forall Q(\sigma) \subset Q(\sigma_0) \subset \Omega \)
\[ |D'u|^2_{p, Q(\sigma)} < c(\sigma, \sigma_0 - \sigma, U) \{ 1 + F^{1 + \theta_0} + |u|^2_{m, q(Q(\sigma_0))} \}. \]

As \( \{\theta_i\} \) is an increasing sequence that converges to 1, (3.32) and (3.33) are proved. Using theorem 2.1, (3.34) follows from (3.32) if we fix \( \theta \) such that
\[ \frac{n}{n + 4 \lambda} < \theta < 1. \]

**Proof of Theorem 1.I.** The previous results enable us to complete the proof of the differentiability theorem (theorem 1.I).

Let \( Q(2\sigma) = Q(x^0, 2\sigma) \subset \Omega \) and \( \psi \in C_0^\infty(\Omega) \) with
\[ 0 < \psi < 1, \quad \psi = 1 \quad \text{in} \ Q(\sigma), \quad \psi = 0 \quad \text{in} \ \Omega \setminus Q(2\sigma). \]

Let \( |h| < \sigma/2. \) From inequality (3.3) and lemma 3.I, in which we assume \( \varepsilon \) small enough and \( p = 4 \), we conclude that
\[ \sum_{i=1}^{n} \int_{Q(\sigma)} ||\tau_{i,h} D'' u||^2 \, dx < c(\sigma, U, \sigma) \frac{n}{1 + F^2 + |u|^4_{m, A, Q(2\sigma)}} + c(K) \sum_{i=1}^{n} \int_{Q(\sigma)} ||D'' u||^2 \cdot ||\tau_{i,h} D'' u||^2 \, dx. \]

Now, using lemma 2.II
\[ \int_{Q(\sigma)} ||D'' u||^2 \cdot ||\tau_{i,h} D'' u||^2 \, dx < c(h^2 \int_{Q(\sigma)} ||D'' u||^4 \, dx)^{\frac{1}{2}} \left( \int_{Q(\sigma)} ||D'' u||^4 \, dx \right)^{\frac{1}{2}} \left( \int_{Q(\sigma)} ||\tau_{i,h} D'' u||^4 \, dx \right)^{\frac{1}{2}} < \]
\[ c(h^2 \int_{Q(\sigma)} ||D'' u||^4 \, dx)^{\frac{1}{2}} \cdot |u|^4_{m, A, Q(2\sigma)} < h^2 c(K) \{ 1 + |u|^4_{m, A, Q(2\sigma)} \}. \]

Then, for \( |h| < \sigma \)
\[ \sum_{i=1}^{n} \int_{Q(\sigma)} ||\tau_{i,h} D'' u||^2 \, dx < c(\sigma, U, \sigma) h^2 \{ 1 + F^2 + |u|^4_{m, A, Q(2\sigma)} \}. \]
Therefore, from lemma 2.1 we get that \( u \in H^{m+1}(Q(\sigma), \mathbb{R}^n) \) and

\[
|u|_{m+1, Q(\sigma)}^2 \leq c(U, \sigma) \left\{ 1 + F^2 + |u|_{m, Q(2\sigma)}^4 \right\}.
\]

The proof of theorem 1.1 is complete.

**Remark 3.1.** In order to prove (1.29) we have only to estimate the term \( |u|_{m, Q(2\sigma)}^4 \).

Let us suppose that \( Q(2\sigma) \subset \Omega \) and \( 0 < \delta < \frac{1}{2}(d(x^0) - 2\sigma) \). Having chosen \( \theta \) such that

\[
\max \left\{ \sqrt{2} - 1, \frac{n}{n + 4\lambda} \right\} \theta < 1
\]

we have

\[
\frac{2(1 + \theta)^n}{n - 20\lambda} > 4.
\]

Then, by theorem 2.1,

\[
\int_{Q(2\sigma)} \| D' u - (D' u)_{Q(2\sigma)} \|^4 \, dx < c(\sigma, U) |D' u|_{4/(1 + \theta), Q(2\sigma)}^4
\]

and so

\[
\int_{Q(2\sigma)} \| D' u \|^4 \, dx < c(\sigma, U) |D' u|_{4/(1 + \theta), Q(2\sigma)}^4 + c(\sigma)|u|_{m, Q(2\sigma)}^4.
\]

Now, by lemma 1 of the appendix

\[
|u|_{m, Q(2\sigma)}^4 \leq c(K) \left\{ 1 + |D' u|_{4/(1 + \theta), Q(2\sigma)}^4 \right\}.
\]

Thus, recalling (3.33)

\[
|u|_{m, Q(2\sigma)}^4 \leq c(\sigma, U) \left\{ |D' u|_{4/(1 + \theta), Q(2\sigma)}^4 + 1 \right\} \leq c(\sigma, U) \left\{ 1 + F^2 + |u|_{m, Q(2\sigma)}^4 \right\}.
\]

Again, by lemma 1 of the appendix and (3.33)

\[
|u|_{m, Q(2\sigma)}^4 \leq c(K) \left\{ 1 + |D' u|_{4/(1 + \theta), Q(2\sigma)}^4 \right\} \leq c(\sigma, U) \left\{ 1 + F^2 + |u|_{m, Q(2\sigma)}^4 \right\}.
\]

According to (3.45) we have \( 2/(1 + \theta)^2 < 1 \) and then

\[
|u|_{m, Q(2\sigma)}^4 \leq c(\sigma, U) \left\{ 1 + F^2 + |u|_{m, Q(2\sigma)}^2 \right\}.
\]

Inequality (1.29) follows from (3.44) and (3.47).
REM ARK 3.11. The functions $f^a$ that appear in condition (1.20) could be assumed to satisfy the following more general hypothesis:

\[(3.48) \quad \mathcal{F} = \sum_{|a| \leq m} D^a f^a \in H^{-m+1}(\Omega)\]

which means (see for instance [5], chap. I, n. 4) that $f^a \in L^{2m}(\Omega)$ with

\[q_a = \max\left\{1, \frac{2m}{n+2(m-1-|a|)}\right\} \quad \text{if } m-1-|a| \neq \frac{n}{2},\]

\[q_a \in (1, 2) \quad \text{if } m-1-|a| = \frac{n}{2}.\]

In this case, under the same hypotheses of lemma 3.1, we get instead of (3.18)

\[(3.49) \quad \left| \sum_{|a| \leq m} \int_D (a^a(x, Du)|\tau_{i,a} D^a(p^{2m} \tau_{i,a} u)) \right| dx \leq \epsilon \int_D \left| \tau_{i,a} D^a(p^{2m} \tau_{i,a} u) \right|^2 dx +
\]

\[+ c(\sigma, \Psi, \ U)|h|^{p-2+\lambda(2-\nu/2)}(1 + \|\mathcal{F}\|^2 + |u|^2_{m,Q(\Omega)}).\]

To do this we note that, if hypothesis (3.48) holds, then $\forall \epsilon > 0$

\[\sum_{|a| \leq m} \int_D |f^a| \cdot |\tau_{i,a} D^a(p^{2m} \tau_{i,a} u)| \cdot dx \leq \epsilon \int_D |\mathcal{F}| \cdot |\tau_{i,a} D^a(p^{2m} \tau_{i,a} u)| \cdot dx\]

\[\leq \epsilon \int_D |\mathcal{F}| \cdot |\tau_{i,a} D^a(p^{2m} \tau_{i,a} u)| \cdot dx \leq c|\mathcal{F}| \cdot |\tau_{i,a} D^a(p^{2m} \tau_{i,a} u)| \cdot dx\]

\[\leq \epsilon \int_D |\mathcal{F}| \cdot |\tau_{i,a} D^a(p^{2m} \tau_{i,a} u)| \cdot dx + c(\Psi, K)|h|^a(1 + \|\mathcal{F}\|^2 + |u|^2_{m,Q(\Omega)}).\]

Hence, inequality (3.25) becomes

\[(3.50) \quad |D'u|^2_{\sigma, Q(\Omega)} \leq c(\sigma, \sigma_0 - \sigma, U) \left\{1 + \|\mathcal{F}\|^2 + |u|^2_{m,Q(\Omega)} + |D'u|^2_{\sigma, Q(\Omega)} \right\}
\]

and inequality (3.33) becomes

\[(3.51) \quad |D'u|^2_{\sigma, Q(\Omega)} \leq c(\sigma, \sigma_0 - \sigma, U) \left\{1 + \|\mathcal{F}\|^2 + |u|^2_{m,Q(\Omega_0)} \right\}, \quad \forall 0 < \theta < 1.
\]

No change is required in the proof of theorem 1.I or inequality (3.44), except from replacing $F^2$ by $|\mathcal{F}|^2$. It is known (see for instance [5]) that
Using (3.51) and repeating the argument of remark 3.I, we can estimate \( |u|_{m,q_0,q(2\sigma)} \) as follows

\begin{equation}
|u|_{m,q_0,q(2\sigma)}^4 < C(\sigma, U) \left( 1 + \|F\|^{4(1+\theta)} + |u|_{m,q_0,q(2\sigma)}^2 \right), \quad \forall 0 < \theta < 1.
\end{equation}

4. Partial Hölder continuity of the derivatives \( D^\alpha u, \ |x| = m. \)

Let \( u \in H^m \cap C^{m-1,1}(\Omega, R^n) \) be a solution of system

\begin{equation}
\int_{\Omega} \sum_{|\alpha| < m} (a^\alpha(x, Du)|D^\alpha \varphi|) \, dx = 0
\end{equation}

\( \forall \varphi \in H^m_0 \cap H^{m-1,\infty}(\Omega, R^n) \)

under the hypotheses (1.9) (1.20) (1.21). We have shown that \( u \in H^{m+1}_0(\Omega, R^n) \) and then, by a standard calculation (see for instance [5], chap. V, n. 4), we obtain that for every open set \( \Omega_0 \subset \subset \Omega \)

\( u \in H^{m+1} \cap C^{m-1,1}(\Omega_0, R^n) \)

is a solution of the quasilinear system of order \( 2(m+1) \)

\begin{equation}
\int_{\Omega} \sum_{|\alpha|-m} \sum_{r,s} (B_{sr,\alpha}(x, Du) D_s D^r u |D_s D^r \varphi|) \, dx =
\end{equation}

\( = \int_{\Omega} \sum_{|\alpha|-m} \sum_{i=1}^n (G^{s,i}(x, Du) D_s D^i \varphi + \sum_{|\beta| < m} \sum_{k=1}^n (a^\beta(x, Du) |D_s D^k \varphi|) \, dx \quad \forall \varphi \in C_0^\infty(\Omega_0, R^n) \)

where \( B_{sr,\alpha} = \{B^{s,k}_{sr,\alpha}\} \) are \( N \times N \) matrices defined in \( \Omega_0 \times R \) as follows

\begin{equation}
B^{s,k}_{sr,\alpha} = \delta_{r,s} \left( \frac{\partial a^\alpha}{\partial p^k} \right)
\end{equation}

and \( G^{s,i}: \Omega_0 \times R \to R^n \) are the following vectors

\begin{equation}
G^{s,i}(x, Du) = - \frac{\partial a^\alpha(x, Du)}{\partial x_i} + \sum_{|\beta| < m} \sum_{k=1}^N D_s D^k u_k \frac{\partial a^\beta(x, Du)}{\partial p^k}.
\end{equation}
System (4.2) is strongly elliptic and, according to (1.21),

\[(4.4) \quad \|G^{\alpha}(x, Du)\| < c(K) \{1 + \|D'u\|^2\}\]

if \(\alpha \in \Omega_\theta\) and \(\|D'u\| < K\).

Assume that

\[\frac{\partial a^\alpha(x, p)}{\partial p_k}, \quad |\alpha| = |\beta| = m \text{ and } 1 < k < N\]

are uniformly continuous in \(\bar{Q} \times \mathbb{R}\) and that in condition (1.20) we have

\[f^\alpha \in L^p(\Omega) \quad \text{with } p > n\]

or, more generally (see also remark 3.11)

\[\mathcal{F} = \sum_{|\alpha| < m} D^\alpha f^\alpha \in H^{-m+1,p}(\Omega) \quad \text{with } p > n\]

which means (see for instance [5], chapter I, n. 4) that \(f^\alpha \in L^{q^\alpha}(\Omega)\) where

\[q^\alpha(p) = \max\left\{1, \frac{pn}{n + p(m - 1 - |\alpha|)}\right\} \quad \text{if } m - 1 - |\alpha| \neq \frac{n}{p^*}, \quad \frac{1}{p} + \frac{1}{p^*} = 1\]

\[q^\alpha(p) \in (1, p) \quad \text{if } m - 1 - |\alpha| = \frac{n}{p^*}\]

Then system (4.2) satisfies the hypotheses of theorem 3.I, chap. IV of [5] (\(^5\)), and from this theorem we get the partial Hölder continuity result contained in theorem 1.11 of the present work.

Once we obtained the Hölder continuity of the \(D^\alpha u, |\alpha| = m\), system (4.2) reduces to a linear system with smooth coefficients and right hand side. Therefore, the higher regularity of \(u\) is a consequence of the theory of linear systems.

**Appendix.**

In this appendix we prove theorem 2.1. The proof is completely analogous to that of theorem 3.III of [3], which deals with the case \(\theta = 1\). We can therefore suppose \(0 < \theta < 1\).

\(^5\) This theorem extends the result of [4] to systems of order \(> 4\).
We note that (2.8) (2.9) are a trivial consequence of (2.7). To see that, if
\[ 1 < s < q = \frac{2(1 + \theta) \mu}{n - 2s} \]
then \( \forall M > 0 \)

\[
\int_0^M \left| D_i u - (D_i u)_0 \right|^s dx = s \int_0^\infty t^{s-1} \text{meas} \{ x \in Q : \| D_i u(x) - (D_i u)_0 \| > t \} dt = \\
= s \int_0^M t^{s-1} \text{meas} \{ x \in Q : \| D_i u(x) - (D_i u)_0 \| > t \} dt + \\
+ s \int_M^\infty t^{s-1} \text{meas} \{ x \in Q : \| D_i u(x) - (D_i u)_0 \| > t \} dt < \\
< (\text{meas} Q) M^{s-1} \frac{s}{q-s} M^{1-s} A^s
\]

where \( A^s = c(n, \theta) |D_i \mu|_Q^{(1+\theta)}[u]_Q^{(1+\theta)} \).

Inequality (2.9) follows choosing \( M = A(\text{meas} Q)^{-1/q} \).

As in [3], we derive (2.7) from some interpolation formulas. Let

\[ Q(\sigma) = Q(x_0, \sigma) = \{ x : |x_\sigma - x_0^0| < \sigma \} . \]

**Lemma 1.** For every \( u \in H^{1+\theta}(Q(\sigma), \mathbb{R}^n) \), \( 0 < \theta < 1 \), the following inequality holds

\[
|u|_{1,Q(\sigma)} < C(n, \theta) \left( \sum_i |D_i u|_{\overline{\partial}_i Q(\sigma)} \right)^{1/(1+\theta)} \cdot |u|_{0,Q(\sigma)}^{\theta/(1+\theta)} + \sigma^{-1} |u|_{0,Q(\sigma)} .
\]

We give a proof for the reader's convenience.

Let \( U \in H^{1+\theta}(\mathbb{R}^n, \mathbb{R}^n) \) be an extension of \( u \) such that (see for instance [1], chapters IV and VII)

\[
|U|_{0,\mathbb{R}^n} < c |u|_{0,Q(\sigma)} , \\
|U|_{1,\mathbb{R}^n} + \left( \sum_i |D_i U|_{\partial_i \mathbb{R}^n} \right)^i < c \left( |u|_{1,Q(\sigma)} + \sum_i |D_i u|_{\partial_i Q(\sigma)} \right)^i .
\]

Via Fourier transform we get

\[
|U|_{1,\mathbb{R}^n} < c(n, \theta) \left( \sum_i |D_i U|_{\partial_i \mathbb{R}^n} \right)^{1/(1+\theta)} \cdot |U|_{0,\mathbb{R}^n}^{\theta/(1+\theta)} .
\]
(1) follows from (2) and (3). The way how constants depend on \( \sigma \) may be easily checked by a homothetical argument.

**Lemma 2.** For every \( u \in H^{1+\theta}(Q(\sigma), \mathbb{R}^n) \), \( 0 < \theta < 1 \), the following inequality holds

\[
\sum_{i=1}^{n} |D_i u - (D_i u)_{Q(\sigma)}|_{0,Q(\sigma)} \leq C(n, \theta) \left( \sum_i |D_i u|^2_{0,Q(\sigma)} \right)^{1/(1+\theta)} \cdot \|u - u_{Q(\sigma)}\|_{0,Q(\sigma)}^{\theta(1+\theta)}. \tag{4}
\]

**Proof.** Let \( \mathcal{P}_1 \) be the class of all the polynomials in \( x \) with degree \(< 1 \) and let \( P_0 = \sum a_i x_i + a_0 \) be the polynomial such that

\[
\int_{Q(\sigma)} \|u - P_0\|^2 \, dx = \inf_{P \in \mathcal{P}_1} \int_{Q(\sigma)} \|u - P\|^2 \, dx.
\]

Let us set, also

\[
P_1(x) = \sum_{i=1}^{n} (D_i u)_{Q(\sigma)}(x_i - a_i^0) + u_{Q(\sigma)}.
\]

According to (1), written for \( u - P_0 \),

\[
|u - P_1|_{1,Q(\sigma)} < |u - P_0|_{1,Q(\sigma)} < c(n, \theta) \left( \sum_i |D_i u|^2_{0,Q(\sigma)} \right)^{1/(1+\theta)} \cdot \|u - u_{Q(\sigma)}\|_{0,Q(\sigma)}^{\theta(1+\theta)} + \sigma^{-1} \|u - P_0\|_{0,Q(\sigma)}^{\theta(1+\theta)}. \tag{5}
\]

Now, by Poincaré's inequality

\[
\|u - P_0\|_{0,Q(\sigma)} < \|u - P_1\|_{0,Q(\sigma)} < c(n) \sigma |u - P_1|_{1,Q(\sigma)} < c(n) \sigma^{1+\theta} \left( \sum_i |D_i u|^2_{0,Q(\sigma)} \right)^{1/(1+\theta)}.
\]

Therefore

\[
\sigma^{-1} \|u - P_0\|_{0,Q(\sigma)} < c(n) \left( \sum_i |D_i u|^2_{0,Q(\sigma)} \right)^{1/(1+\theta)} \cdot \|u - P_0\|_{0,Q(\sigma)}^{\theta(1+\theta)}. \tag{6}
\]

Furthermore

\[
\|u - P_0\|_{0,Q(\sigma)} < \inf_{c \in \mathcal{B}} \|u - c\|_{0,Q(\sigma)} = \inf_{c \in \mathcal{B}} \|u - u_{Q(\sigma)}\|_{0,Q(\sigma)}. \tag{7}
\]

Thus, (4) follows from (5) (6) (7).

We recall also the following lemma, due to John-Nirenberg [6]:

**Lemma 3.** Let \( Q \) be a cube of \( \mathbb{R}^n \) and let \( Q = \bigcup Q_k \) be a subdivision of \( Q \) into a denumerable number of cubes \( Q_k \), no two having a common interior point.
Let \( u \) be integrable in \( Q \) and assume that for fixed \( p, 1 < p < +\infty \),

\[
K(u) = \left\{ \frac{\sup_{Q \cap Q} \sum (\text{meas } Q_k)^{1-p} \left( \int_{Q_k} \| u - u_{Q_k} \| \, dx \right)^p}{\| u \|_{L^p(Q)}} \right\}^{1/p} < +\infty.
\]

Then \( \forall t > 0 \)

\[
\text{meas} \{ x \in Q : \| u(x) - u_Q \| > t \} < c(n, p) \left( \frac{K_p(u)}{t} \right)^p.
\]

We can now give the proof of theorem 2.1.

Let \( Q = \bigcup_{k} Q_k \) be a subdivision of \( Q \) into cubes \( Q_k \), no two having a common interior point. Let \( u \in H^{1+\theta} \cap C^{0,\lambda}(\overline{Q}, R^N) \), \( 0 < \theta < 1 \) and \( 0 < \lambda < 1 \); let finally \( q = 2(1 + \theta) n/(n - 2\theta\lambda) \).

From inequality (4) we conclude that

\[
\left( \frac{\sum |D_i u|_{L^q(Q)}^q}{(\text{meas } Q)^{q/(1+\theta)}} \right)^{1/q} \leq c(n, \theta) \left( \sum |D_i u|_{L^q(Q)}^{q/(1+\theta)} \right)^{1/q}.
\]

As

\[
\| u - u_{Q_k} \|_{L^q(Q)}^{q/(1+\theta)} \leq (\text{meas } Q_k)^{q/(1+\theta)}(q/(1+\theta)) \left[ u \right]_{L^q_{1,q}}^{q/(1+\theta)},
\]

from (10) we get

\[
(\text{meas } Q_k)^{1/q} \left( \sum |D_i u|_{L^q(Q)}^{q/(1+\theta)} \right)^{1/q} \leq c(n, \theta) \left( \sum |D_i u|_{L^q(Q)}^{q/(1+\theta)} \right)^{1/q} \leq c(n, \theta) \left( \sum |D_i u|_{L^q(Q)}^{q/(1+\theta)} \right)^{1/q}.
\]

Then

\[
K_q(u) \leq c(n, \theta) \left( \sum |D_i u|_{L^q(Q)}^{q/(1+\theta)} \right)^{1/q} \cdot [u]_{L^q_{1,q}}^{q/(1+\theta)}.
\]

The result (2.7) of theorem 2.1 follows from (11) by virtue of the John-Nirenberg lemma (lemma 3).

BIBLIOGRAPHY


Istituto Matematico dell'Università
via Buonarroti 2
56100 Pisa

Scuola Normale Superiore
piazza dei Cavalieri 7
56100 Pisa