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Versal deformations for two-dimensional pseudoconvex manifolds


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Versal Deformations
for Two-Dimensional Pseudoconvex Manifolds.

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Let $M$ be a strictly pseudoconvex manifold with a one-dimensional exceptional set $A$. Let $\Theta$ be the holomorphic tangent sheaf to $M$. The general Kodaira-Spencer [11] theory shows that $H^1(M, \Theta)$ corresponds to first order infinitesimal deformations of $M$ and that $H^2(M, \Theta)$ represents the obstructions to formally extending deformations to higher order. $H^1(M, \Theta)$ is finite dimensional since $M$ is strictly pseudoconvex [1]. $H^2(M, \Theta) = 0$ essentially because $A$ is one-dimensional. But it is known [6], [5] that there is no finite-dimensional deformation theory for $M$ if one keeps track of the boundary. So in order to stay within the Kodaira-Spencer framework, given a deformation of $M$ and a compact set $K$ in $M$, we shall only worry about the deformation near $K$. Then $M$ has a versal deformation $\omega : M \to Q$ with $Q$ a manifold of dimension $\dim H^1(M, \Theta)$ in case either (i) $M$ is of arbitrary dimension and is a sufficiently small neighborhood of $A$ (Definition 1, Theorem 2 and Theorem 5 below) or (ii) $M$ is of dimension two (Theorem 8 below). The existence of $\omega$ was proved for arbitrary Stein $M$ by Andreotti and Vesentini [2]. Openness of versality holds (Theorem 3 and Theorem 8 below).

Some applications of this paper are given in [16] and [17]. In [17], the dimension two analogue of [7] and [23, Theorem 2.1 and Proposition 2.3] is proved, i.e. if all of the fibers of a deformation are isomorphic, then the deformation is trivial.

Most of the results of this paper have been announced in [15].

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**Definition 1.** Let $M$ be a strictly pseudoconvex manifold. A special cover $\mathcal{U} = \{U_i\}$, $0 < i < m$, is a finite cover of $M$ such that each $U_i$ is Stein and such that $\overline{U_i} \cap \overline{U_j} \cap \overline{U_k} = \emptyset$ for $i \neq j \neq k$. ($\overline{\cdot}$ denotes closure in $M$.)

**Theorem 2.** Let $M^*$ be a strictly pseudoconvex manifold with a one-dimensional exceptional set $A$. Then there is a strictly pseudoconvex neighborhood $M$ of $A$, a special cover $\mathcal{U}$ of $M$, and a deformation $\omega: M \to Q$ of $\omega = \omega^{-1}(0)$, with $Q$ a manifold, such that the Kodaira-Spencer map $\partial_\omega: T_{\theta} \to H^1(M, \Theta)$ is an isomorphism. $\omega$ may be chosen to be a 1-convex holomorphic map.

**Proof.** We first construct a larger cover $\mathcal{B} = \{V_i\}$, $0 < i < m$. Let the $V_i$, $1 < i < m$, be small balls in local coordinates for $M^*$ centered about the singular points $s_i$ of $A$. Choose $\overline{V_i} \cup \overline{V_j} = \emptyset$ for $i \neq j$; closure is in $M^*$. Should a connected component of $A$ be non-singular, also choose such a $V_i$ about some points $s_i$ in the component. So we get points $s_i$, $1 < i < m$, lying in all irreducible components $A_k$ of $A$. Let $S = \cup s_i$, $1 < i < m$. Let $T \subset \cup V_i$, $1 < i < m$, be a closed neighborhood of $S$ in $A$. We choose $V_0$ to be a Stein neighborhood of $A - T$ as follows. Each $A_k - S$ is an open Riemann surface and thus Stein [9, Theorem IX. C. 10, p. 270]. Let $f_k$ be a $C^\infty$ strictly plurisubharmonic function on $A_k - S$ such that $f_k(z) \to \infty$ as $z \to S$, $z \in A_k$. By [18, Satz 3.3, p. 275], there is a neighborhood $W_k$ of $A_k - S$ in $M^*$ such that $f_k$ has a $C^\infty$ plurisubharmonic extension, also denoted by $f_k$, to $W_k$. Let $g$ be a $C^\infty$ function defined in a neighborhood $W$ of the connected component $A'$ of $A$ containing $A_k$ such that $f_k(z) = 0$ for $z \in A'$, $g > 0$ off $A'$, $g$ is plurisubharmonic on $W$, and $g$ is strictly plurisubharmonic on $W - A'$. Let $N$ be sufficiently large so that $f_k(z) < N - 1$ for $z \in A_k - T$. Then for $r$ sufficiently large real number, $V_{\alpha k} = \{z \in W \cap W_k | f_k(z) + + rg(z) < N\}$ is a Stein neighborhood of $A_k - T$. Moreover, for large $r$ the various $V_{\alpha k}$ will be disjoint. Let $V_0 = \cup V_{\alpha k}$, all $k$. Then $\overline{V_i} \cap \overline{V_j} \cap \overline{V_k} = \emptyset$ for $i \neq j \neq k$.

Let $M_i$ be a strictly pseudoconvex neighborhood of $A$ contained in $\cup V_i$, $0 < i < m$. Replace $\{V_i\}$ by $\{V_i \cap M_i\}$, which we will also denote by $\{V_i\} = \mathcal{B}$. Since $\mathcal{B}$ is a Leray cover of $M_i$, $H^1(M_i, \Theta) \approx H^1(N(\mathcal{B}), \Theta)$. Let $\theta_1, \ldots, \theta_n$ be vector fields on $\{V_i \cap V_j\}$ which represent a basis of $H^1(M_i, \Theta)$. If $M_i \supset M$, also a strictly pseudoconvex neighborhood of $A$, then the restriction map $H^1(M_i, \Theta) \to H^1(M, \Theta)$ is an isomorphism [13, Lemma 3.1, p. 599]. So $\{\theta_i\}$ will also be a basis for $H^1(M, \Theta)$ for $M$ smaller than $M_i$ and for refinements $\mathcal{U}$ of $\mathcal{B}$.

Using just the specialness of the cover $\mathcal{B}$, we shall construct $\mathcal{M}$ via coordinate patches. These patches will be modified in the course of the construction. Let $\mathcal{B}' = \{V_i\}$, $0 < i < m$ with $V_i \subset V_j$ be a refinement of $\mathcal{B}$. Given
any compact set $K$ in $M_1$, we may choose $B'$ to be a cover of $K$. Now let $\overline{\cdot}$ denote closure in $M_1$. Let $K = \overline{M}$. Take an initial $Q$ to be a poly-disc of dimension $n = \dim H^1(M_1, \Theta)$. Start with patches $V'_i \times Q$, $0 < i < m$.

We must give the $g_{ij}$, the transition functions for $\mathcal{M}$. For each small $t = (t_1, \ldots, t_n)$ in $Q$, integration along $t_1\theta_1 + \ldots + t_n\theta_n$ for time 1 gives a map $h_{ij}(t): \overline{V'_i \cap V'_j} \to V_i \cap V_j$. Restrict $Q$ to these small values of $t$ and define an initial $g_{ij}: (V'_i \cap V'_j) \times Q \to (V_i \cap V_j) \times Q$ by $g_{ij} = (h_{ij}(t), t)$. There will be no compatibility conditions to verify for these changes of coordinates since no three coordinate patches intersect. However, for these changes of coordinates to define a manifold and in particular to insure that the space is Hausdorff, we still must modify the domains and ranges of the $g_{ij}$.

Let $B$ be the set of non-interior points of $V'_i - V'_j$. Then the points of $V'_i \times Q$ which might not be separated from points in $V'_j \times Q$ (which are not identified by $g_{ij}$) lie in $B \times Q$. $\overline{B}$ is disjoint from the compact set $\mathcal{C} = (V'_i - V'_j) \cap V_i \cap K$. Let $D$ be a neighborhood of $\mathcal{C}$ such that $\overline{D}$ is compact and $\overline{D} \cap \overline{B} = \emptyset$. Then for small $Q$, $h_{ij}(B \times Q) \cap \overline{D} = \emptyset$. So far, $g_{ij}$ maps $(V'_i \cap V'_j) \times Q \subset V'_i \times Q$ biholomorphically to an open subset $R_{ij}$ of $V_i \times Q$. $R_{ij}$ lies near to $(V'_i \cap V'_j) \times Q$, as a subset of $V'_i \times Q$. In the cover for $\mathcal{M}$, replace $V_i \times Q$ by the subset $[(V'_i - V'_j) \cup D] \times Q \cup R_{ij} = T_i$. This modifies $V'_i \times Q$ only near $V_i$ and makes Hausdorff the space $(V'_i \times Q) \cup T_i$ with points identified under $g_{ij}$.

Since $V_i \cap V_j \cap V_k = \emptyset$ for $i \neq j \neq k$, the construction of the above paragraph leaves $V_i \cap V_k$ and $V_j \cap V_k$ unchanged. Thus to complete the construction of coordinate patches for $\mathcal{M}$, we look at an unordered pair $(i, j)$, $i \neq j$. We favor one element of the unordered pair, say $i$, and form $T_i$ as in the previous paragraph. This changes the range of $g_{ij}$ and the domain of $g_{ij} = [g_{ij}]^{-1}$ to $R_{ij}$. We then consider a different unordered pair and repeat the construction of the previous paragraph. After considering all unordered pairs, we have a Hausdorff space $\mathcal{M}'$ and a projection map $\omega': \mathcal{M}' \to Q$ which shows that $\mathcal{M}'$ is a family of deformations of $M' = (\omega')^{-1}(0)$. $K \subset M'$. $\mathcal{M}$, the interior of $K$, is the desired strictly pseudoconvex manifold.

Let $U_i = M \cap V_i$. $U = \{U_i\}$ is then a special cover. Let $\mathcal{M}$ be a neighborhood of $M$ in $\mathcal{M}'$ such that $\mathcal{M} \cap (\omega')^{-1}(0) = M$. Then, after possibly shrinking $Q$, $\omega = \omega'|\mathcal{M}$ is the desired deformation. By construction, $\omega_6: \mathcal{T}_6 \to H^1(M, \Theta)$ is an isomorphism. Recall [19, Satz 1, p. 547]:

Let $\pi: Z \to S$ be a holomorphic mapping of complex spaces with strictly pseudoconvex special fiber $X = \pi^{-1}(s_0)$, $s_0 \in S$ fixed. Then for every compact set $K \subset X$, there exist open sets $U \subset Z$ and $V \subset S$, with $K \subset U$, $s_0 \in V$, $\pi(U) \subset V$, such that $\pi|U: U \to V$ is a 1-convex map.
We shall use this result several times in this paper. In particular, $\omega$ can be chosen to be 1-convex. This completes the proof of the Theorem.

**Theorem 3.** Let $\omega: \mathcal{M} \to Q$ be a deformation of a strictly pseudoconvex manifold $M_0 = \omega^{-1}(0)$ which has a special cover $\mathcal{U}$. Let $\Theta$ be the tangent sheaf on $M_0 = \omega^{-1}(q)$. Suppose that $\omega$ is 1-convex, $Q$ is a manifold and $\varphi_0: qT \to H^1(M_0, \Theta_0)$ is surjective. Then $\varphi_q: qT \to H^1(M_q, \Theta_q)$ is surjective for all small $q$.

**Proof.** Let $\Theta$ be the sheaf of germs of vector fields on $\mathcal{M}$ which lie in the direction of the fibers. Let $\omega^*(\Theta)$ be the first direct image sheaf of $\Theta$ under the map $\omega$. Then $\omega^*(\Theta)$ is a coherent analytic sheaf on $Q$ [21, Main Theorem (i), p. 213].

Using $\omega$, we may shrink $\mathcal{M}$ along the fibers and not change any map $\varphi_q$ for small $q$. Then, as in the proof of Theorem 2, we may use [18] to extend the special cover $\mathcal{U}$ on $M$ to a special cover on the shrunken $\mathcal{M}$. Without loss of generality, we may thus assume that $\mathcal{M}$ has a special cover. Then $\omega^*_q(F) = 0$ for $r > 1$ and $F$ any coherent sheaf on $\mathcal{M}$. In particular, $\omega^*_q(F)$ is $\Theta$-flat. $\Theta$ is locally free and so is $\omega$-flat. Let $m_q$ be the ideal sheaf of $q \in Q$. Then [22, Proposition 2.2, p. 208] $H^1(M_q, \Theta_q) \approx \omega^*_q(\Theta)/m_q \omega^*_q(\Theta)$. Let $\mathcal{C}$ be the tangent sheaf on $Q$. Then the Kodaira-Spencer map $[11, e: 13 \to \omega^*_q(\Theta)$ is a map of coherent analytic sheaves. Since $qT \approx \mathcal{C}/m_q \mathcal{C}$, the given hypothesis that $\varphi_0$ is surjective says that $\varphi_0: \mathcal{C}/m_q \mathcal{C} \to \omega^*_q(\Theta)/m_q \omega^*_q(\Theta)$ is surjective. By Nakayama's Lemma, $\varphi$ is surjective at 0. Then $\varphi$ is surjective near 0 by coherence. Then $\varphi_q$ is surjective for $q$ near 0.

To deal with non-reduced parameter spaces, we need the following easy strengthening of [2].

**Theorem 4.** Let $M$ be a Stein manifold and $\omega: \mathcal{M} \to S$ a deformation of $M = M_0 = \omega^{-1}(0)$ with $S$ a possibly non-reduced analytic space. Then given any compact set $K \subset M$, there is a neighborhood $\mathcal{M}_1$ or $K$ in $\mathcal{M}$ such that $\omega|_{\mathcal{M}_1}$ is a trivial deformation.

**Proof.** $\omega$ is given to be locally trivial. As in [9, p. 266-269], we may use a $C^\infty$ strictly plurisubharmonic exhaustion function on $M$ to write $M = \bigcup M^{(i)} 1 \leq i < \infty$, with $M^{(i)} \subset M^{(i+1)}$, $M^{(i)}$ a strictly pseudoconvex Stein manifold, and $M^{(i+1)} = M^{(i)} \cup N^{(i)}$ with $N^{(i)}$ a Stein manifold near which $\omega$ is a trivial deformation. We may assume that $\omega$ is a trivial deformation near $M^{(i)}$.

We can now prove the theorem by induction on $i$. The case $i = 1$ is given. $M^{(i+1)} = M^{(i)} \cup N^{(i)}$. After shrinking a little, we may assume by induction that $\omega$ is trivial near $M^{(i)}$ and $N^{(i)}$. Then near $M^{(i+1)}$, $\omega$ may be
defined by giving just one transition map \( g_{12} : U_1 \cap U_2 \rightarrow U_1 \cap U_2 \) with \( U_1 \approx M^{(0)} \times S \) and \( U_2 \approx N^{(0)} \times S \). Shrinking \( M^{(\epsilon + 1)} \) a little more, we shall extend \( \omega \) to a (non-singular) ambient neighborhood \( \Delta \) of \( 0 \in S \). The theorem will then follow from the original formulation in \([2]\).

To extend \( \omega \), let \( M' \subseteq M^{(0)} \) and \( N' \subseteq N^{(0)} \) with \( M', N', N' \) Stein. Then for \( T \) a sufficiently small neighborhood of \( 0 \) in \( S \), \( g_{12} \) restricts to give a map \( (h_{12}(s), s) : (M' \cap N') \times T \rightarrow (M' \cap N') \times T \). Here, in the domain of \( h_{12} \), we are using the product structure on \( U_2 \). In the range of \( h_{12} \), we are using the product structure on \( U_1 \). \( h_{12}(0) \) is the inclusion map. So that \( h_{12}(s) \) may be given by a set of functions, embed the Stein manifold \( M' \cap N' \) in \( C^n \) for some \( n \). By \([9\text{, Theorem VIII, C. 8, p. 257]}\), there is a neighborhood \( V \) of \( M' \cap N' \) in \( C^n \) and a holomorphic retraction map \( q : V \rightarrow M' \cap N' \). Let the initial ambient neighborhood \( \Delta' \) of \( 0 \) in \( S \) be Stein with \( \Delta' \cap T \) a subvariety of \( \Delta' \). Then the functions defining \( h_{12}(s) \) extend to functions on \( (M' \cap N') \times \Delta' \). By restricting to a smaller neighborhood \( \Delta'' \), we may assume that the image of the extended \( h_{12}(s) \) lies in \( V \). Composing with \( q \) gives \( (f_{12}(s), s) : (M' \cap N') \times \Delta'' \rightarrow (M' \cap N') \times \Delta'' \). Since \( f_{12}(0) = h_{12}(0) \) is the identity map onto its image, \( f_{12}(s) \) is a biholomorphic map onto its image for all sufficiently small \( s \in \Delta'' \). Proceeding as in the proof of Theorem 2, we may shrink \( M^{(\epsilon + 1)} \) a little more and form the desired deformation which extends \( \omega \). This completes the proof of Theorem 4.

**Theorem 5.** Let \( M \) be a strictly pseudoconvex manifold with a special cover \( U \). Let \( \Theta_0 \) be the tangent sheaf to \( M \). Let \( \omega : \mathcal{K} \rightarrow Q \) be a deformation of \( M = M_s = \omega^{-1}(0) \) such that \( Q \) is a manifold and \( \partial \omega : \mathcal{K}_0 \rightarrow H^1(M_s, \Theta_0) \) is surjective. Let \( \lambda : \mathcal{R} \rightarrow S \) be any deformation of \( M = M_s = \lambda^{-1}(0) \) with \( S \) a possibly non-reduced analytic space. Then, given any compact set \( K \) in \( M \), there are neighborhoods \( \mathcal{K}_1 \) and \( \mathcal{R}_1 \) of \( 0 \) in \( \mathcal{K} \) and \( \mathcal{R} \) respectively, neighborhoods \( Q_1 \) and \( S_1 \) of \( 0 \) in \( Q \) and \( S \) respectively, and a holomorphic map \( f : S_1 \rightarrow Q_1 \) such that \( \omega|_{\mathcal{K}_1} = \omega_1 : \mathcal{K}_1 \rightarrow Q_1 \) and \( \lambda|_{\mathcal{R}_1} = \lambda_1 : \mathcal{R}_1 \rightarrow S_1 \) are deformations with \( \lambda_1 \) induced by \( f \). If \( \omega_0 \) is also injective, then the tangent map of \( f \) at the origin is uniquely determined.

**Proof.** Shrinking \( M \) and \( U \) a little, we may assume by Theorem 4 that \( \lambda \) is trivial near \( 0 \) on each \( U_i \). As in the proof of Theorem 4, we may shrink \( M \) further and extend \( \lambda \) to a non-singular ambient neighborhood \( \Delta \) of \( 0 \) in \( S \).

So, without loss of generality, we shall now assume that \( S \) is non-singular. Let the transition maps for \( \lambda \) be given by \( g_{i}(s), s \in S \). Let the transition maps for \( \omega \) be given by \( h_{i}(q), q \in Q \). Let \( U'_i \subseteq U'_i \subseteq U_i \) be two refinements of \( U \). Choose \( Q_1 \) and \( S_1 \), small so that \( h_{i}(q) \circ g_{i}(s) = k_{i}(q, s) : U'_i \cap U'_j \rightarrow U'_i \cap U'_j \) is well defined for \( (q, s) \in Q_1 \times S_1 \). Then, as in the proof of The-
Theorem 2, the \( k_{ij} \) may be used to construct a deformation \( \tau : \mathcal{Y} \to B \) of a slightly shrunk \( M \). \( B \) is a Cartesian product \( Q_1 \times S_1 \) of neighborhoods \( Q_1 \) and \( S_1 \) of 0 in \( Q \) and \( S \) respectively. Above \( 0 \times S_1 \), \( \tau \) coincides with \( \lambda \). Above \( Q_1 \times 0 \), \( \tau \) coincides with \( \omega \). Let \( \mathcal{E} \) be the tangent sheaf of \( B \). Let \( \mathcal{E} \) be the subsheaf of \( \mathcal{E} \) of germs of vector fields on \( B \) in the \( Q_i \) directions. Choose \cite{[19, Satz 1, p. 547]} \( \tau \) to be a 1-convex map. Then, by the proof of Theorem 2, \( \varrho : \mathcal{E} \to \tau^*_s(\Theta) \) is surjective near \( 0 \times 0 = 0 \). Let \( v_1, \ldots, v_n \) be vector fields on \( B \) such that \( v_i(0), \ldots, v_n(0) \) project onto a basis of \( \tau^*_sT_0 \). Since \( \varrho \) is surjective near 0, we may modify \( v_1, \ldots, v_n \) by sections of \( \varrho \mathcal{E} \) and assume that \( \varrho(v_i) = 0 \) in \( \tau^*_s(\Theta) \) for all \( i \) and small \( B \). Then, for sufficiently small \( B \), \( \varrho(v_i) = 0 \) in \( H^1(\tau^*(B), \Theta) \). Then, by the nature of \( \varrho \), for each \( i \) there exists a vector field \( \theta_i \) on \( \tau^{-1}(B) \) such that at each point \( b \) of \( \tau^{-1}(B) \), \( \varrho_s \) maps \( \theta_i(b) \) to \( v_i(\tau(b)) \). Let \( (t_1, \ldots, t_n) \) be near \( (0, \ldots, 0) \). Then, integrating along \( t_1 \theta_1 + \ldots + t_n \theta_n \) and \( t_1 v_1 + \ldots + t_n v_n \) for time 1 and for small \( (t_1, \ldots, t_n) \) gives a Cartesian product structure \( \mathcal{Y} \approx \mathcal{M} \times S_1 \) with a projection map \( \omega \times \text{id} : \mathcal{M} \times S_1 \to Q_1 \times S_1 \) which shows that \( \mathcal{Y} \) is a deformation of a slightly smaller \( M \). There is also an automorphism of \( B = Q_1 \times S_1 \) near \( 0 \times 0 \) which shows that \( \tau \) and \( \omega \times \text{id} \) are equivalent deformations. \( \lambda : \mathcal{M} \to S_1 \) is a subspace of \( \tau : \mathcal{Y} \to B \). Projecting \( \mathcal{Y} \) onto \( \mathcal{M} \) via the Cartesian product structure gives the desired map \( f : S_1 \to Q_1 \).

This concludes the proof of Theorem 5 except for the last sentence. But the tangent map of \( f \) at the origin just agrees with the infinitesimal Kodaira-Spencer map in this case.

Let \( M \) be as in Theorem 5. Then \( H^1(M, 0) = 0 \). \cite{[19, Satz 5, p. 562]} says that under such circumstances we can form its simultaneous-blow-down subspace \( T \) of \( Q \), as in Definition 9 below. The versality result of Theorem 5 implies versality for deformations of germs of \( M \) near \( A \). Blow down \( M \) to \( V \). Let \( p \) be the singular point of \( V \). Then \cite{[19, Satz 7, p. 562]} says that the simultaneous blow-down over \( T \) is versal for deformations which can be simultaneously resolved.

The following corollary about the rigidity of exceptional curves of the first kind is known. For example, use \cite{[10, Theorem 3, p. 85]}, which says that \( A \) lists above \( S \), and \cite{[19, Satz 2, p. 547]}, which says that one can simultaneously blow down near the lifting. We shall use it to strengthen our results in the two-dimensional case.

**Corollary 6.** Let \( M \) be a two-dimensional manifold. Let \( A \) be a submanifold of \( M \) which is a compact Riemann surface of genus 0 with \( A \cdot A = -1 \). Let \( \lambda : \mathcal{M} \to S \) be a deformation of \( M = \lambda^{-1}(0) \). Then in a neighborhood of \( A \) in \( \mathcal{M} \), \( \lambda \) is the trivial deformation.
Proof. It suffices to see that for any small strictly pseudoconvex neighborhood \(N\) of \(A\) in \(M\), \(H^1(N, \Theta) = 0\).

Since \(A\) is in fact an exceptional curve of the first kind, \(H^1(N, \Theta)\) can be directly computed via a Leray cover to give 0. Or, one may use [8, Satz 1, p. 355] and [14, (3.9), p. 85].

Proposition 7. Let \(M\) be a strictly pseudoconvex two-dimensional manifold. Let \(A\) be the exceptional set. Then there are a finite number of points \(p_i \in \mathcal{M} - A\) such that the manifold \(M'\) obtained from \(M\) by quadratic transformations at the \(p_i\) can be written \(M' = U_1 \cup U_2\) with \(U_1\) and \(U_2\) open Stein subsets of \(M'\).

Proof. Let \(\mathcal{M}'\) be a strictly pseudoconvex manifold with \(\mathcal{M}' \subseteq M\) and also with the same exceptional set \(A\). Let \(\mathfrak{g}\) be the ideal sheaf of \(A\). By [12, Lemma 4.10, p. 61], we can find a divisor \(D\) on \(A\) with \(A_i \cdot D\) arbitrarily negative for all irreducible components \(A_i\) of \(A\). Let \(\mathfrak{g}\) be the ideal sheaf corresponding to \(D\). Then, by [12, Lemma 6.19, p. 117] (and its proof in case \(A\) lacks normal crossings), for the \(A_i \cdot D\) sufficiently negative, \(H^1(\mathcal{M}'', \mathfrak{g}) = H^1(\mathcal{M}'', \mathfrak{g}) = 0\). Then \(\Gamma(\mathcal{M}'', \mathfrak{g}) \to \Gamma(\mathcal{M}'', \mathfrak{g})\) and \(\Gamma(\mathcal{M}'', \mathfrak{g}) \to \Gamma(\mathcal{M}'', \mathfrak{g})\) are surjective. Then we can find \(f_1, f_2 \in \Gamma(\mathcal{M}'', \mathfrak{g})\) such that \((f_1) - D\) and \((f_2) - D\) contain no \(A_i\), and also if \(p \in \text{supp}((f_1) - D) \cap \text{supp}((f_2) - D)\) then \(p \not\in A\) and \(p\) is a point of normal crossing for \((f_1) - D\) and \((f_2) - D\). There are only a finite number of such \(p_i\). Let \(M'\) be obtained from \(M\) by quadratic transformations at the \(p_i\). Let \(D_1\) and \(D_2\) be the proper transforms on \(M'\) of \((f_1) - D\) and \((f_2) - D\) respectively. Let \(U_i = \mathcal{M}' - \text{supp} D_i, i = 1, 2\). Then \(U_1\) and \(U_2\) are the desired Stein subsets of \(M'\). One may construct the needed holomorphic functions on the \(U_i\) by considering \(f_2/f_1\), with \(f_2 \in \Gamma(\mathcal{M}'', \mathfrak{g})\) or \(f_2 \in \Gamma(\mathcal{M}'', \mathfrak{g})\). Then \(U_i\) is holomorphically convex and the \(f_2/f_1\) will give local coordinates. This concludes the proof of Proposition 7.

Theorem 8. Let \(M\) be a strictly pseudoconvex two-dimensional manifold. Then there exists a deformation \(\omega: \mathcal{M} \to Q\) of \(M = \omega^{-1}(0)\) such that \(\omega\) is 1-convex, \(Q\) is a manifold and the Kodaira-Spencer map \(q_\omega: qT_q \to H^1(M, \Theta_q)\) is an isomorphism. Let \(M_q = \omega^{-1}(q)\). \(q_\omega: qT_q \to H^1(M_q, \Theta_q)\) is surjective for all small \(q \in Q\). Let \(\lambda: \mathcal{R} \to S\) be any deformation of \(M = M_0 = \lambda^{-1}(0)\) with \(S\) a possibly non-reduced analytic space. Then, given any compact set \(K\) in \(M\), there are neighborhoods \(\mathcal{M}_1\) and \(\mathcal{R}_1\) of \(K\) in \(\mathcal{M}\) and \(\mathcal{R}\) respectively, neighborhoods \(Q_1\) and \(S_1\) of 0 in \(Q\) and \(S\) respectively, and a holomorphic map \(f: S_1 \to Q_1\) such that \(\omega|\mathcal{M}_1 = \omega_1: \mathcal{M}_1 \to Q_1\) and \(\lambda|\mathcal{R}_1 = \lambda_1: \mathcal{R}_1 \to S_1\) are deformations with \(\lambda_1\) induced from \(\omega_1\) by \(f\). The tangent map of \(f\) at 0 is uniquely determined.
PROOF. For any coherent sheaf $\mathcal{F}$ on $M$, $H^1(M, \mathcal{F})$ is determined by small neighborhoods of the exceptional set. If $N$ is a small holomorphically convex neighborhood of an exceptional curve of the first kind, then $H^1(N, \Theta) = 0$. Hence quadratic transformations off the exceptional set have no effect on $H^1(M, \Theta)$.

To construct $\omega$, let $M^*$ be a strictly pseudoconvex manifold with $M \subset M^*$. Let $M^*$ be obtained from $M^*$ by a finite number of quadratic transformations and have a special cover (Proposition 7). Let $\pi : M^* \to M^*$. By the proof of Theorem 2, there is a deformation $\omega : \mathcal{M}' \to Q$ of $M' = \pi^{-1}(M)$ with $\varphi_q : q^{-1}T \to H^1(M', \Theta)$ an isomorphism and $\omega'$ a 1-convex map. By Corollary 6, the exceptional curves of the first kind in $M'$ which are the result of quadratic transformations in $M$ have neighborhoods on which $\omega'$ is a trivial deformation. Simultaneously blow down the exceptional curves of the first kind in these neighborhoods. This gives a deformation $\omega : \mathcal{M} \to Q$ of $M$. $\omega$ is 1-convex. $\varphi_q$ is an isomorphism by the observation of the previous paragraph. $\varphi_q'$ is surjective for small $q$ by Theorem 3. With $A_{\omega}'$, the exceptional set in $M_{\omega}'$, is the subvariety of $\mathcal{M}'$ where the Remmert reduction is not an isomorphism. $H^1(M_{\omega}', \Theta)$ is the exceptional set in $M_{\omega}'$ which are the result of quadratic transformations in $M$ have neighborhoods on which $\omega'$ is a trivial deformation. Simultaneously blow down the exceptional curves of the first kind in these neighborhoods. This gives a deformation $\omega : \mathcal{M} \to Q$ of $M$. $\omega$ is 1-convex. $\varphi_q$ is an isomorphism by the observation of the previous paragraph. $\varphi_q'$ is surjective for small $q$ by Theorem 3. $\pi(q) = \pi^{-1}(q)$ near 0 such that, letting $\mathcal{A} = \omega^{-1}(T)$, the family $\omega_a = \omega|A : A \to T$ simultaneously blows down to a flat deformation $\pi_a : \mathcal{X} \to T$ of the blow down $V = X_a = \pi_a^{-1}(0)$ of $M$. $T = \{q \in Q | \dim H^1(M_{\omega}, \Theta) = \dim H^1(M_{\omega}, \Theta)\}$.

DEFINITION 9. Let $\omega : \mathcal{M} \to Q$ be a 1-convex deformation of $M = M_\omega = \omega^{-1}(0)$. Let the reduced space $T$ be given by $T = \{q \in Q | \dim H^1(M_{\omega}, \Theta) = \dim H^1(M_{\omega}, \Theta)\}$. Then $T$ is the simultaneous blow-down subspace of $Q$. 
THEOREM 10. Let $M$ be a strictly pseudoconvex two-dimensional manifold with exceptional set $A$. Let $w: M \to Q$ be as in Theorem 8. Suppose that $M$ is the minimal resolution of the normal two-dimensional analytic space $V$. Let $T$ be the simultaneous-blow-down subspace of $Q$. Then the blow-down $\pi_a: X \to T$ of $w$ over $T$ is the unique deformation of $V$ which is versal for deformations with reduced parameter spaces that can be simultaneously resolved, i.e. given any deformation $\pi: Y \to S$ of $V = X_0 = \pi^{-1}(0)$ with $S$ reduced such that $\pi$ may be simultaneously resolved and any compact set $K \subset V$, then there exist neighborhoods $X_1$ and $Y_1$ of $K$ in $X$ and $Y$ respectively, neighborhoods $T_1$ and $S_1$ of 0 in $T$ and $S$ respectively, and a holomorphic map $f: S_1 \to T_1$ such that $\pi_a|X_1: X_1 \to T_1$ and $\pi = \pi|Y_1: Y_1 \to S_1$ are deformations with $\pi_a$ induced by $f$. The induced map $f^*$ on the Zariski tangent space of $S$ at 0 to the Zariski tangent space of $T$ at 0 is unique.

For all points $t \in T$ sufficiently near to 0, $\pi_a$ is versal near $t$ except for the uniqueness of the map $f^*$.

If $X', open in X$, has $\pi_a = \pi|X': X' \to T$ a deformation with $V' = (\pi_a')^{-1}(0)$ being a strictly pseudoconvex neighborhood of the singular points of $V$, then $\pi_a'$ is the unique deformation of $V'$ which is versal for deformations with reduced parameter spaces which can be simultaneously resolved.

PROOF. Let $\lambda: R \to S$ be a simultaneous resolution of $\pi_a$. Then $R = \lambda^{-1}(0)$ is a resolution of $V = \pi^{-1}(0)$. Suppose that $A_i \subset R$ is an exceptional curve of the first kind. Then by Corollary 6, we can simultaneously blow down $A_i$ and nearby exceptional curves of the first kind and still have a deformation of the blown down $R$. Thus, without loss of generality, we may assume that $R$ is the minimal resolution of $V$. Since minimal resolutions are unique [20], [12, pp. 87-88], $R \cong M$. Let $\tau_a: M \to V$ be the resolving map.

Apply Theorem 8, using the compact set $\tau_a^{-1}(K)$. We need that $f(S_1) \subset T$. But since $\lambda$ may be simultaneously blown down, for $s \in S$, $\dim H^1(R_s, 0) = \dim H^1(M, 0)$. Hence $f(s) \in T$. The first paragraph of the Theorem now follows by letting $X_a$ and $Y_a$ be the blow downs of $M \cap w^{-1}(T_1)$ and $R$ respectively. (The uniqueness of $\pi_a$ is proved in the usual way from the uniqueness of $f^*$.)

The second paragraph of the Theorem follows from Theorem 8 and the above argument, which proved the first paragraph.

Let $M' = \tau_a^{-1}(V')$. Let $K'$ be a compact set in $M'$ with $A \subset K'$. By [19, Satz 1, p. 547], there is a neighborhood $\mathcal{M}'$ of $K'$ in $\mathcal{M}$ and a neighborhood $Q'$ of 0 in $Q$ such that $\omega' = \omega|\mathcal{M}' : \mathcal{M}' \to Q'$ is a 1-convex map. Since in $\mathcal{M}$ the union of the exceptional sets of $M_a$ is the subvariety of $\mathcal{M}$ where the Remmert reduction is not an isomorphism [19, p. 553], $M_a = \omega^{-1}(q)$ and $M_a' = (\omega')^{-1}(q)$ have the same exceptional set for all small $q$. Then
[13, Lemma 3.1, p. 599] the restriction map \( H^q(M_q, \mathcal{O}) \to H^q(M'_q, \mathcal{O}) \) is an isomorphism for all small \( q \). Thus \( \omega \) and \( \omega' \) have the same simultaneous-blow-down subspace \( T \) of \( Q \) for small \( q \). This concludes the proof of Theorem 10.

REFERENCES

VERSAL DEFORMATIONS FOR TWO-DIMENSIONAL PSEUDOCONVEX MANIFOLDS


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