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Extreme positive functionals

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Extreme Positive Functionals.

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1. – Introduction.

In 1948 Gelfand and Naimark using the techniques of representation theory for C^* algebras characterized the positive homomorphisms of a commutative Banach *-algebra A with identity as the extreme points of an appropriate compact convex subset of the topological dual A' [2, ch. VIII]. Bucy and Maltese [1] in 1966 gave a nice proof of this result using only the inherent properties of the algebra A. That is, they were able to avoid the techniques of representation theory which tend to obscure the geometric picture, especially in the commutative situation.

In this note we characterize the multiplicative functionals for an appropriate class of commutative Banach *-algebras and obtain the Gelfand-Naimark result as a consequence. Since our results are based on geometric properties of certain compact convex subsets of the cone of positive functionals, as in [1], we avoid the use of representation theory.

2. – Notation.

Let A denote a complex Banach algebra with isometric involution. We call such an algebra a *Banach *-algebra*. Let A' denote the topological dual of A. Then

$$K = \{f \in A' \colon f(xx^*) \geqslant 0 \ \text{ for all } \ x \in A\}$$

is the usual *cone of positive functionals* defined on A. Further, we let

$$\begin{aligned} P &= \{f \in K \colon \|f\| \leqslant 1\} \quad \text{and} \\ M &= \{f \in P \colon f \ \text{ is multiplicative}\}\,. \end{aligned}$$

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The nonempty convex set P is weak* compact, hence *ext* P, the set of extreme points of P, is nonempty. Finally, we let *extr* K denote the union of the extreme rays of K.

In Theorem 3 we give a sufficient condition on K to insure that K have an extreme ray.

3. – The main result.

In the work below we always assume that the algebra A is a Banach *-algebra with dimension greater than one. We first recall that a cone is *proper* if and only if it contains no line (i.e., if $\pm f \in K$ then $f = 0$.) As the following lemma indicates, there is a relationship, perhaps well known, between $A^2 = \{xy | x, y \in A\}$ and the cone K.

LEMMA 1. *The set A^2 is total if and only if K is proper.*

PROOF. K is not proper

$\Leftrightarrow$ there exists $f \neq 0$ such that $\pm f \in K$

$\Leftrightarrow$ there exists $f \neq 0$ such that $f(xx^*) = 0$ for all $x \in A$

$\Leftrightarrow$ there exists $f \neq 0$ such that $f(A^2) = \{0\}$

$\Leftrightarrow$ A^2 is not total in A.

In the above proof we have used only the Hahn-Banach Theorem and the known fact that every product can be expressed as a linear combination of elements of the form xx^*, i.e., $4xy = \sum_{k=0}^{3} (x^* + i^{-k} y)^*(x^* + i^{-k} y)$. Thus, A^2 total is equivalent to elements of the form xx^* total, and the two ideas are used interchangeably.

To obtain the desired representation of the extremal rays the restriction that K be proper is necessary. For instance, consider C^2 (2-dimensional Hilbert space) endowed with the multiplication $(x, y)(a, b) = (xa, 0)$ and involution $(x, y)^* = (\bar{x}, \bar{y})$ for elements (x, y), $(a, b) \in C^2$. Then the line $L = \{(0, \lambda i) \colon \lambda \text{ real}\}$ lies in K and is not generated by a homomorphism. Conversely, the homomorphism $h(a, b) = a$ does not lie on an extreme ray of the cone K.

Although, in general, positive functionals need not be *real*, i.e., $f(x^*) = \overline{(fx)}$, we recall that positive homomorphisms are real. Note that in the preceding example no element of the line L is real. Further, if the algebra A has an identity, it is well known that positive functionals are real. The following lemma analogizes this result.

LEMMA 2. *If A^2 is total then every element of K is real.*

PROOF. Let $f \in K$, $a \in A$ and since A^2 is total, let $\{p_n\}$ be a sequence in A with $\lim_n p_n = a$ where each p_n is a linear combination of elements of the form xx^*. Then $f(p_n^*) = \overline{f(p_n)}$ and by the continuity of f it follows that f is real.

THEOREM 3. *Suppose A^2 is total. If $f \neq 0$ and $f \in M$, then f lies on an extreme ray of K.*

PROOF. If $f \in M$ and $f \neq 0$ assume $0 \leqslant g \leqslant f$ with $g \neq 0$. To show $f \in \operatorname{extr} K$ it is sufficient to find $\beta > 0$ so that $\beta f = g$. The inequality

$$|g(xy^*)|^2 \leqslant g(xx^*)g(yy^*) \leqslant |f(x)|^2 f(yy^*) \tag{1}$$

for all x, $y \in A$ shows that if $f(x) = 0$, then $g(xy^*) = 0$. Thus, for each y the linear functional $\varphi(x) = g(xy^*)$ is a multiple of f and we write

$$g(xy^*) = \alpha(y^*)f(x) \qquad \text{for all } x\,. \tag{2}$$

Since A^2 is total and $g \neq 0$ there exists some $x \in A$ with $g(xx^*) > 0$ and consequently by (1), $f(x) \neq 0$. Thus define a continuous linear functional of y by $\alpha(y) = g(xy)(f(x))^{-1}$, which is independent of x from (2). If $f(y) = 0$ then from (1) it follows that $\alpha(y) = 0$ so $\alpha = \beta f$ for some complex constant β.

Now, by our choice of x,

$$0 < g(xx^*) = f(x)\alpha(x^*) = \beta f(x) f(x^*) = \beta |f(x)|^2$$

so $\beta > 0$. Thus,

$$g(xy) = \beta f(xy)\,, \qquad \text{for } x,\ y \in A,\ f(x) \neq 0\,. \tag{3}$$

Finally, when $f(x) = 0$ from (1) we conclude that (3) holds and the proof is complete.

COROLLARY 4. *If $\|f\| = 1$ and $f \in M$ then $f \in \operatorname{ext} P$.*

An application of the Cauchy-Schwarz inequality for positive functionals yields the following lemma which will be used in the proof of Theorem 6.

LEMMA 5. *Suppose A^2 is total and $f \in K$. Then, if $f(xyx^*) = 0$ for all x, $y \in A$ it follows that $f = 0$.*

PROOF. From the hypotheses on f it follows that $|f(abcd)|^2 \leqslant f(aa^*) \cdot f(b(cd)(cd)^* b^*) = 0$ for all $a, b, c, d \in A$ and, if $A^4 = \{abcd: a, b, c, d \in A\}$ then $f(A^4) = 0$. Since every product $xy \in A^2$ is the limit of a sequence of elements from the span of A^4, the continuity of f, the totality of A^2 and the above inequality imply that $f = 0$.

The following "square root lemma" for Banach *-algebras will be used in Theorem 6. Let $z \in A$ with $\|z\| < 1$, then in A_1, the algebra with identity adjoined, the element $(e - zz^*)^{\frac{1}{2}} = a$ exists and $a = a^*$. Since A is a maximal ideal in A_1, for any $x \in A$ it follows that $(xa)(xa)^* = xaa^*x^* = x(e - zz^*)x^* = xx^* - (xz)(xz)^* \in A$.

For the first time in our work we now require that the algebra A be commutative. The necessity of this hypothesis in Theorem 5 is indicated by the following example.

Let Ω denote the algebra of 2×2 matrices (with complex entries) with the usual matrix multiplication. The norm of a matrix χ in Ω is given by the sum of the absolute values of its entries and χ^* is the usual conjugate transpose. The topological dual Ω' is again the set of 2×2 matrices, the norm of an element of Ω' is the maximum of the absolute values of its entries. The positive cone $\mathcal{K} \subset \Omega'$ is the set

$$\left\{\begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} : \alpha \geqslant 0, \quad \gamma \geqslant 0, \quad \beta \text{ real}, \quad \beta^2 \leqslant \alpha\gamma \right\}.$$

The following elements, none of which are multiples of multiplicative functionals, generate extreme rays of K:

$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right\}.$$

THEOREM 6. *Let A be a commutative Banach *-algebra with A^2 total. If f lies on an extreme ray of K then there exists a constant $\beta > 0$ such that βf is multiplicative.*

PROOF. If $f = 0$ the result follows, hence we assume $f \neq 0$ and $f \in \operatorname{extr} K$. Choose $z \in A$ with $\|z\| < 1$ and $z \neq 0$ and define $g \in A'$ by $g(x) = f(xzz^*)$ for $x \in A$. Then $g \in K$ and $(f - g) \in K$ since from preceding remarks, for each $x \in A$ there exists $y \in A$ such that $xx^* - xx^*zz^* = yy^*$. Since $f \in \operatorname{extr} K$ and K is proper, there exists a constant $\alpha(z) \geqslant 0$ so that $\alpha(z) f = g$. Therefore, from a routine computation, it follows that for every $z \in A$, $z \neq 0$, there exists $\alpha(z) \geqslant 0$ such that

$$f(xzz^*) = \alpha(z) f(x), \quad x \in A, \tag{4}$$

Consequently for $y, z \in A$ $(z \neq 0)$ it follows that

$$\alpha(z) f(yy^*) = f(yy^* zz^*) = \alpha(y) f(zz^*) . \tag{5}$$

From Lemma 5 and the fact that $f \neq 0$ there exist $x, z \in A$ with $f(xzz^*) \neq 0$ and hence, from (4) we have $\alpha(z) > 0$ and $f(x) \neq 0$. Further, since

$$|\alpha(z) f(x)|^2 = |f(xzz^*)| \leqslant f\big((xz)(xz)^*\big) f(zz^*) , \tag{6}$$

we conclude that $f(zz^*) > 0$. Finally, from (5) and the fact that $f(zz^*) > 0$ and $\alpha(z) > 0$ it follows that $f(yy^*) = 0$ if and only if $\alpha(y) = 0$ for all $y \in A$.

Let $\alpha = \big(\alpha(z)\big)^{-1} f(zz^*)$ which, from (5) is independent of z for $\alpha(z) > 0$. Then,

$$f(x) f(yy) = \alpha f(xyy^*) , \qquad x,\ y \in A,\ f(yy^*) \neq 0 . \tag{7}$$

With α fixed as above, it follows that (7) is also valid when $f(yy^*) = 0$ since then, (6) implies that $f(xyy^*) = 0$ for every $x \in A$. Thus (7) holds for all $x, y \in A$.

With elements of the form yy^* total in A we conclude that $f(x) f(y) = \alpha f(xy)$ for $x,\ y \in A$ and consequently, that $\alpha^{-1} f$ is multiplicative.

For a commutative algebra, we note that whenever A contains an approximate identity bounded by one, the extreme points of P are exactly the homomorphisms. Further, when A has an identity of norm one we obtain the well known result that the extreme points of $P_1 = \{f \geqslant 0 : f(e) = 1\}$ are exactly the multiplicative elements of the set [1]. In fact $\operatorname{ext} P_1 = (\operatorname{extr} K) \cap P_1$ since P_1 is the intersection of K and the hyperplane $\{f \in A' : f(e) = 1\}$ (see, [3, p. 337].)

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