G. D’Este

The $\otimes_c$-topology on abelian $p$-groups


<http://www.numdam.org/item?id=ASNSP_1980_4_7_2_241_0>
The $\oplus_c$-Topology on Abelian $p$-Groups (*).

G. D'ESTE

Introduction.

In this paper we investigate the topology of an abelian $p$-group $G$ which admits as a base of neighborhoods of 0 all the subgroups $X$ of $G$ such that $G/X$ is a direct sum of cyclic groups. We call this topology the $\oplus_c$-topology of $G$. If $G$ with the $\oplus_c$-topology is a complete Hausdorff topological group, then $G$ is said to be $\oplus_c$-complete. The Hausdorff completion of $G$ with respect to the $\oplus_c$-topology is called the $\oplus_c$-completion of $G$ and is denoted by $\hat{G}$.

In section 1 we prove that the $\oplus_c$-completion $\hat{G}$ of a $p$-group $G$ is a $\oplus_c$-complete group; moreover the completion topology of $\hat{G}$ and its own $\oplus_c$-topology are the same. The group $\hat{G}$ coincides with the completion of $G$ with respect to the inductive topology if and only if $G$ is thick.

In section 2 we study the class of $\oplus_c$-complete groups. This class of separable $p$-groups is very large, containing the groups which are direct sums of torsion-complete $p$-groups, as well as the groups which are the torsion part of direct products of direct sums of cyclic $p$-groups. But the most interesting result in this direction perhaps is that every separable $p^\tau$-projective $p$-group is $\oplus_c$-complete. There are a lot of these groups: in fact Nunke proved in [12] that, for every ordinal $\sigma$, there exists a $p^\tau$-projective $p$-group which fails to be $p^\tau$-projective for every $\tau < \sigma$. Moreover the class of $\oplus_c$-complete groups has many closure properties typical of both the classes of $p^\omega$-projective and $p^\omega$-injective $p$-groups.

In section 3 we study the $\oplus_c$-completion with respect to basic subgroups and we prove the inadequacy of the socle in determining the $\oplus_c$-complete groups; finally we give some applications in connection with the class of thick groups.

I would like to express my gratitude to Dr. L. Salce for his many helpful suggestions.

1. The \( \oplus \)-completion.

All groups considered in the following are abelian groups. Notations and terminology are those of [4]. In particular \( p \) is a prime number and the symbol \( \oplus \) denotes a direct sum of cyclic \( p \)-groups. If \( G \) is any group and \( G' \) is a pure subgroup of \( G \), then we write \( G' \leq G \). A \( p \)-group \( G \) may be equipped with various topologies. The \( p \)-adic topology has the subgroups \( p^n G \) with \( n \in \mathbb{N} \) as a base of neighborhoods of 0; the inductive topology has the family of large subgroups as a base of neighborhoods of 0. Throughout the paper, for every \( p \)-group \( G \), the group \( \hat{G} \) stands for the completion of \( G \) with respect to the inductive topology. If \( \lambda \) is a limit ordinal, then the generalization of the \( p \)-adic topology is the \( \lambda \)-adic topology. This topology, studied by Mines in [11], has the subgroups \( p^n G \) with \( \sigma < \lambda \) as a base of neighborhoods of 0. In [13] Salce has studied the \( \lambda \)-inductive topology introduced by Charles in [3]; a base of neighborhoods of 0 for this topology consists of all subgroups \( G(u) \) where \( G(u) = \{ x \in G : h(p^n x) > \sigma_n, \ n \in \mathbb{N} \} \) and \( u = (\sigma_n)_{n \in \mathbb{N}} \) is an increasing sequence of ordinals \( \sigma_n < \lambda \) for all \( n \in \mathbb{N} \). In the following, unless otherwise indicated, every \( p \)-group \( G \) is endowed with the \( \oplus \)-topology. If we are dealing with some other topology, then the group \( G \) equipped with its \( \oplus \)-topology is denoted by \( (G, \oplus_e) \).

Let \( G \) be a \( p \)-group and let \( L \) be a large subgroup of \( G \). Since \( G/L = \oplus_e \) ([4] Proposition 67.4), \( L \) is open with respect to the \( \oplus_e \)-topology of \( G \) and so the \( \oplus_e \)-topology is finer than the inductive topology. The next statement immediately follows from this result and the fact that a \( p \)-group \( G \) is thick if and only if \( G/X = \oplus_e \) implies \( L < X \) for some large subgroup \( L \) of \( G \).

**Proposition 1.1.** Let \( G \) be a \( p \)-group. Then \( G \) is thick if and only if the \( \oplus_e \)-topology coincides with the inductive topology and a thick group \( G \) is \( \oplus_e \)-complete if and only if it is torsion-complete.

Since quasi-complete groups are thick ([4] Theorem 74.1, Corollary 74.6; [1] Theorem 3.2), the quasi-complete and non torsion-complete group constructed by Hill and Megibben ([7] Theorem 7) is an example of a group which is not \( \oplus_e \)-complete. Let us note the following facts.

1) A \( p \)-group \( G \) is discrete in the \( \oplus_e \)-topology if and only if \( G = \oplus_e \) and \( G \) is Hausdorff if and only if \( p^n G = 0 \).
2) Every homomorphism \( f: G \to H \) with \( G \) and \( H \) p-groups is continuous with respect to the \( \oplus_c \)-topologies. In fact if \( H/X = \oplus_c \), the same holds for \( G/f^{-1}(X) \).

3) For every p-group \( G \) the \( \oplus_c \)-topology of \( G/p^mG \) coincides with the quotient topology of the \( \oplus_c \)-topology of \( G \). By property 2, it is enough to observe that the natural homomorphism \( G \to G/p^mG \) is open.

Therefore in the study of the \( \oplus_c \)-completion it is not restrictive to confine ourselves to separable non thick groups. In order to show that the \( \oplus_c \)-completion of a p-group is \( \oplus_c \)-complete, we need two lemmas.

**Lemma 1.2.** Let \( G \) be a p-group. Then the \( \oplus_c \)-completion \( \hat{G} \) of \( G \) is a p-group.

**Proof.** By definition \( \hat{G} = \lim G/X \) where \( X \) ranges over the subgroups \( X \) of \( G \) such that \( G/X = \oplus_c \). Let \( \hat{G} \) denote the p-adic completion of \( G \). Since \( \hat{G} = \lim G/p^nG \) where \( n \in \mathbb{N} \), there is a canonical homomorphism \( \varphi: \hat{G} \to \hat{G} \) such that \( \varphi((g_x + X)_x) = (g_{x^n} + p^nG)_n \) for all \( (g_x + X)_x \in \hat{G} \). Since the completion of \( G \) in the inductive topology is the group \( \hat{G} = \lim G/L \) with \( L \) running over the large subgroups of \( G \), there exists a natural homomorphism \( \psi: \hat{G} \to \hat{G} \) that takes \( (g_x + X)_x \) to \( (g_L + L)_L \) for all \( (g_x + X)_x \in \hat{G} \). To show that \( \hat{G} \) is a p-group, it suffices to check that \( \psi \) is an embedding, and this clearly holds if \( \varphi \) is injective. We shall now prove that if \( (g_x + X)_x \in \text{Ker} \varphi \), then \( g_x \in X \) for all \( X \). To see this, fix \( X \). Let \( m \in \mathbb{N} \); if \( Y = X \cap p^mG \), then \( G/Y = \oplus_c \). By hypothesis \( g_{p^mG} \in p^mG \) and, by the choice of \( Y \), \( g_x + p^mG = g_{p^mG} + p^mG \); consequently \( g_x \in p^mG \). On the other hand \( g_x + X = g_x + X \) and so the height of \( g_x + X \) in \( G/X \) is at least \( m \). Since \( m \) is any natural number and \( G/X = \oplus_c \), we conclude that \( g_x \in X \), as claimed. This completes the proof that \( \hat{G} \) is a p-group.

From now on we shall identify \( \hat{G} \) with the subgroup \( \psi(\hat{G}) \) of \( \hat{G} \) and, if \( G \) is separable, then we shall view \( G \) as a subgroup of \( \hat{G} \).

**Lemma 1.3.** Direct summands of \( \oplus_c \)-complete groups are \( \oplus_c \)-complete.

**Proof.** Let \( G' \) be a direct summand of a \( \oplus_c \)-complete group \( G \). Since the inclusion \( G' \to G \) is continuous, every Cauchy net in \( G' \) is a Cauchy net in \( G \). Therefore the hypothesis that \( G \) is \( \oplus_c \)-complete and the continuity of the projection of \( G \) onto \( G' \) assure that \( G' \) is \( \oplus_c \)-complete.

We are now ready to establish the main result of this section.

**Theorem 1.4.** Let \( G \) be a p-group. Then the \( \oplus_c \)-completion \( \gamma \) of \( G \) is \( \oplus_c \)-complete.
PROOF. Without loss of generality we may assume that \( G \) is separable. For every ordinal \( \lambda \) we define a group \( G_\lambda \) as follows: if \( \lambda = 0 \), then \( G_0 = G \); if \( \lambda > 0 \) and \( \lambda \) is not a limit ordinal, then \( G_\lambda \) is the \( \oplus_c \)-completion of \( G_{\lambda-1} \); if \( \lambda \) is a limit ordinal, then \( G_\lambda = \bigcup_{\sigma < \lambda} G_\sigma \). To prove the theorem, we shall use three facts:

(i) The \( \oplus_c \)-topology of \( \bar{G} \) is finer than the completion topology.

Let \( \mathcal{B} \) be the family of all subgroups \( X \) of \( G \) such that \( G/X = \oplus_c \). Then \( \bar{G} = \lim_{\mathcal{B}} G/X \) and \( \bar{G} \) with the completion topology is a topological subgroup of the group \( \prod_{X \in \mathcal{B}} G/X \) equipped with the product topology of the discrete topologies on every \( G/X \). Thus a base of neighborhoods of \( 0 \) for the completion topology of \( \bar{G} \) consists of all subgroups \( U_F = \bar{G} \cap \prod_{X \in \mathcal{B} \setminus F} G/X \) where \( F \) is a finite subset of \( \mathcal{B} \). Since every \( U_F \) is a neighborhood of \( 0 \) for the \( \oplus_c \)-topology of \( \bar{G} \), and so (i) is proved.

(ii) \( G_\lambda \) is a subgroup of \( \bar{G} \) for all \( \lambda \).

We shall prove by transfinite induction that \( G_\lambda \leq \bar{G} \) for all \( \lambda \). If \( \lambda = 0 \) the assertion is obvious. Let \( \lambda > 0 \) and assume \( G_\sigma \leq \bar{G} \) for every \( \sigma < \lambda \). If \( \lambda \) is a limit ordinal, then evidently \( G_\lambda \leq \bar{G} \). If \( \lambda \) is not a limit ordinal and \( \lambda = \sigma + 1 \), then the hypothesis that \( G < G_\sigma \leq \bar{G} \) implies that \( G_\sigma < G_\lambda < \bar{G} = \bar{G} \). Since \( G_\sigma \leq G_\lambda \), we get \( \bar{G}_\sigma = \bar{G} < \bar{G}_\lambda \) and therefore \( G_\lambda \leq \bar{G} \), as required.

(iii) \( G_\lambda \) is a direct summand of \( G_\lambda \) for all \( \lambda > 1 \).

Assume by transfinite induction that \( G_1 \) is a summand of \( G_\sigma \) for all \( 1 < \sigma < \lambda \). Write \( G_\sigma = G_1 \oplus G'_\sigma \) for all \( 1 < \sigma < \lambda \). If \( \lambda \) is a limit ordinal, \( G_1 \) is a direct summand of \( G_\lambda \), because \( G_\lambda = \bigcup_{\sigma < \lambda} G_\sigma = G_1 \oplus \bigcup_{1 < \sigma < \lambda} G'_\sigma \). If \( \lambda \) is not a limit ordinal and \( \lambda = \sigma + 1 \) then, by the induction hypothesis, \( G_\sigma = G_1 \oplus G'_\sigma \). Let \( \pi: (G_\sigma, \oplus_c) \to (G_1, \mathcal{C}) \) be the canonical projection where \( (G_1, \mathcal{C}) \) is the \( \oplus_c \)-completion of \( G \). To check that \( \pi \) is continuous, let \( U \) be an open subgroup of \( (G_1, \mathcal{C}) \). Then, by property (i), there is some \( W < U \) such that \( G_\sigma/W = \oplus_c \). Since \( G_\sigma/W = G_1 \oplus G'_\sigma/W \cong G_1/W = \oplus_c \), we see that \( \pi \) is continuous. This result guarantees the existence of a homomorphism \( \bar{\pi} \) making the following diagram commute:

\[ \begin{array}{ccc}
(G_\sigma, \oplus_c) & \xrightarrow{\pi} & (G_1, \mathcal{C}) \\
\downarrow & & \downarrow \\
(G_\lambda, \mathcal{C}) & \xrightarrow{\bar{\pi}} & (G_1, \mathcal{C})
\end{array} \]
where the vertical maps are the natural ones and \((G, \mathcal{C})\) is the \(\Theta_c\)-completion of \((G, \oplus_c)\). Consequently \(G_1 = G_1 \oplus \text{Ker}\varphi\) and so \(G_1\) is a direct summand of \(G_2\), as claimed.

We can now show that \(\mathcal{G} = \mathcal{G}_1\) is \(\oplus_c\)-complete. Suppose this were not true. Then, from Lemma 1.3 and property (iii), we deduce that \(G_2\) is not \(\oplus_c\)-complete for any \(\lambda\), and therefore the groups \(G_2\) are all distinct. But this is clearly impossible, because, by property (ii), they are all subgroups of \(\mathcal{G}\). This contradiction establishes that \(\mathcal{G}\) is \(\oplus_c\)-complete and the theorem is proved. \(\square\)

The next proposition describes the topological structure of the \(\Theta_c\)-completions.

**Proposition 1.5.** For every \(p\)-group \(G\) the \(\Theta_c\)-topology \(\mathcal{G}\) coincides with the completion topology.

**Proof.** It is not restrictive to assume \(p^nG = 0\). As before \(\mathcal{C}\) denotes the completion topology of \(\mathcal{G}\). By property (i) of Theorem 1.4 we know that the \(\Theta_c\)-topology of \(\mathcal{G}\) is finer than \(\mathcal{C}\). On the other hand, by a well known result of general topology ([2] Chapter III §3, No. 4 Proposition 7), a base of neighborhoods of 0 for the completion topology \(\mathcal{C}\) is formed by the closures in \(\mathcal{G}\) with respect to \(\mathcal{C}\) of the neighborhoods of 0 for the \(\Theta_c\)-topology of \(\mathcal{G}\). Therefore, to end the proof, it is enough to show that if \(U\) is an open subgroup of \((\mathcal{G}, \oplus_c)\) and \(U' = U \cap \mathcal{G}\), then the closure \(V\) of \(U'\) in \((\mathcal{G}, \mathcal{C})\) is a subgroup of \(U\). To prove this, let \(\{g_i\}\) be a Cauchy net in \((\mathcal{G}, \oplus_c)\) with \(g_i \in U'\) for all \(i\). Since the natural embedding \(G \to \mathcal{G}\) is continuous with respect to the \(\Theta_c\)-topologies, \(\{g_i\}\) is a Cauchy net in \((\mathcal{G}, \oplus_c)\). Thus, by Theorem 1.4, it converges to some \(x\) in \((\mathcal{G}, \oplus_c)\) and clearly \(x \in U\), because \(U\) is closed in \((\mathcal{G}, \oplus_c)\) and \(g_i \in U\) for all \(i\). Since \(\mathcal{C}\) is smaller than the \(\Theta_c\)-topology of \(\mathcal{G}\), the given net converges to \(x\) in \((\mathcal{G}, \mathcal{C})\); so \(x \in V\), by the definition of \(V\). This means that \(V < U\) and therefore the \(\Theta_c\)-topology of \(\mathcal{G}\) coincides with the completion topology, as claimed. \(\square\)

**Corollary 1.6.** Let \(G\) be a separable \(p\)-group. Then \(G\) is a pure topological subgroup with divisible cokernel of a \(\Theta_c\)-complete group.

**Proof.** By Theorem 1.4 and Proposition 1.5, \(G\) is a pure dense topological subgroup of the \(\Theta_c\)-complete group \(\mathcal{G}\). Consequently \(G\) is a dense subgroup of \(\mathcal{G}\) equipped with the \(p\)-adic topology. Hence \(\mathcal{G}/G\) is divisible and the proof is complete. \(\square\)

Before comparing the \(\Theta_c\)-completion and the completion with respect to the inductive topology, we prove the following lemma.
**Lemma 1.7.** Let $G$ be a separable $p$-group and let $G < X < ar{G}$. Then $G < \bar{X}$ and $\bar{X} < \bar{G}$.

**Proof.** Since $G < X$, we may assume $\bar{G} < \bar{X}$. To show that $G < \bar{X}$, select $g \in G$. Then, by Proposition 1.5, there exists a net $\{g_i\}$ with $g_i \in G$ for all $i$ which converges to $\bar{g}$ in $(\bar{G}, \oplus)$. Since $\{g_i\}$ is also a Cauchy net in $(X, \oplus)$ and all the canonical maps $\bar{G} \to G$, $\bar{G} \to \bar{X}$, $\bar{X} \to \bar{G}$ are continuous with respect to the $\oplus$-topologies, $\bar{g}$ is the limit of $\{g_i\}$ in $(\bar{X}, \oplus)$ and so $\bar{g} \in \bar{X}$. This proves the inclusion $G < \bar{X}$. To see that $\bar{X} < \bar{G}$, take $x \in \bar{X}$. As before, there is a net $\{x_i\}$ with $x_i \in X$ for all $i$ which converges to $x$ in $(\bar{X}, \oplus)$. Since $\{x_i\}$ is a Cauchy net in $(\bar{G}, \oplus)$ and all the natural embeddings $\bar{X} \to \bar{X}$, $\bar{G} \to \bar{X}$ are continuous with respect to the $\oplus$-topologies, $\bar{x}$ is the limit of $\{x_i\}$ in $(\bar{G}, \oplus)$ and so $\bar{x} \in \bar{G}$. Consequently $\bar{X} < \bar{G}$ and the lemma is proved.

**Proposition 1.8.** Let $G$ be a separable $p$-group. The following facts hold:

(i) If $G$ is not thick, then the group $G/G$ has uncountable rank.

(ii) If $G$ is not $\oplus$-complete, then the group $G/G$ may have finite rank.

**Proof (i).** We first show that $\bar{G} \neq \bar{G}$. Since $\bar{G}$ is thick, it has the same inductive and $\oplus$-topologies. Moreover, by ([13] Theorem 2.3), the inductive topology of $\bar{G}$ induces on $\bar{G}$ its own inductive topology. On the other hand, by Proposition 1.5, the $\oplus$-topology of $\bar{G}$ induces on $G$ its own $\oplus$-topology. Therefore, if $G$ is not thick, then $\bar{G}$ must be a proper subgroup of $\bar{G}$. We now prove that $G/\bar{G}$ is uncountable. Suppose this were not true. Since $\bar{G}$ is a pure subgroup of $\bar{G}$ with countable divisible cokernel, we deduce from ([10] Theorem 3.5) that $\bar{G}$ is thick, and this is impossible. In fact $\bar{G}$ is $\oplus$-complete, but it is not torsion-complete. This contradiction shows that $\bar{G}/\bar{G}$ is uncountable.

(ii) Assume the rank of $G/\bar{G}$ is not finite. Choose a pure subgroup $H$ of $\bar{G}$ such that $G < H$ and $\bar{G}/H \cong \mathbb{Z}(p^\infty)$. Then Lemma 1.7 tells us that $\bar{H} = \bar{G}$. Since the rank of $\bar{H}/H$ is 1, the proof is complete.

2. $\oplus$-complete groups.

In this paragraph we study the $\oplus$-complete groups. As the results of section 1 suggest, the class of $\oplus$-complete groups is very large.

First we prove a statement that we shall often use.

**Proposition 2.1.** Direct sums of $\oplus$-complete groups are $\oplus$-complete.
Proof. Let $G = \bigoplus G_i$ where $G_i$ is $\oplus_e$-complete for all $i$. To show that $G$ is $\oplus_e$-complete, we notice the following properties:

(i) The groups $X = \bigoplus X_i$ where $X_i \lhd G_i$ and $G_i/X_i = \oplus_e$ for every $i$ are a base of neighborhoods of 0 for the $\oplus_e$-topology of $G$.

This assertion is obvious.

(ii) $G$ is a closed topological subgroup of the group $\prod G_i$ equipped with the box topology of the $\oplus_e$-topology on each component.

We recall that the box topology considered on $\prod G_i$ admits the subgroups of the form $\prod X_i$ with $X_i \lhd G_i$ and $G_i/X_i = \oplus_e$ for all $i$ as a base of neighborhoods of 0. Thus the conclusion that $G$ is a topological subgroup of $\prod G_i$ follows from (i). To complete the proof, let $\tilde{g} = (g_i)_{i \in I}$ with $g_i \in G_i$ for every $i$ be an element of the closure of $G$ in $\prod G_i$. Let $S$ be the support of $\tilde{g}$, that is let $S = \{i \in I : g_i \neq 0\}$. Then for each $i \in S$ we can choose a subgroup $X_i$ of $G_i$ such that $g_i \notin X_i$ and $G_i/X_i = \oplus_e$. Our assumption on $g$ assures that consequently $S$ is finite and so $\tilde{g} \in G$. This proves that $G$ is a closed subgroup of $\prod G_i$, as required.

The hypothesis that every $G_i$ is $\oplus_e$-complete implies that $\prod G_i$ with the box topology is complete ([4] Proposition 13.3). Hence, by property (ii), $G$ is $\oplus_e$-complete.

**Corollary 2.2.** Direct sums of torsion-complete $p$-groups are $\oplus_e$-complete.

**Proof.** Since torsion-complete $p$-groups are $\oplus_e$-complete, the corollary follows from Proposition 2.1.

We shall obtain another large class of $\oplus_e$-complete groups by means of the next lemmas.

**Lemma 2.3.** Let $G$ be a separable $p$-group and let $G'$ be a subgroup of $G$ with bounded cokernel. Then $G$ is $\oplus_e$-complete if and only if $G'$ is $\oplus_e$-complete.

**Proof.** We first show that $G'$ is a topological subgroup of $G$. Let $X$ be a subgroup of $G'$ such that $G'/X = \oplus_e$. Since $(G/X)/(G'/X) \cong G/G'$ is bounded and $G'/X = \oplus_e$, we have $G/X = \oplus_e$. This proves that the restriction to $G'$ of the $\oplus_e$-topology of $G$ is finer than the $\oplus_e$-topology of $G'$. Therefore the two topologies coincide, because the natural injection $G' \to G$
is continuous. Assume now that $G$ is $\oplus_{\varepsilon}$-complete. Since $G'$ is a closed topological subgroup of $G$, we conclude that $G'$ is $\oplus_{\varepsilon}$-complete. Conversely, suppose $G'$ is $\oplus_{\varepsilon}$-complete. Since $G'$ is an open complete topological subgroup of $G$, evidently $G$ is $\oplus_{\varepsilon}$-complete and the proof is finished. 

**Lemma 2.4.** Let $G$ be a separable $p$-group and let $P$ be a bounded subgroup of $G$ with separable cokernel. Then $G$ is $\oplus_{\varepsilon}$-complete if and only if $G/P$ is $\oplus_{\varepsilon}$-complete.

**Proof.** Assume first $G/P$ is $\oplus_{\varepsilon}$-complete and choose $n \in \mathbb{N}$ such that $p^n P = 0$. Let us verify that $\mathcal{G} \triangleleft G + \mathcal{G} [p^n]$. Take $g \in \mathcal{G}$; then there is a net $(g_i)$ in $\mathcal{G}$ which converges to $g$ in $\mathcal{G}$. The hypothesis that $G/P$ is $\oplus_{\varepsilon}$-complete guarantees that $(g_i + P)$ has a limit $g + P \in G/P$. Since the canonical homomorphism $G/P \to p^n G$ and $p^n G \to G$ are continuous, $(p^n g_i)$ converges to $p^n g$ in $G$ and obviously $p^n g = p^n g$. Thus $g \in G + \mathcal{G} [p^n]$ and therefore $\mathcal{G} \triangleleft G + \mathcal{G} [p^n]$. By Theorem 1.4 and Lemma 2.3, this implies that $G$ is $\oplus_{\varepsilon}$-complete. Conversely, suppose $G$ is $\oplus_{\varepsilon}$-complete; then Lemma 2.3 says that $p^n G$ is $\oplus_{\varepsilon}$-complete. Since $(G/P)/(G[p^n]/P) \cong p^n G$, the first part of the proof assures that $G/P$ is $\oplus_{\varepsilon}$-complete and the lemma follows. 

Observe that the class of $\oplus_{\varepsilon}$-complete groups is a full $p^\omega$-class in the sense of [6]. Indeed, by Proposition 2.1 and Lemma 2.3, the class of $\oplus_{\varepsilon}$-complete groups is a $p^\omega$-class. Moreover, if $G$ is separable and $G/P$ is $\oplus_{\varepsilon}$-complete for some $P \triangleleft G[p]$, then, by Lemma 2.4, $G$ is $\oplus_{\varepsilon}$-complete.

We can now prove the following

**Theorem 2.5.** Let $\sigma$ be any ordinal. If $G$ is a $p^\sigma$-projective separable $p$-group, then $G$ is $\oplus_{\varepsilon}$-complete.

**Proof.** The proof is by induction on $\sigma$. If $\sigma < \omega$ the assertion is obvious, because $G = \oplus_{\varepsilon}$. Let $\sigma > \omega$ and assume the assertion is true for all $\lambda < \sigma$. By ([4] § 82 Ex. 13), $G$ is a summand of the group Tor $(H_\sigma, G)$, where $H_\sigma$ is the generalized Prüfer group of length $\sigma$. To see that $G$ is $\oplus_{\varepsilon}$-complete, we first suppose $\sigma$ is a limit ordinal. Then, by ([4] § 82 Ex. 2 and 8; Lemma 64.1) and by the induction hypothesis, $G$ is a summand of a direct sum of $\oplus_{\varepsilon}$-complete groups. Hence the conclusion that $G$ is $\oplus_{\varepsilon}$-complete follows from Lemma 1.3 and Proposition 2.1. Assume now $\sigma$ is not a limit ordinal. From the exact sequence

$$0 \to p^{\sigma-1} H_\sigma \cong \mathbb{Z}(p) \to H_\sigma \to H_\sigma / p^{\sigma-1} H \cong H_{\sigma -1} \to 0,$$
one obtains the long exact sequence

\[ 0 \rightarrow \text{Tor} (\mathbb{Z}(p), G) \cong G[p] \rightarrow \text{Tor} (H_\sigma, G) \xrightarrow{\varphi} \text{Tor} (H_{\sigma^{-1}}, G) \xrightarrow{\psi} \rightarrow \mathbb{Z}(p) \otimes G \cong G/pG \rightarrow H_\sigma \otimes G \rightarrow H_{\sigma^{-1}} \otimes G \rightarrow 0. \]

Thus the following sequences are exact:

1. \[ 0 \rightarrow G[p] \rightarrow \text{Tor} (H_\sigma, G) \rightarrow \text{Im} \varphi \rightarrow 0, \]
2. \[ 0 \rightarrow \text{Im} \varphi \rightarrow \text{Tor} (H_{\sigma^{-1}}, G) \rightarrow \text{Im} \psi \rightarrow 0. \]

Evidently in (2) the group Tor (H_{\sigma^{-1}}, G) is $\bigoplus$-complete, by the induction hypothesis, and Im $\psi$ is bounded; therefore, by Lemma 2.3, Im $\varphi$ is $\bigoplus$-complete. From Lemma 2.4 and the exactness of (1), we deduce that Tor (H_\sigma, G) is $\bigoplus$-complete and, by Lemma 1.3, the same applies to its summand G.  

Proposition 2.1 indicates that the class of $\bigoplus$-complete groups has a closure property analogous to a closure property of the class of direct sums of cyclic groups. This projective property can be regarded as dual of the following injective property, which is similar to a closure property of the class of torsion-complete groups ([4] Corollary 68.6).

**Proposition 2.6.** The torsion part of a direct product of $\bigoplus$-complete groups is $\bigoplus$-complete.

Proof. Let \( G = t\left( \prod_{i \in I} G_i \right) \) where \( G_i \) is $\bigoplus$-complete for all \( i \). Since \( G_i \leq \bar{G}_i \) for every \( i \), it is easy to check that \( G \) is a pure subgroup of the torsion-complete group \( T = t\left( \prod_{i \in I} \bar{G}_i \right) \). Therefore, by the first part of Lemma 1.7, we may assume \( \bar{G} < T \). Let now \( t = (t_i)_{i \in I} \in \bar{G} \) with \( t_i \in \bar{G}_i \) for all \( i \). Then \( t \) is the limit of a net \( \{g_i\}_{i \in J} \) where \( g_i = (g_{ij})_{j \in J} \in G \) and \( g_{ij} \in G_i \) for all \( i \in I, j \in J \). Fix \( i \in I \); to end the proof, it is enough to show that \( t_i \in G_i \). Since \( \bar{G} < T \) and the canonical projection \( T \rightarrow \bar{G}_i \) is continuous, \( \{g_{ij}\}_{j \in J} \) converges to \( t_i \in \bar{G}_i \). From the hypothesis that \( G_i \) is $\bigoplus$-complete and \( g_{ij} \in G_i \) for all \( j \), we get \( t_i \in G_i \). This completes the proof.  

As the next corollary shows, Proposition 2.6 gives some information about $\bigoplus$-complete groups which is not contained in Corollary 2.2 and Theorem 2.5.

**Corollary 2.7.** There is a $\bigoplus$-complete group which is not a direct sum of torsion-complete $p$-groups and $p^\infty$-projective separable $p$-groups.
Proof. Let \( G = \bigoplus_{n \in \mathbb{N}} G_n \) where \( G_n = \bigoplus_{k>n} \mathbb{Z}(p^k) \) for all \( n \). By Proposition 2.6, \( G \) is \( \oplus_c \)-complete. We observe now the following facts:

(i) \( G \) is not a direct sum of torsion-complete \( p \)-groups.

This can be easily proved.

(ii) A proper \( p^n \)-projective separable \( p \)-group \( G' \) with \( \sigma > \omega \) cannot be a direct summand of \( G \).

Assume the contrary. Then \( G' = \bigoplus_{n \in \mathbb{N}} C_n \) where \( C_n = \bigoplus_c \mathbb{Z}(p^k) \) for all \( n \) \((\text{[9] Theorem 3})\). Since \( \sigma > \omega \), there is no \( k \in \mathbb{N} \) such that \( p^k C_n = 0 \) for almost all \( n \). Hence, by \((\text{[8] Proposition 1.6})\), \( G' \) has an unbounded torsion-complete group \( T \) as a summand, but this is impossible. Indeed, by \((\text{[12] Proposition 6.7})\), a \( p^n \)-projective \( p \)-group cannot contain an unbounded torsion-complete group. This contradiction shows that (ii) holds.

The corollary is now obvious. \( \square \)

Let us note that the group \( G \) defined in the proof of Corollary 2.7 is pure-complete \((\text{[9] Theorem 2})\). Another application of Proposition 2.6 enables us to characterize all the \( \oplus_c \)-complete groups.

**Theorem 2.8.** Let \( G \) be a \( p \)-group. The following statements are equivalent:

(i) \( G \) is \( \oplus_c \)-complete.

(ii) \( G \) is a closed topological subgroup of the torsion part of a direct product of a direct sums of cyclic \( p \)-groups.

Proof (i) \( \Rightarrow \) (ii). By hypothesis \( G = \lim_{\longrightarrow} G/X \) where \( \mathcal{B} \) is a base of neighborhoods of \( 0 \) for \( G \) and \( G/X = \bigoplus_c \) for all \( X \in \mathcal{B} \). Let \( \Pi = \prod_{X \in \mathcal{B}} G/X \) and let \( T = t(\Pi) \). If \( G \) and \( T \) are equipped with the \( \bigoplus_c \)-topology and \( \Pi \) is regarded as the topological product of the discrete groups \( G/X \), then all the natural inclusions in the commutative diagram

\[
\begin{array}{ccc}
G & \xrightarrow{i} & T \\
\downarrow{j} & & \downarrow{t} \\
\Pi & \rightarrow & \Pi
\end{array}
\]

are continuous. Evidently the groups of the form \( j^{-1}(U) \) where \( U \) ranges over the open subgroups of \( \Pi \) are a base of neighborhoods of \( 0 \) for \( G \). Thus the same holds for the groups \( i^{-1}(V) \) with \( V \) running over the open subgroups
of $T$. Hence $G$ is a topological subgroup of $T$. Since $G$ is $\oplus_c$-complete, $G$ must be closed in $T$ and (ii) is proved.

(ii) $\Rightarrow$ (i). This immediately follows from Proposition 2.6. $\square$

It is now clear that the class of $\oplus_c$-complete groups is the smallest class of separable $p$-groups $C$ with the following properties:

1. $0 \in C$ and a group isomorphic to a member of $C$ belongs to $C$.
2. If $S \leq G[p]$ and $G/S \in C$, then $G \in C$.
3. $C$ is closed under direct sums and the torsion part of a direct product of groups of $C$ belongs to $C$.
4. $C$ contains every group that, endowed with its $\oplus_c$-topology, is a closed topological subgroup of a group determined by the above conditions.

3. Some applications.

In this last section we discuss some consequences of the preceding results. The next proposition investigates the connection between $\oplus_c$-complete groups and basic subgroups.

**Proposition 3.1.** The following facts hold:

(i) If two separable $p$-groups have isomorphic $\oplus_c$-completions, then they have isomorphic basic subgroups.

(ii) There exist $2^{\aleph_0}$ pairwise nonisomorphic $\oplus_c$-complete groups with isomorphic basic subgroups.

**Proof (i).** Let $G$ and $H$ be separable $p$-groups such that $G \cong H$. Since $G$ is isomorphic to $H$, we conclude that $G$ and $H$ have isomorphic basic subgroups.

(ii) Let $B = \bigoplus_{n \geq 1} \mathbb{Z}(p^n)$. We want to prove that there exist $2^{\aleph_0}$ pairwise nonisomorphic $\oplus_c$-complete subgroups of $B$ whose basic subgroup is $B$. To see this, let $I$ be a set of cardinality $2^{\aleph_0}$ and let $\{X_i\}_{i \in I}$ be a family of subsets of positive integers such that if $i \neq j$ then $(X_i \setminus X_j) \cup (X_j \setminus X_i)$ is not finite. Let $G_i = \left( \prod_{n \in X_i} \mathbb{Z}(p^n) \right) \oplus \left( \bigoplus_{n \not\in X_i} \mathbb{Z}(p^n) \right)$ for all $i$; then every $G_i$ is a $\oplus_c$-complete group admitting $B$ as a basic subgroup. To complete the proof, it is enough to show that if $|X_i \setminus X_j| = \aleph_0$, then $G_i$ is not isomorphic
to $G_i$. Suppose this were not true. Then, by ([4] Theorem 73.6; Lemma 71.1), there exist an isometry $q : G_i[p] \to G_j[p]$, a finite subset $F \subseteq \mathbb{N} \setminus X_i$ and some $k \in \mathbb{N}$ such that $q\left(\prod_{n \in F} \mathbb{Z}(p^n)\right) \subseteq \left(\prod_{n \in X_j} \mathbb{Z}(p^n)\right) \oplus \left(\bigoplus_{n \in \mathbb{F}} \mathbb{Z}(p^n)\right)$. Consequently there is a finite subset $F' \subseteq X_i$ such that $X_i \setminus F' \subseteq F \cup X_j$, while, by hypothesis, $X_i \setminus X_j$ is not finite. This contradiction proves that $G_i$ is not isomorphic to $G_j$, as claimed.

The following statement shows that socles, viewed as valued vector spaces, do not give much information in the study of $\oplus_c$-complete groups.

**Proposition 3.2.** The following facts are true:

(i) There exists a $\oplus_c$-complete group whose socle is isometric to the socle of a non $\oplus_c$-complete group.

(ii) There exist nonisomorphic $\oplus_c$-complete groups with isometric socles.

**Proof** (i). Let $G$ be a separable $p$-group which is neither $\oplus_c$-complete nor thick (for instance, let $G$ be an infinite direct sum of quasi-complete non torsion-complete $p$-groups) and let $S = 6[p]$. Since $G \subseteq \mathbb{G}$, we can choose $x \in G[p] \setminus \mathbb{G}$. Let $y$ be an element of order $p^2$ of $G$ and let $z = x + y$. Take a subgroup $A$ of $G$ such that $A \leq G$ and $A[p] = S$. Since $A \leq G$ and $A[p] = S$, there exists a pure subgroup $H$ of $G$ such that $A \leq H$ and $H[p] = S$. We want to prove that $H$ is not $\oplus_c$-complete. Assume the contrary. Since $G < H < G$ and, by hypothesis, $H$ is $\oplus_c$-complete, Lemma 1.7 implies that $\mathbb{G}$ is a pure subgroup of $H$. Using this fact and the equality $H[p] = S = \mathbb{G}[p]$, one obtains $\mathbb{G} = H$. This is a contradiction, because $x \notin \mathbb{G}$, $y \in H$ and $z = x + y \in H$. Hence $H$ is not $\oplus_c$-complete and (i) is proved.

(ii) A result of ([5] Corollary 1) guarantees the existence of two nonisomorphic groups $G_1$ and $G_2$ with isometric socles such that $G_1$ is a direct sum of torsion-complete $p$-groups and $G_2$ is a $p^{\omega+1}$-projective $p$-group. On the other hand, by Corollary 2.2 and Theorem 2.5, $G_1$ and $G_2$ are $\oplus_c$-complete. Therefore (ii) holds and the proof is finished.

Now we point out some relations between $\oplus_c$-complete groups and thick groups. As Proposition 1.8 suggests and as we shall see in the following, these two classes have completely different properties.

**Proposition 3.3.** Let $G$ be a separable $p$-group which is neither $\oplus_c$-complete nor thick. The following facts hold:

(i) There exists a thick group $K$ such that $K + \mathbb{G} = G$; $K \cap \mathbb{G} = G$ and $K \subseteq \mathbb{G}$. 
(ii) Condition (i) does not necessarily determine the group $K$ up to isomorphisms.

**Proof (i).** By Corollary 1.6 we can choose a subgroup $K$ of $\bar{G}$ such that $\bar{G}/G \cong K/G$. Therefore $K + \bar{G} = \bar{G}; K \cap \bar{G} = G$ and $K \leq \bar{G}$. It remains to check that $K$ is thick. Since $G < K < \bar{G}$, Lemma 1.7 implies that $\bar{G} \leq K$ and clearly $\bar{G} = K$. Consequently $\bar{K} = K$ and so, by Proposition 1.8, $K$ is thick, as required.

(ii) Let $G = G_1 \oplus G_2$ where $G_1 = \bigoplus_{n \geq 1} \mathbb{Z}(p^n)$ and $G_2$ is a quasi-complete non-torsion-complete group whose basic subgroup is $G_1$ ([4] § 74 Example). Then $G$ is neither $\oplus$-complete nor thick, $\bar{G} = G_1 \oplus G_2$ and a suitable choice for $K$ is that of $\oplus_{J} = 0$, (D G). We shall prove that there exist $2^{|J|}$ pairwise nonisomorphic subgroups of $\bar{G}$ satisfying condition (i). To see this, let $J$ be a set of cardinality $2^{|J|}$. Take a subgroup $H$ of $\bar{G}$ with $G < H$ and elements $\alpha_n, \beta_n \in \bar{G}/G$, where $n \in \mathbb{N}$, with the following properties:

1. $\bar{G}/G = \langle \alpha_n : n \in \mathbb{N} \rangle \oplus H/G$ where $\langle \alpha_n : n \in \mathbb{N} \rangle \cong \mathbb{Z}(p^n)$, $p \alpha_n = 0$ and $p \alpha_{n+1} = \alpha_n$ for all $n \in \mathbb{N}$.
2. $K/G = \bigoplus_{i \in I} \langle \beta_{i+1} : n \in \mathbb{N} \rangle$ where $\langle \beta_{i+1} : n \in \mathbb{N} \rangle \cong \mathbb{Z}(p^n)$, $p \beta_{i+1} = 0$ and $p \beta_{i+1} = p \beta_{i+1}$ for all $n \in \mathbb{N}$; $i \in I$.

For every $J \subseteq I$, let $K_J$ be the subgroup of $\bar{G}$ such that $G < K_J$ and

$$K_J/G = \bigoplus_{i \in J} \langle \beta_{i+1} : n \in \mathbb{N} \rangle \oplus \bigoplus_{i \in I \setminus J} \langle \beta_{i+1} : n \in \mathbb{N} \rangle.$$

We claim that every $K_J$ satisfies condition (i). In fact let $J$ be any subset of $I$. Since $K_J/G + \bar{G}/G = \langle \alpha_n, \beta_{i+1} : n \in \mathbb{N}, i \in I \rangle + H/G = K/G \oplus \bar{G}/G = \bar{G}/G$, evidently $K_J + \bar{G} = \bar{G}$. The definition of $K_J$ guarantees that $K_J \leq \bar{G}$, because $G < K_J$ and $K_J/G$ is a divisible subgroup of $\bar{G}/G$. To verify that $K_J \cap \bar{G} = G$, select $z \in K_J \cap \bar{G}$. Then there exist $v \in H/G; n, n_i \in \mathbb{N}$ and $a, a_i \in \mathbb{Z}$ where $0 < a, a_i < p$ and $a_i = 0$ for almost all $i$ such that

$$z = \sum_{i \in J} a_i (\beta_{i+1} + \alpha_n) + \sum_{i \in I \setminus J} a_i \beta_{i+1} = ax_n + v.$$

Since $K_J/G \cap \bar{G}/G = 0$, we get $a_i = 0$ for all $i$; hence $z = 0$. Therefore $K_J/G \cap \bar{G}/G = 0$ and so $K_J \cap \bar{G} = G$. Consequently every $K_J$ satisfies condition (i) and it is easy to show that the groups $K_J$ are all distinct. To end the proof, we apply an argument similar to that used in the last part of ([4] Theorem 66.4). Let $B$ be a basic subgroup of $G$; then $B$ is countable.
Since $|\text{Hom}(K, \tilde{G})| < |\text{Hom}(B, \tilde{G})| = 2^{\aleph_1}$, the subgroups of $\tilde{G}$ isomorphic to $K$ are at most $2^{\aleph_1}$. This means that $2^{2^{\aleph_1}}$ groups of the form $K_j$ are pairwise nonisomorphic and so (ii) holds. □

From Propositions 1.8 and 3.3 we deduce that, if $G$ is neither $\oplus_\tau$-complete nor thick, then there exist a lot of thick groups between $G$ and $\tilde{G}$. The following result indicates that, under the same hypotheses on $G$, there exist also a lot of $\oplus_\tau$-complete groups between $G$ and $\tilde{G}$.

**Proposition 3.4.** Let $G$ be a separable group which is neither $\oplus_\tau$-complete nor thick and let $K$ be as in Proposition 3.3. The following are true:

(i) There exists an increasing sequence of $\oplus_\tau$-complete non thick groups $\{X_n\}$ with $\tilde{G} < X_n$ for all $n$ such that $\tilde{G} = \bigcup_{n \in \mathbb{N}} X_n$.

(ii) There exists an increasing sequence of non $\oplus_\tau$-complete non thick groups $\{Y_n\}$ with $G < Y_n$ for all $n$ such that $K = \bigcup_{n \in \mathbb{N}} Y_n$.

**Proof (i).** Let $X_0 = \tilde{G}$ and let $X_n = \tilde{G} + [p^n]$ for all $n > 1$. Then, by Lemma 2.3, every $X_n$ is $\oplus_\tau$-complete and the other properties clearly hold.

(ii) Under the hypotheses of (i), let $Y_n = K \cap X_n$ for all $n$. Then the sequence $\{Y_n\}$ is an increasing sequence, $G < Y_n$ for every $n$, and $K = \bigcup_{n \in \mathbb{N}} Y_n$. To prove (ii), fix $n \in \mathbb{N}$. Since $G < Y_n$ and $\tilde{G} < Y_n$, Lemma 1.7 assures that $Y_n$ is not $\oplus_\tau$-complete. Moreover, since $Y_n/G$ is bounded and, by hypothesis, $G$ is not thick, it is easily seen that $Y_n$ is not thick. This completes the proof. □

The next statement gives some properties of $\oplus_\tau$-complete groups and thick groups with respect to intersection and group union.

**Proposition 3.5.** Let $G$ be a separable $p$-group. The following facts hold:

(i) Let $X$ be a pure subgroup of $G$ and let $X = \bigcap_i X_i$ where $X_i < G$ for all $i$. If every $X_i$ is $\oplus_\tau$-complete, then $X$ is $\oplus_\tau$-complete; if every $X_i$ is thick, then $X$ is not necessarily thick.

(ii) Group unions of $\oplus_\tau$-complete or thick subgroups of $G$ are not necessarily $\oplus_\tau$-complete or thick.

**Proof (i).** Suppose $X_i$ is $\oplus_\tau$-complete for every $i$. Then, by the first part of Lemma 1.7, $\tilde{X} < X_i$ for all $i$; hence $X$ is $\oplus_\tau$-complete, as claimed. Let now $X$ be a $\oplus_\tau$-complete group which is not thick and let $G = \tilde{X}$. Evidently $X$ is the intersection of all $X_i < G$ such that $X < X_i$ and $G/X_i \cong \mathbb{Z}(p^\infty)$. Since all these groups are thick ([10] Theorem 3.5), (i) is proved.
(ii) First assume $G$ is not $\oplus_c$-complete. Then $G = \bigcup_{n \geq 1} G[p^n]$ and every $G[p^n]$ is $\oplus_c$-complete. Finally let $G = \bigoplus_{n \geq 1} \mathbb{Z}(p^n)$. Since $G[p^n]$ is thick for all $n \geq 1$ and $G$ is not thick, (ii) follows.

This last proposition shows that in a $\oplus_c$-complete group the cardinality of nondiscrete $\oplus_c$-complete subgroups and that of non $\oplus_c$-complete thick subgroups may be as large as possible.

**Proposition 3.6.** There exists a $\oplus_c$-complete group $G$ with the following properties:

(i) $G$ has $2^{|G|}$ $\oplus_c$-complete nondiscrete non thick subgroups.

(ii) $G$ has $2^{|G|}$ non $\oplus_c$-complete thick subgroups.

**Proof.** We claim that the $\oplus_c$-complete group where

$$
\forall \mathcal{U} \subseteq N \text{ free ultrafilter} \quad \mathfrak{U} = \{ f \in G : Z(f) \subseteq \mathcal{U} \}
$$

satisfies the above conditions.

(i) View $G$ as the group of all bounded maps $f : N \to \bigcup_{n \in \mathbb{N}} G_n$ such that $f(n) \in G_n$ for every $n \in \mathbb{N}$. If $f \in G$, let $Z(f) = \{ n \in \mathbb{N} : f(n) = 0 \}$. For every free ultrafilter $\mathcal{U}$ on $N$, let $G_{\mathcal{U}}$ be the $\oplus_c$-completion of the group

$$
\mathfrak{U} = \{ f \in G : Z(f) \subseteq \mathcal{U} \}
$$

Fix $\mathcal{U}$; since $\mathfrak{U} \subseteq G$, the first part of the proof of Lemma 1.7 guarantees that $G_{\mathcal{U}} \leq G$ and clearly $G_{\mathcal{U}} \neq \oplus_c$, because $\mathfrak{U} \neq \oplus_c$. Also note that every $G_n$ is a summand of $G_{\mathcal{U}}$; consequently $G_{\mathcal{U}}$ is not thick. The next step is to show that if $\mathcal{U} \neq \mathcal{V}$, then $G_{\mathcal{U}} \neq G_{\mathcal{V}}$. To this end, pick $F \in \mathcal{U} \setminus \mathcal{V}$. Let $g$ be an element of $G[p]$ with the property that $Z(g) = F$ and $g(n) \in G_n \setminus \mathbb{P}G_n$ for every $n \in \mathbb{N} \setminus F$. Obviously $g \in G_{\mathcal{U}}$, but we claim that $g \notin G_{\mathcal{V}}$. Suppose this were not true. Then, from the hypothesis that $g \in G_{\mathcal{V}}$, we deduce that $g$ belongs to the closure of $\mathfrak{V}$ with respect to the $\oplus_c$-topology of $G$. Hence $g = g_1 + pg_2$ for some $g_1 \in \mathfrak{V}$ and $g_2 \in G$. Since $Z(g) \notin \mathcal{V}$ and $Z(g_1) \notin \mathcal{V}$, there exists $n \in Z(g_1) \setminus Z(g)$ and so $0 \neq g(n) = pg_2(n)$, contrary to the choice of $g$. This contradiction proves that $g \notin G_{\mathcal{V}}$; thus the groups $G_{\mathcal{U}}$ are all distinct. Since $\mathbb{N}$ has $2^{2^{|\mathbb{N}|}}$ free ultrafilters and $|G| = 2^{|\mathbb{N}|}$, (i) holds.

(ii) This property immediately follows from the proof of (ii) in Proposition 3.3.

Indeed all the groups $K_j$ used in that proof can be embedded in the group $T = \bigoplus_{n \geq 1} (\mathbb{Z}(p^n) \oplus \mathbb{Z}(p^n))$ and it is easy to see that $G$ has a direct summand isomorphic to $T$. □
REFERENCES


Seminario Matematico
Università di Padova
via Belzoni 7
35100 Padova