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One attempt to the $K3$ modular function I

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One Attempt to the $K3$ Modular Function I.

HIRONORI SHIGA (*)

0. – Introduction.

In this note the author reconstructs the Picard's modular function as a modular function for a family of algebraic $K3$ surfaces with two complex parameters.

In 1883 Picard has constructed an analytic function of two variables analogous to the elliptic modular function (see [1]). He started from the following integral containing two complex parameters x and y,

$$I(x, y) = \int_{\infty}^{1} \frac{dt}{\sqrt[3]{t(t-1)(t-x)(t-y)}} \,. \tag{0.0}$$

The function $I(x, y)$ of x and y is a multivalued analytic function on the domain $\Lambda = \{(x, y) : xy(x-1)(y-1)(x-y) \neq 0\}$ in $\boldsymbol{C}^2$. This integral plays a similar role as the following integral,

$$I'(x) = \int_{\infty}^{1} \frac{dt}{\sqrt{t(t-1)(t-x)}}$$

which induces the elliptic modular function $\lambda(\zeta)$. This integral is a multivalued analytic function of x on $\boldsymbol{C} - \{0, 1\}$ and it is a solution of the following hypergeometric differential equation,

$$x(x-1)z'' + (2x-1)z' + \tfrac{1}{4}z = 0 \,. \tag{0.1}$$

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Let $\omega_1(x)$ and $\omega_2(x)$ be two independent solutions of (0.1). And let us consider the ratio $\zeta(x) = \omega_2(x)/\omega_1(x)$, this is the s-function of Schwartz.

The inverse function $x = \lambda(\zeta)$ of $\zeta(x)$ becomes a single valued automorphic function defined on the upper half plane. The fundamental region of $\lambda(\zeta)$ is defined by the inequalities $(\zeta_1 - \frac{1}{2})^2 + \zeta_2^2 \geqq \frac{1}{4}$, $(\zeta_1 + \frac{1}{2})^2 + \zeta_2^2 \geqq \frac{1}{4}$ and $-1 \leqq \zeta_1 \leqq 1$, where $\zeta = \zeta_1 + \sqrt{-1}\zeta_2$. We note that $\lambda(\zeta)$ realizes the universal covering space of the domain $\boldsymbol{C} - \{0, 1\}$. Similarly the function $z = I(x, y)$ satisfies the following differential equation:

$$(0.2)\quad \begin{cases} 9x(1-x)(x-y)r = 3(5x^2 - 4xy - 3x + 2y)p + 3y(1-y)q + (x-y)z \\ 3(x-y)s = p - q \\ 9y(1-y)(y-x)\,t = 3x(1-x)p + 3(5y^2 - 4xy - 3y + 2x)q + (y-x)z\,, \end{cases}$$

where we use the conventional notations $p = \partial z/\partial x$, $q = \partial z/\partial y$, $r = \partial^2 z/\partial x^2$, $s = \partial^2 z/\partial x\,\partial y$ and $t = \partial^2 z/\partial y^2$. And easily we can rewrite (0.2) in the form of a total differential equation;

$$(0.3)\qquad d\begin{pmatrix} z \\ p \\ q \end{pmatrix} = \Omega \begin{pmatrix} z \\ p \\ q \end{pmatrix},$$

where Ω is a matrix of rational 1-forms. The differential equation (0.2) is the equation for the Appell's hypergeometric function $F_1(\alpha, \beta, \beta', \gamma; x, y)$ (see [2]), in our case the parameters take the values $(\alpha, \beta, \beta', \gamma) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1)$. And the equation (0.3) is completely integrable. Hence the dimension of the solution space of (0.3) (and also of (0.2)) is equal to three.

Let ω_1, ω_2 and ω_3 be the three independent solutions of (0.3). And let us consider the ratios $\zeta_1(x, y) = \omega_2/\omega_1$ and $\zeta_2(x, y) = \omega_3/\omega_1$. Also these are multivalued analytic functions on Λ. And we obtain single valued holomorphic functions $x = \varphi_1(\zeta_1, \zeta_2)$ and $y = \varphi_2(\zeta_1, \zeta_2)$, as the inverse mapping of $(\zeta_1, \zeta_2) = (\zeta_1(x, y),\ \zeta_2(x, y))$. If we choose ω_i $(i = 1, 2, 3)$ adequately, then $\varphi_i(\zeta_1, \zeta_2)$ $(i = 1, 2)$ is defined on the hyperball $\{(\zeta_1, \zeta_2) \colon |\zeta_1|^2 + |\zeta_2|^2 < 1\}$. In such a way Picard constructed his modular function. But unfortunately the mapping (φ_1, φ_2) does not realize the universal covering of the domain Λ.

We shall study this mapping. At first we shall translate the original integral (0.0) to a double integral on an algebraic surface $S(\lambda, \mu)$ containing two complex parameters ((1.4) in the section 1). The surface $S(\lambda, \mu)$ is defined by the equation

$$w^3 - uv^2(1 - u - v)(1 - \lambda u - \mu v) = 0\,,$$

where (u, v, w) is an affine coordinate of $\boldsymbol{P}^3$ and the parameters (λ, μ) move on the domain Λ. It will be shown that the minimal nonsingular model $\tilde{S}(\lambda, \mu)$ of $S(\lambda, \mu)$ is a $K3$ surface (in the section 2). Hence there is only one independent holomorphic 2-form ψ on $\tilde{S}(\lambda, \mu)$ and it does not vanish. And the second homology group $H_2(\tilde{S}(\lambda, \mu), \boldsymbol{Z})$ is a free Abelian group of rank 22 (see [4]). The surface $\tilde{S}(\lambda, \mu)$ will be characterized as an elliptic surface (X, p, Δ) (in the section 2) satisfying the following conditions:

i) the base space Δ is equal to $\boldsymbol{P}^1$,

ii) the general fibre $p^{-1}(v)$ (v is a point on $\Delta = \boldsymbol{P}^1$) is an elliptic curve defined by the lattice $\{m \cdot \exp(2\pi i/3) + n\colon m \text{ and } n \text{ are integers}\}$,

iii) there are five singular fibres $p^{-1}(v_i)$ $(i = 1, 2, 3, 4)$ and $p^{-1}(v_\infty)$. The fibre $p^{-1}(v_i)$ $(i = 1, 2, 3, 4)$ consists of three rational curves intersecting at one point. The singular fibre $p^{-1}(v_\infty)$ consists of seven nonsingular rational curves $\Theta_0, \dots, \Theta_6$. And these components have the following intersection multiplicities,

$$\Theta_0\Theta_1 = \Theta_0\Theta_2 = \Theta_0\Theta_3 = \Theta_1\Theta_4 = \Theta_2\Theta_5 = \Theta_3\Theta_6 = 1$$

and any other intersection multiplicity is equal to zero.
The former singular fibre is of type IV and the latter is of type IV* according to the study of Kodaira (see [3]),

iv) the total space X has a holomorphic section.

Next we shall study the surface $\tilde{S}(\lambda, \mu)$ and we shall obtain the following properties.

a) We find a subgroup $A(\tilde{S})$ of $H_2(\tilde{S}, \boldsymbol{Z})$ which is composed of algebraic cycles with rank 16, and this subgroup coincides with the Neron-Severi group (that is the subgroup of all algebraic cycles) for almost all (λ, μ) on Λ (in the section 3).

b) We construct a basis system $\Gamma_1, \dots, \Gamma_{22}$ of $H_2(\tilde{S}(\lambda, \mu), \boldsymbol{Z})$ such that $\Gamma_1, \dots, \Gamma_6$ induces a generator system of the quotient group

$$H_2(\tilde{S}(\lambda, \mu), \boldsymbol{Z})/A(\tilde{S}(\lambda, \mu)) \quad \text{(in the section 4)}.$$

c) If we set

$$\eta_i(\lambda, \mu) = \int_{\Gamma_i} \psi \quad (i = 1, \dots, 6),$$

there occurs a relation

$$\eta_{i+1}(\lambda, \mu) = \exp(4\pi i/3)\eta_i(\lambda, \mu) \quad \text{for } i = 1, 3, 5$$

((4.1) in the section 4).

d) We construct the dual basis system $G_1, \ldots, G_{22}$ of $H_2(\tilde{S}, \boldsymbol{Q})$ such that $G_i\Gamma_j = \delta_{ij}$ $(1 \leq i, j \leq 22)$ (in the section 3 and 4), and we determine the matrix of the intersection multiplicities of this system (in the section 3).

e) We describe the generator system $\tilde{\delta}_1, \ldots, \tilde{\delta}_5$ of the monodromy transformation group of $H_2(\tilde{S}(\lambda, \mu), \boldsymbol{Z})$ induced from the fundamental group $\pi_1(\Lambda)$ (in the section 5).

Finally we obtain the following results using the above consideration·

CONCLUSION:

1) *The inverse mapping* $(\lambda, \mu) = (\varphi_1(\zeta_1, \zeta_2), \varphi_2(\zeta_1, \zeta_2))$ *of the period mapping* $(\zeta_1, \zeta_2) = (\eta_3(\lambda, \mu)/\eta_1(\lambda, \mu), \eta_5(\lambda, \mu)/\eta_1(\lambda, \mu))$ *for the family* $\tilde{S}(\lambda, \mu)$ *coincides with the Picard's modular function stated above.*

2) *The defining domain* Ω *for the function* φ_i $(i = 1, 2)$ *is determined by the Riemann-Hodge relation* ((4.8) *in the section* 4).

3) *The generator system of the transformation group of* Ω *which corresponds to the automorphic functions* φ_i $(i = 1, 2)$ *is given as the table in the part III of the section* 5.

The author wishes to know whether it is possible to obtain other significant analytic functions of several variables in the same manner.

1. – Translation to a double integral.

Here we reduce the integral (0.0) to a double integral. There are two integral representation formulas for the Appell's hypergeometric function $F_1(\alpha, \beta, \beta', \gamma; x, y)$ (see [2]), F_1 is defined as the following

$$F_1(\alpha, \beta, \beta', \gamma; x, y) = \sum_{m,n} \frac{(\alpha, m+n)(\beta, m)(\beta', n)}{(\gamma, m+n)m!n!} x^m y^n,$$

where (λ, k) indicates the product $\lambda(\lambda+1) \ldots (\lambda+k-1)$.

[Line integral representation formula]

If the parameters satisfy the condition $\mathrm{Re}\,\alpha > 0$ *and* $\mathrm{Re}\,(\gamma - \alpha) > 0$, *we have*

$$(1.1)\qquad F_1(\alpha, \beta, \beta', \gamma; x, y) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-xu)^{-\beta}(1-yu)^{-\beta'}\,du$$

for any point (x, y) *on the polydisk* $\{|x| < 1,\ |y| < 1\}$, *where* Γ *indicates the gamma function.*

[Double integral representation formula]

If the parameters satisfy the condition

$$\mathrm{Re}\,\beta > 0\,, \qquad \mathrm{Re}\,\beta' > 0 \qquad \textit{and} \qquad \mathrm{Re}\,(\gamma - \beta - \beta') > 0\,,$$

then we have

$$(1.2)\qquad F_1(\alpha, \beta, \beta', \gamma; x, y) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta-\beta')} \cdot \iint_A u_1^{\beta-1} v_1^{\beta'-1}(1-u_1-v_1)^{\gamma-\beta-\beta'-1}(1-xu_1-yv_1)^{-\alpha}\,du_1\,dv_1$$

for any point (x, y) *on the polydisk* $\{|x| < 1,\ |y| < 1\}$, *where* A *is a triangle in the* $(\mathrm{Re}\,u_1, \mathrm{Re}\,v_1)$*-space defined by inequalities* $u_1 \geqq 0$, $v_1 \geqq 0$ *and* $1 - u_1 - v_1 \geqq 0$.

If we use the variable $t' = 1/t$, then it follows that

$$I(x, y) = -\int_0^1 \frac{dt'}{\sqrt[3]{t'^2(1-t')(1-t'x)(1-t'y)}}\,.$$

By the formula (1.1) the right hand side of the above equality is equal to $-\Gamma(1/3)\Gamma(2/3)F_1(1/3, 1/3, 1/3, 1; x, y)$. Then by the formula (1.2) we have

$$F_1(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, 1; x, y) = \frac{1}{(\Gamma(1/3))^3} \iint_A u_1^{-\frac{2}{3}} v_1^{-\frac{2}{3}}(1-u_1-v_1)^{-\frac{2}{3}}(1-ux_1-yv_1)^{-\frac{1}{3}}\,du_1\,dv_1\,.$$

And if we set $w_1^3 = u_1v_1(1-u_1-v_1)(1-xu_1-yv_1)^2$, it follows that

$$(1.3)\qquad \iint_A \frac{1-xu_1-yv_1}{w_1^2}\,du_1\,dv_1 = -\frac{(\Gamma(1/3))^2}{\Gamma(2/3)}\,I(x, y)\,.$$

Here we consider the transformation:

$$\begin{cases} u = xu_1 , \\ v = 1 - xu_1 - yv_1 , \\ w = (xy^2(y-1))^{-1/3} w_1 , \\ \lambda = \dfrac{y-x}{x(y-1)} , \\ \mu = 1/(1-y) . \end{cases}$$

After this transformation we obtain the following:

$$\mu^{5/3}(\lambda-\mu)^{5/3}(\mu-1)^4 \iint_D \frac{v}{w^2}\, du\, dv = (\Gamma(1/3))^2 (\Gamma(2/3))^{-1} I(x, y) , \tag{1.4}$$

where D is a real 2-dimensional triangle defined by inequalities $u/x \geqq 0$, $(1-u-v)/y \geqq 0$ and $1-\lambda u-\mu v \geqq 0$. And also we have the following:

$$w^3 - uv^2(1-u-v)(1-\lambda u-\mu v) = 0 . \tag{1.5}$$

Hence we know that the Picard's original integral is represented as the double integral of the left hand side of (1.4). Then in the following we study the property of this integral.

2. – Minimal nonsingular model of $S(\lambda, \mu)$.

In this section we study the minimal nonsingular model of the algebraic surface (1.5). We define the compactification of this surface in $\boldsymbol{P}\times\boldsymbol{P}^2$ as follows:

$$\begin{cases} \xi_2^3 - v^2\xi_1(\xi_0-\xi_1-v\xi_0)(\xi_0-\lambda\xi_1-\mu v\xi_0) = 0 \\ v'^4\xi_2^3 - \xi_1(\xi_0 v'-\xi_1 v'-\xi_0)(\xi_0 v'-\lambda\xi_1 v'-\mu\xi_0) = 0 , \end{cases} \tag{2.1}$$

where $[\xi_0, \xi_1, \xi_2]$ is a homogeneous coordinate of $\boldsymbol{P}^2$ and we set $v' = 1/v$. In the following the parameters (λ, μ) move on Λ. We denote the surface (2.1) by $S(\lambda, \mu)$ or simply S. We use the following notations,

$\tilde{S}$: the minimal nonsingular model of S,

Δ: the compactified Riemann sphere of v-space,

p': the projection mapping from S to Δ,

p: the projection mapping from $\tilde{S}$ to Δ,

$$v_0 = 0 , \quad v_1 = 1 , \quad v_2 = (\lambda-1)/(\lambda-\mu) , \quad v_3 = 1/\mu , \quad v_\infty = \infty .$$

The fibre $p'^{-1}(v)$ is a nonsingular elliptic curve for every value v except v_i $(i = 0, 1, 2, 3, \infty)$. Hence $\tilde{S}$ is an elliptic surface. The fibre $p'^{-1}(v_i)$ $(i = 1, 2, 3)$ is a rational curve with one isolated singularity which is locally isomorphic to the singularity $w^3 - uv = 0$. When we resolute this singularity there occur two rational curves with self intersection number -2, and the self intersection number of the proper image of $p'^{-1}(v_i)$ is also equal to -2. These three curves meet transversally at one point. Hence we get a singular fibre of type IV as $p^{-1}(v_i)$ $(i = 1, 2, 3)$ (see [3] section 6). Next we consider $p^{-1}(v_0)$. The surface S has cusp singularity along this curve. When we proceed the σ-process along $p'^{-1}(v_0)$ there occur three rational curves with self intersection number -3. Any curve of them intersects the proper image of $p'^{-1}(v_0)$ transversally at one point, and these intersection points are different. The self intersection number of the proper image of $p'^{-1}(v_0)$ is equal to -1, that is exceptional. After the blow down process of this curve we obtain a singular fibre of type IV as $p^{-1}(v_0)$.

The general fibre has the canonical form $y^2 = 4x^3 - 0x - g_3$, hence the invariant $j = g_2^3/(g_2^3 - 27g_3^2)$ of this curve is equal to 0. Consequently the functional invariant J of the elliptic surface $\tilde{S}$ is the constant function 0. For the elliptic surface with the functional invariant constant zero there are seven possibilities as its fibre:

the regular fibre of the invariant 0,

the singular fibre of type II, IV, I_0^*, IV*, II* and

the multiple singular fibre of type ${}_mI_0$ (see [3] section 9).

We note that the Euler number of these fibres are equal to 0, 2, 4, 6, 8, 10 and 0, respectively.

The surface $(\tilde{S}, p, \Delta)$ has a holomorphic section L given by $\{\xi_1 = \xi_2 = 0\}$ in (2.1). Namely $\tilde{S}$ is a basic member. According to the calculation we know that L meets with every singular fibre on an irreducible component with multiplicity 1. We can describe the singular fibre as the decomposition with its irreducible components:

$$p^{-1}(v_i) = \Theta_{i0} + \Theta_{i1} + \Theta_i \qquad (i = 0, 1, 2, 3),$$

$$p^{-1}(v_\infty) = 3\Theta_{\infty 0} + 2\Theta_{\infty 1} + 2\Theta_{\infty 2} + 2\Theta_{\infty 3} + \Theta_{\infty 4} + \Theta_{\infty 5} + \Theta_\infty,$$

where Θ_i $(i = 0, 1, 2, 3, \infty)$ is the component intersecting L.

According to Kodaira we have the following canonical bundle formula for a basic member (see [3] section 12).

THEOREM 2.1 (**Kodaira**). *Suppose an elliptic surface (X, Φ, Δ) is a basic* *member. Then we have*

$$K_X = \Phi^*(K_\Delta - F) \quad (K \text{ indicates the canonical bundle}),$$

where F *is a certain line bundle on* Δ *with* $c(F) = -p_a - 1$.

According to this theorem we have $c_1^2 = 0$ for such a surface. Using the Noether's formula:

$$p_g - q + 1 = \frac{1}{12}(c_1^2 + c_2),$$

we obtain $c_2 \equiv 0 \pmod{12}$. For any elliptic surface X the Euler number $\chi(X)$ is equal to the summation of the Euler numbers of all singular fibres. Hence we have

$$c_2 = \chi(\tilde{S}) = \sum_{i=0}^{3} \chi(p^{-1}(v_i)) + \chi(p^{-1}(v_\infty)).$$

Already we have $(p^{-1}(v_i)) = 4$ for $i = 0, 1, 2, 3$. Then $p^{-1}(v_\infty)$ must be a singular fibre of type IV*, consequently we obtain

$$c_2 = 24.$$

Now we show that $\tilde{S}$ is a $K3$ surface, that is a minimal nonsingular compact complex surface with $K = 0$ and the irregularity $q = 0$ (it is equivalent with the condition $b_1 = c_1 = 0$).

Proposition 2.1. *Suppose an elliptic surface* (X, Φ, Δ) *is a basic member. Then* X *is a* $K3$ *surface if and only if* $c_2 = 24$ *and* $\Delta = \boldsymbol{P}$.

Proof. (Necessity) Because $K = 0$ we know $p_g = 1$. According to the Noether's formula it follows $c_2 = 24$. From Theorem 2.1 we know $c(F) = -2$. And $c(K_\Delta)$ is equal to $2g - 2$, where g indicates the genus of Δ. Hence g must be 0.

(Sufficiency) By the assumption $c_2 = 24$ it follows $p_g - q + 1 = 2$. We have $c(K_\Delta - F) = 0$ because of Theorem 2.1. Then the assumption $\Delta = \boldsymbol{P}$ assures $K_\Delta - F = 0$. This implicates $K_X = 0$, consequently we have $q = 0$.

According to this proposition we can conclude that $\tilde{S}$ is a $K3$ surface.

Conclusion 1. *The surface* $\tilde{S}(\lambda, \mu)$ *is an elliptic surface satisfying the condition* (i)-(iv) *in the section* 0 *and is a* $K3$ *surface, where the parameters* (λ, μ) *lie on* Λ.

REMARK. The condition (i)-(iv) induces unique homological invariant which belongs to the functional invariant $J = 0$. Then it characterizes the surface $\tilde{S}$ as an elliptic surface.

Let us consider the 2-form ψ on $\tilde{S}$ which is canonically induced from the 2-form $vw^{-2}du\wedge dv$, the integrand of (1.4), on S. By the calculation we can obtain that ψ is the holomorphic 2-form on $\tilde{S}$. In the rest of this paper we shall study the range of the integral (1.4).

3. – Basis of $H_2(\tilde{S}, \boldsymbol{Q})$.

In this section we construct a basis system of $H_2(\tilde{S}, \boldsymbol{Q})$. We consider the fixed surface $\tilde{S}(-\frac{1}{2}, \frac{1}{2})$ till the end of this section. Because the Euler number $\chi(\tilde{S})$ is equal to 24 and $b_1 = 0$, then $H_2(\tilde{S}, \boldsymbol{Q})$ is a 22-dimentional vector space over $\boldsymbol{Q}$.

(I) Transcendental cycles. Let v be a point on Δ different from v_i $(i = 0, 1, 2, 3, \infty)$. And let us consider a closed arc $a_i(v)$ which starts from v and goes around the critical point v_i in the positive sense $(i = 0, 1, 2, 3, \infty)$. Then $a_i(v)$ induces a monodromy transformation (A_i) of the first homology group $H_1(p^{-1}(v), \boldsymbol{Z}) \cong \boldsymbol{Z} \oplus \boldsymbol{Z}$ of the general fibre $p^{-1}(v)$. Let us choose a basis system $(\gamma_1(v), \gamma_2(v))$ of $H_2(p^{-1}(v), \boldsymbol{Z})$ so that the intersection multiplicity $\gamma_1(v)\gamma_2(v)$ is equal to -1 and so that we have

$$\int_{\gamma_1}\omega = \exp\,[2\pi i/3]\int_{\gamma_2}\omega\,,$$

where ω is the Abelian differential on $p^{-1}(v)$. According to the study of Kodaira (see [3] section 9) we know that (A_i) is determined as the following:

$$(A_i) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \quad \text{for } i = 0, 1, 2, 3\,, \tag{3.1}$$

$$(A_\infty) = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.2}$$

We note that these transformations are of order 3. Let us consider a line segment l_i $(i = 0, 1, 2, 3)$ connecting v_i and v_∞ in the lower half v-plane. We denote the open set $\Delta - \bigcup_{i=0}^{3} l_i$ by Δ_0. We can determine the basis $(\gamma_1(v), \gamma_2(v))$ of $H_1(p^{-1}(v), \boldsymbol{Z})$ so as to vary continuously while v moves on Δ_0.

If we determine the basis $(\gamma_1(v), \gamma_2(v))$ at one point, then for every value v on Δ_0 the basis $(\gamma_1(v), \gamma_2(v))$ is uniquely determined up to the homotopic equivalence. We shall give a concrete construction of $\gamma_1(v)$ and $\gamma_2(v)$ in the next section, then we leave them indeterminate for the moment. Now we construct 2-cycles G_i $(i=1, \ldots, 6)$ of $\tilde{S}$ with the following procedure.

Let us make an oriented Jordan arc α_1 from v_0 to v_1. And let us make a closed oriented Jordan arc g_1 which goes around the line segment l_1 in the negative sense and intersects α_1, l_2, l_3 and l_0 in this order. We denote the intersecting point of g_1 and α_1 by r_1. Let us take the 1-cycle $\gamma_2(r_1)$ of the fibre $p^{-1}(r_1)$. And let us make a continuation of $\gamma_2(r_1)$ along the arc g_1 till arriving at the intersecting point with l_2. According to (3.1) we can proceed the continuation taking $-(\gamma_1(v)+\gamma_2(v))$ from here. And we shall arrive at the intersecting point with l_3. Similarly we can proceed the continuation along g_1 changing the 1-cycle according to (3.1). The arc g_1 intersects $\bigcup_{i=0}^{3} l_i$ exactly three times, then this continuation determines a 2-cycle G_1 in $\tilde{S}$. And we make other five 2-cycles in the same manner.

Let us make oriented Jordan arcs α_i $(i=1,2,3)$ from v_0 to v_i in the upper half plane, where we suppose that these arcs do not intersect each other. Next we make oriented closed Jordan arcs g_i $(i=1,2,3)$ which goes around l_i in the negative sense, where we make g_i so that any one of them intersects α_j $(j=1,2,3)$ and l_k $(k=0,1,2,3)$ at most one time. We denote the intersecting point of g_i and α_i by r_i. We define six 2-cycles $G_1, \ldots, G_6$ as in the diagram (3.1), where always we define the orientation of G_i as the ordered pair of the orientation of the base arc and the orientation of the 1-cycle of the fibre.

Diagram 3.1

defined 2-cycle	base arc	starting 1-cycle
G_1	g_1	$\gamma_2(r_1)$
G_2	$-g_1$	$\gamma_1(r_1)$
G_3	g_2	$\gamma_2(r_2)$
G_4	$-g_2$	$\gamma_1(r_2)$
G_5	g_3	$\gamma_2(r_3)$
G_6	$-g_3$	$\gamma_1(r_3)$

(II) Intersection multiplicities of $G_1, \ldots, G_6$. At first we define some notations. We denote the restriction of G_i to a fibre of one point $*$ on a base arc by $G_i(*)$. And we denote the intersection multiplicity of two cycles C and C' at a intersecting point $*$ by $(CC')(*)$.

(a) We know that $G_iG_i = 0$ ($i = 1, \ldots, 6$) by changing the base arc g_j ($j = 1, 2, 3$) to a homologous one in $\Delta' = \Delta - \bigcup_{i=0}^{3} v_i - \{v_\infty\}$. And we have $G_iG_{i+1} = 0$ ($i = 1, 3, 5$) by the same reason.

(b) The base arc g_1 of G_1 and the base arc g_2 of G_3 meet each other at two points a_1 and a_2. We have

$$G_1G_3 = (G_1G_3)(a_1) + (G_1G_3)(a_2) = -\sum_{i=1,2} \{(g_1g_2)(a_i) \times G_1(a_1)G_3(a_i)\} .$$

On the other hand we have:

$$(g_1g_2)(a_1) = 1 , \quad (g_1g_2)(a_2) = -1 ,$$

$$G_1(a_1) = \gamma_2(a_1) , \quad G_3(a_1) = \gamma_2(a_1) , \quad G_1(a_2) = -(\gamma_1 + \gamma_2)(a_2) , \quad G_3(a_2) = \gamma_2(a_2) .$$

Hence we have $G_1G_3 = 1$. And also we know the following by the same reason;

$$G_3G_5 = G_2G_4 = G_4G_6 = 1 .$$

(c) Also the base arc g_1 of G_1 and the base arc $-g_2$ of G_4 meet each other at two points a_1 and a_2. Then we have:

$$G_1G_4 = \sum_{i=1,2} G_1G_4(a_i) = -\sum_{i=1,2} \{g_1(-g_2)(a_i) \times G_1(a_i)G_4(a_i)\} .$$

On the other hand we have:

$$\big(g_1(-g_2)\big)(a_1) = -1 , \quad \big(g_1(-g_2)\big)(a_2) = 1 , \quad G_4(a_1) = \gamma_1(a_1) , \quad G_4(a_2) = \gamma_1(a_2) .$$

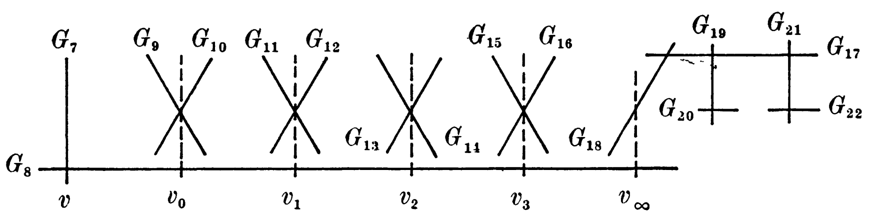
Hence we have $G_1G_4 = 2$. By the same argument we have $G_3G_6 = 2$.

In the same manner we can calculate all the intersection multiplicities $a_{ij} = G_iG_j$ ($1 \leqq i, j \leqq 6$). Consequently we obtain the intersection matrix $A = (a_{ij})$ as the following:

$$A = \begin{pmatrix} 0 & 0 & 1 & 2 & -2 & -1 \\ 0 & 0 & -1 & 1 & -1 & -2 \\ 1 & -1 & 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 & -1 & 1 \\ -2 & -1 & 1 & -1 & 0 & 0 \\ -1 & -2 & 2 & 1 & 0 & 0 \end{pmatrix} . \tag{3.3}$$

(III) Algebraic cycles. We choose 16 divisors $G_7, \ldots, G_{22}$ on the fiber space $(\tilde{S}, p, \boldsymbol{P})$ as the following:
$G_7 =$ one of the general fibres, $G_8 =$ the global section L, $G_9 = \Theta_{00}$, $G_{10} = \Theta_{01}$, $G_{11} = \Theta_{10}$, $G_{12} = \Theta_{11}$, $G_{13} = \Theta_{20}$, $G_{14} = \Theta_{21}$, $G_{15} = \Theta_{30}$, $G_{16} = \Theta_{31}$, $G_{17} = \Theta_{\infty 0}$, $G_{18} = \Theta_{\infty 1}$, $G_{19} = \Theta_{\infty 2}$, $G_{20} = \Theta_{\infty 3}$, $G_{21} = \Theta_{\infty 4}$, $G_{22} = \Theta_{\infty 5}$. We note that we eliminated every component of the singular fibres which intersects the global section L. And we have $G_i^2 = -2$ for every i $(i = 7, \ldots, 22)$, because they occur from a rational double singularities. We know the intersection multiplicities $G_i G_j$ $(7 \leqq i, j \leqq 22)$ by observing their geometric situation. Here we give the configuration diagram (diag. 3.2).

Diagram 3.2



Consequently we obtain the matrix $B = (b_{ij})$ of the intersection multiplicities $b_{ij} = G_{i+6} G_{j+6}$ $(1 \leqq i, j \leqq 16)$ as the following:

$$(3.4) \qquad B = \begin{vmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2
\end{vmatrix}$$

(IV) Intersection matrix. We can easily show that the determinants of A and B are not equal to zero. Next we consider the intersection of transcendental cycles $G_1, \ldots, G_6$ and algebraic cycles $G_7, \ldots, G_{22}$. It is apparent that the transcendental cycle does not intersect all algebraic cycles except G_8, the global section. Then we examine the intersection of G_i $(i = 1, \ldots, 6)$ and G_8.

Let δ_j be a sufficiently small disk in the v-sphere which has the center v_j $(j = 0, 1, 2, 3, \infty)$. Let $v = c$ be a fixed point on Δ', and let β_j be a positively oriented circle in δ_j which goes around v_j. Let us consider a loop $\tilde{\beta}_j$ which starts from c and goes to the initial point of β_j and goes around v_j along β_j and finally returns to c along the former path. Then the base arc g_1 of G_1 is represented as $-\tilde{\beta}_1 - \tilde{\beta}_\infty = \tilde{\beta}_0 + \tilde{\beta}_2 + \tilde{\beta}_3$. And also the other g_i can be represented as same. The restriction of the fibre space $(\tilde{S}, p, \boldsymbol{P})$ over Δ_0 is biholomorphically equivalent to the trivial one. Then it is sufficient to observe that the restriction of G_i over β_j does not intersect L. According to Kodaira (see [3]) all of $p^{-1}(\delta_j)$ $(j = 0, 1, 2, 3)$ are biholomorphically equivalent, then we examine only the case when we have $j = 0$.

We have a following representation of $(\tilde{S}|\delta_0, p, \delta_0)$, that is the restriction of $\tilde{S}$ over δ_0:

$$\eta_0^3 v - \eta_1(\eta_2^2 - \eta_1^2) = 0 \,, \tag{3.5}$$

where $[\eta_0, \eta_1, \eta_2]$ is the homogeneous coordinate of P^2. Set $s = \eta_1/\eta_0$ and $t = \eta_2/\eta_0$, then we obtain an affine representation

$$v - s(t^2 - s^2) = 0 \,. \tag{3.5$'$}$$

And we know that the correspondence between (3.5) and (2.1) is given by setting $\eta_1/\eta_0 = v\xi_1/\xi_2$ and $\eta_2/\eta_0 = v\xi_0/\xi_2$. Hence L is given by $\eta_0 = \eta_1 = 0$ in (3.5). Let us regard the fibre $p^{-1}(v)$ in (3.5$'$) as a two sheeted Riemann surface over s-sphere.

Then we obtain four ramified points $s = 0$, $\sqrt[3]{-v}$, $\omega\sqrt[3]{-v}$ and $\omega^2\sqrt[3]{-v}$, where $\omega = \exp(2\pi i/3)$. Here we define two circles $\underline{\gamma}_1(v)$ and $\underline{\gamma}_2(v)$ on the s-sphere as follows:

$$\begin{aligned} &\underline{\gamma}_1(v)\colon\ s = (\tfrac{1}{2})(1 + 2e^{i\theta})\sqrt[3]{-v}\,, \quad 0 \leqq \theta \leqq 2\pi\,, \\ &\underline{\gamma}_2(v)\colon\ s = (\tfrac{1}{2})(1 + 2e^{i\theta})\omega\sqrt[3]{-v}\,, \quad 0 \leqq \theta \leqq 2\pi\,. \end{aligned} \tag{3.6}$$

We consider the closed Jordan arc on the Riemann surface $p^{-1}(v)$ which has the projection $\underline{\gamma}_i(v)$ $(i = 1, 2)$. We denote them by $\bar{\gamma}_i(v)$. Then we obtain a canonical homology basis of $p^{-1}(v)$ because we have $\bar{\gamma}_1\bar{\gamma}_2 = -1$.

Now let u tend to zero by fixing the value v, then the value of w tends to zero with the order $w = O(|u|^{\frac{1}{3}})$. Remember that we determined $s = uv/w$ and $t = v/w$, then we know that the intersecting point of $p^{-1}(v)$ and the global section L corresponds to a point at infinity $(s, t) = (0, \infty)$.

Hence it is apparent that $\bar{\gamma}_i(v)$ does not intersect L for every v on $\delta_i - \{v_j\}$. Consequently we have

$$G_i G_k = 0 \quad \text{for } i = 1, \ldots, 6 \text{ and } k = 7, \ldots, 22\,. \tag{3.7}$$

Set $X = \{\tilde{S}(\lambda, \mu) : (\lambda, \mu) \in \Lambda\}$. And let $\tilde{\Lambda}$ be the universal covering space of Λ. The fibre space X over $\tilde{\Lambda}$ is topologically trivial. Because of this trivialization we can define the basis system $G_1(\tilde{\lambda}, \tilde{\mu}), \ldots, G_{22}(\tilde{\lambda}, \tilde{\mu})$ of $H_2(S(\lambda, \mu), \boldsymbol{Q})$ for every $(\tilde{\lambda}, \tilde{\mu})$ on $\tilde{\Lambda}$.

Because of (3.3), (3.4) and (3.7) we obtain the following.

CONCLUSION 2. *A basis system of $H_2(S(\lambda, \mu), \boldsymbol{Q})$ is given by $\{G_1, \ldots, G_{22}\}$, and the intersection matrix $M = (G_i G_j)_{1 \leq i, j \leq 22}$ is the direct sum $M = A \oplus B$.*

4. – Basis of $H_2(\tilde{S}, \boldsymbol{Z})$ and the Riemann-Hodge relation.

(I) In this section we construct the basis of $H_2(\tilde{S}, \boldsymbol{Z})$ using the basis $G_1, \ldots, G_{22}$ of $H_2(\tilde{S}, \boldsymbol{Q})$. And for the moment we consider the surface $\tilde{S}(\lambda, \mu)$ with fixed parameters $(\lambda, \mu) = (-\frac{1}{2}, \frac{1}{2})$.

Let us consider the following automorphism ϱ_1 of the surface S defined by (1.5):

$$\varrho_1 \begin{cases} u' = u\,, \\ v' = v\,, \\ w' = w \cdot \exp(2\pi i/3)\,. \end{cases}$$

We can examine that ϱ_1 can be extended to an automorphism ϱ of $\tilde{S}$. And ϱ is of order three on every simple component of the fibre and it is the identity on every multiple component of the fibre.

(II) Construction of transcendental 2-cycles. We consider a general fibre $p^{-1}(v)$ of the fibre space $(\tilde{S}, p, \boldsymbol{P})$, where we suppose $0 < v < 1$. We regard this general fibre as a three sheeted covering Riemann surface over u-sphere represented in (1.5). And we denote this Riemann surface by $R(v)$. We consider the following arcs d_i $(i = 1, 2, 3)$ on u-sphere:

d_1: the line segment connecting two points $u = 0$ and $u = \infty$,

d_2: the line segment connecting two points $u = 1 - v$ and $u = \infty$,

d_3: the line segment connecting two points $u = -2 + v$ and $u = \infty$.

When u satisfies the inequality $0 < u < 1 - v$ there are three different values of

$$w = \{v^2 u(1 - u - v)(1 - \mu v - \lambda u)\}^{\frac{1}{3}}\,,$$

and their arguments are $0, 2\pi i/3$ and $4\pi i/3$. We can define single sheet of our covering Riemann surface by the continuation of one of these branches of w over u-sphere — $\{d_1, d_2, d_3\}$. Then we define the first, second and third sheets of the cut Riemann surface $R(v)-\{u^{-1}(d_1), u^{-1}(d_2), u^{-1}(d_3)\}$ as the continuation of the value w which satisfies $\arg w = 0$, $\arg w = 2\pi i/3$ and $\arg w = 4\pi i/3$ over the open arc $0 < u < 1-v$, respectively.

Next we choose a real valued continuous function $\varepsilon(v)$ defined on the open arc $0 < v < 1$ which satisfies the inequality

$$\text{Min}\left((1-v)/2, (-2+v)/2\right) > \varepsilon(v) > 0 .$$

And we make following arcs β_1, g_1 and β_2 on u-sphere:

g_1: starts from ε and goes to the end point $1-v-\varepsilon$ along a straight line,

β_1: goes around the origin according to the parametrization

$$\varepsilon e^{i\theta}(0 \leqq \theta \leqq 2\pi) ,$$

β_2: goes around the point $1-v$ according to the parametrization $1-v-\varepsilon e^{i\varphi}(0 \leqq \varphi \leqq 2\pi)$. We denote the composite closed arc $g_1\beta_2 g_1^{-1}\beta_1^{-1}$ by c_1. Let $\gamma_1(v)$ $(\gamma_2(v))$ be the lift of c_1 to the Riemann surface $R(v)$ which take the second (the first) sheet, respectively, along the arc g_1. According to the continuation along the base arc β_i $(i=1, 2)$ it occurs a permutation $(1, 2, 3)$ of sheets of $R(v)$. Hence we know that $\gamma_1(v)$ and $\gamma_2(v)$ are closed arcs and that they have the intersection multiplicity $\gamma_1(v)\gamma_2(v) = -1$.

Here we consider the union

$$\Gamma_i' = \bigcup_{0<v<1} \gamma_i(v) \qquad (i=1, 2) .$$

Let us examine that Γ_i' tends to one point when v tends to 0 or 1. In the section 3 (IV) we already obtained a homology basis $(\bar{\gamma}_1(v), \bar{\gamma}_2(v))$ of a general fibre $p^{-1}(v)$ of the fibre space $(\tilde{S}, p, \boldsymbol{P})$ with respect to the local representation (3.5). Then we consider again the representation (3.5). Let p be a point of $\bar{\gamma}_i(v)$ $(i=1, 2)$. The point p is determined by v and the parameter θ. According to (3.5) and (3.6) we have

$$s = a(\theta)\sqrt[3]{-v} ,$$
$$t = \sqrt{1-(a(\theta))^3/a(\theta)}\,\sqrt[3]{-v} ,$$

where $a(\theta) = (1+2e^{i\theta})/2$.

We know that s and t tends to zero as v tends to zero, because $a(\theta)$ satisfies the condition $1 \leqq |a(\theta)| \leqq 2$. Hence $\tilde{\gamma}_i(v)$ $(i = 1, 2)$ tends to the origin of (s, v, t)-space when v tends to 0. Consequently $\gamma_i(v)$ tends to one point when v tends to 0. And this limit point is the intersection of three components of the singular fibre $p^{-1}(0)$, those are Θ_{00}, Θ_{01} and Θ_0.

By the same argument we know that $\gamma_i(v)$ tends to the intersecting point of three components of the singular fibre $p^{-1}(1)$ when v tends to 1.

Hence if we attach these two limit points to Γ_i', we obtain a 2-cycle on $\tilde{S}$ which is homeomorphic to a sphere. We denote them by Γ_i $(i = 1, 2)$. By deforming the base arc $\{0 \leqq v \leqq 1\}$ of Γ_i we may consider that Γ_i is situated over the arc α_1 defined in the preceding section. So we define the orientation of Γ_i as the ordered pair of the orientation of α_1 and the orientation of $\gamma_i(v)$.

Here we make following oriented arcs g_2, g_3, β_1' and β_3 on u-plane:

g_2: starts from $\varepsilon - 2 + v$ and goes to the end point $1 - v - \varepsilon$ in the upper half plane along a Jordan arc,

g_3: starts from $\varepsilon - 2 + v$ and goes to the end point $-\varepsilon$ along the real line,

β_1': goes around the origin according to a parametrization $-\varepsilon e^{i\theta}$ $(0 \leqq \theta \leqq 2\pi)$,

β_3: goes around the point $-2 + v$ according to a parametrization $-2 + v + \varepsilon e^{i\theta}$ $(0 \leqq \theta \leqq 2\pi)$.

The restriction of the fibre space $(\tilde{S}, p, \Delta)$ over Δ_0 is biholomorphically equivalent to the direct product space. Then we can define $\gamma_i(v)$ uniquely (up to the homotopic equivalence) for every value v on Δ_0. By the same procedure as the above we can define the following 2-cycles:

$$\Gamma_3 = \bigcup_{v \in \alpha_2} \gamma_1(v), \qquad \Gamma_4 = \bigcup_{v \in \alpha_2} \gamma_2(v),$$

$$\Gamma_5 = \bigcup_{v \in \alpha_3} \gamma_1(v), \qquad \Gamma_6 = \bigcup_{v \in \alpha_3} \gamma_2(v),$$

where α_i is the arc defined in the section 3 (I). By the construction we have

$$\varrho^{-1}\Gamma_i = \Gamma_{i+1} \quad \text{for } i = 1, 3, 5. \tag{4.1}$$

Then it follows

$$\int_{\Gamma_{i+1}} \psi = \omega^2 \int_{\Gamma_i} \psi \quad (\omega = \exp(2\pi i/3)) \qquad \text{for } i = 1, 3, 5. \tag{4.1'}$$

Let us examine the intersection multiplicity of Γ_i and G_j $(1 \leqq i, j \leqq 6)$. The base arc α_1 of Γ_1 and Γ_2 intersects the base arc g_1 of G_1 at one point r_1. We know the following:

$$\alpha_1 g_1 = 1\,, \quad \Gamma_1(r_1) = \gamma_1(r_1)\,, \quad \Gamma_2(r_1) = \gamma_2(r_1) \quad \text{and} \quad G_1(r_1) = \gamma_2(r_1)\,.$$

Hence we obtain

$$\Gamma_1 G_1 = -(\alpha_1 g_1)\times\gamma_1(r_1)\gamma_2(r_1) = 1\,,$$

$$\Gamma_2 G_1 = -(\alpha_1 g_1)\times\gamma_2(r_1)\gamma_2(r_1) = 0\,.$$

By the same argument we obtain $\Gamma_1 G_2 = 0$ and $\Gamma_2 G_2 = 1$. And it is easily shown that $\Gamma_1 G_j = \Gamma_2 G_j = 0$ for $j = 3, 4, 5, 6$. We can discuss about Γ_i $(i = 3, 4, 5, 6)$ in the same manner. Consequently we have

$$\Gamma_i G_j = \delta_{ij} \quad (1 \leqq i, j \leqq 6)\,, \tag{4.2}$$

where δ_{ij} indicates the Kronecker's delta.

Let us consider the subgroup $A(\tilde{S})$ of $H_2(\tilde{S}, \boldsymbol{Z})$ which is generated by $G_7, \ldots, G_{22}$. The subgroup $A(\tilde{S}) \otimes \boldsymbol{Q}$ of $H_2(\tilde{S}, \boldsymbol{Z})$ is the one which is generated by $G_7, \ldots, G_{22}$. And already we obtained the direct sum decomposition

$$H_2(\tilde{S}, \boldsymbol{Q}) = \{G_1, \ldots, G_6\} \oplus \{G_7, \ldots, G_{22}\}\,.$$

Let C be an arbitrary element of $H_2(\tilde{S}, \boldsymbol{Z})$ and put

$$C' = \sum_{j=1}^{6} a_j \Gamma_j\,, \quad \text{where } a_j = CG_j\,.$$

According to (4.2) we obtain

$$(C - C')G_j = 0 \quad \text{for } j = 1, \ldots, 6\,.$$

Then $C - C'$ belongs to the orthogonal complement of $G_1, \ldots, G_6$, namely it belongs to $A(\tilde{S}) \otimes \boldsymbol{Q}$. Since $C - C'$ is an element of $H_2(\tilde{S}, \boldsymbol{Z})$ it must belong to $A(\tilde{S})$. Let $\Gamma_7, \ldots, \Gamma_{22}$ be a basis system of $A(\tilde{S})$. By the above argument we know that $\{\Gamma_1, \ldots, \Gamma_6, \Gamma_7, \ldots, \Gamma_{22}\}$ is a basis system of $H_2(\tilde{S}, \boldsymbol{Z})$.

(III) Now we consider the Riemann-Hodge relation. Set

$$\eta_k = \int_{\Gamma_k} \psi \quad \text{for } k = 1, \ldots, 22\,.$$

And we consider the cohomology group $H^2(\tilde{S}, \boldsymbol{R})$ and we regard this group as the cohomology group of real 2-forms by the de Rham correspondence. So we choose the basis $(\omega_1, \dots, \omega_{22})$ of $H^2(\tilde{S}, \boldsymbol{R})$ so that

$$\int_{\Gamma_j} \omega_k = \delta_{jk} \quad \text{for } 1 \leqq j, k \leqq 22 .$$

Let M' be a matrix of $a_{jk} = \int_{\tilde{S}} \omega_j \wedge \omega_k$ $(1 \leqq j, k \leqq 22)$. Then we can write the Riemann-Hodge relation as follows (see [4]):

$$\text{(4.5)} \qquad \begin{aligned} &\tilde{\eta} M' {}^t\tilde{\eta} = 0 , \\ &\tilde{\eta} M' {}^t\bar{\tilde{\eta}} > 0 , \end{aligned}$$

where $\tilde{\eta} = (\eta_1, \dots, \eta_{22})$.

Let us consider the dual basis $G_1^*, \dots, G_{22}^*$ of $H_2(\tilde{S}, \boldsymbol{Q})$ such that $\Gamma_i G_j^* = \delta_{ij}$, where $G_j^* = G_j$ for $j = 1, \dots, 6$. Then ω_j is cohomologous to G_j^* as a current. Hence we have $M' = M$. Since the period $\int_D \psi$ is equal to zero for a divisor D on $\tilde{S}$, we have $\eta_i = 0$ for $i = 7, \dots, 22$. Consequently we can write (4.5) using the matrix A of the section 3:

$$\text{(4.6)} \qquad \eta A {}^t\eta = 0 ,$$

$$\text{(4.7)} \qquad \eta A {}^t\bar{\eta} > 0 ,$$

where $\eta = (\eta_1, \dots, \eta_6)$.

Here we consider the universal covering space $\tilde{\Lambda}$ of the domain Λ of parameters. And let $(\tilde{\lambda}, \tilde{\mu})$ be a point on $\tilde{\Lambda}$ which corresponds to the point (λ, μ) on Λ. Then the totality of the surfaces $\tilde{S}(\tilde{\lambda}, \tilde{\mu})$ can be regarded as a fibre space over $\tilde{\Lambda}$ and it is topologically trivial. Then we can define the homology basis $\Gamma_1(\tilde{\lambda}, \tilde{\mu}), \dots, \Gamma_{22}(\tilde{\lambda}, \tilde{\mu})$ of $H_2(\tilde{S}(\tilde{\lambda}, \tilde{\mu}), \boldsymbol{Z})$. When we make the continuation of $\Gamma_i(\lambda, \mu)$ along a closed arc in Λ, it occurs a monodromy transformation. Any how we obtained a basis system of $H_2(\tilde{S}(\lambda, \mu), \boldsymbol{Z})$, where (λ, μ) varies on Λ. And the intersection matrix M of their dual basis does not depend on the parameters (λ, μ). Then we obtain the relation (4.6) and (4.7) for every (λ, μ). From the relation (4.1′) we can reduce the relation (4.7) as follows:

$$\text{(4.8)} \qquad (\eta_1, \eta_3, \eta_5) \begin{pmatrix} 0 & \omega^2 & 1 \\ \omega & 0 & \omega^2 \\ 1 & \omega & 0 \end{pmatrix} \begin{pmatrix} \bar{\eta}_1 \\ \bar{\eta}_3 \\ \bar{\eta}_5 \end{pmatrix} < 0 .$$

Let σ_1, σ_2 and σ_3 be three eigen values of the matrix in this relation. Then they are given as the solutions of $t^3 - 3t + 1 = 0$, and they satisfy the relation $\sigma_1 < 0 < \sigma_2 < \sigma_3$.

If we consider $[\eta_1, \eta_3, \eta_5]$ as a homogeneous coordinate in $\boldsymbol{P}^2$, according to (4.8) we obtain a domain Ω in $\boldsymbol{P}^2$ which is biholomorphically equivalent to a hyperball.

REMARK. Let $NS(\tilde{S})$ be the Neron-Severi group of $\tilde{S}$ (that is the group of all divisors under the algebraic equivalence). The rank of $NS(\tilde{S})$ is 16 for almost all (λ, μ) on Λ. And it exceeds 16 if and only if $[\eta_1, \eta_3, \eta_5]$ is a rational point on $\boldsymbol{P}^2$.

From the argument of this section we obtained the following.

CONCLUSION 3. *The period mapping* $[\eta_1, \eta_3, \eta_5]$ *for the family* $\tilde{S}(\lambda, \mu)$ *defines a multivalued analytic mapping from* Λ *to a hyperball* Ω *in* $\boldsymbol{P}^2$.

5. – Monodromy transformation and the reduction to the Picard's mapping.

In this section we relate the monodromy transformation of $H_2(\tilde{S}(\lambda, \mu), \boldsymbol{Z})$ which is induced from an element of $\pi_1(\Lambda)$. And we show that the mapping $\eta_1(\lambda, \mu), \eta_3(\lambda, \mu), \eta_5(\lambda, \mu)]$ coincides with the mapping which is constructed by Picard. And we give the generators of the transformation group of Ω which is induced from the monodromy transformation.

(I) *Reduction to the original integral.* From the construction of Γ_i $(i = 1, 2)$ these 2-cycles depend on the function $\varepsilon(v)$. But all of them are homotopic. Hence the period

$$\eta_i = \int_{\Gamma_i} \psi \qquad (i = 1, 2)$$

does not depend on $\varepsilon(v)$. So we can consider the limit value of η_i as $\varepsilon(v)$ tends to zero, and that value is also equal to η_i. Here we consider a real 2-dimensional triangle D_1 on S as follows:

$$D_1 = \{(u, v) | u \geqq 0,\ v \geqq 0,\ 1 - u - v \geqq 0\}.$$

This is the projection of the limit cycle of Γ_i $(i = 1, 2)$ to the (u, v)-space. Then we obtain

$$\eta_1 = (\omega^2 - \omega) \iint_{D_1} \frac{v}{w^2}\, du \wedge dv , \tag{5.1}$$

where we take the first sheet of w, that takes a real value on D_1, and

$\omega = \exp(2\pi i/3)$. Let us consider the other triangles as the followings:

$$D_2 = \{(u, v) | (\lambda - \mu)v/(\lambda - 1) \geqq 0,\ \lambda(1 - u - v)/(\lambda - 1) \geqq 0,$$
$$(1 - \lambda u - \mu v)/(1 - \lambda) \geqq 0\},$$
$$D_3 = \{(u, v) | \lambda u \geqq 0,\ \mu v \geqq 0,\ 1 - \lambda u - \mu v \geqq 0\}.$$

The triangle $D_2(D_3)$ is the projection of the limit cycle of Γ_3 and Γ_4 (Γ_5 and Γ_6) to the (u, v)-space, respectively. By the same argument we obtain

$$\eta_3 = (1 - \omega) \iint_{D_2} \frac{v}{w^2}\, du \wedge dv, \tag{5.2}$$

$$\eta_5 = (1 - \omega^2) \iint_{D_3} \frac{v}{w^2}\, du \wedge dv.$$

Because of the equality $D = D_1 + D_2 - D_3$, D is the original triangle in (1.5), it holds that

$$\iint_D \frac{v}{w^2}\, du \wedge dv = \frac{1}{1 - \omega^2} (\omega\eta_1 + \eta_3 + \omega^2 \eta_5). \tag{5.3}$$

Hence we obtain the representation of the original Picard's integral (0.0) in terms of the period on $\tilde{S}$. Namely, according to (1.4) and (5.3) we have

$$\int_\infty^1 \frac{dt}{\sqrt[3]{t(t-1)(t-x)(t-y)}} = c\mu^{5/3}(\mu - \lambda)^{5/3}(\mu - 1)^4 \frac{\omega\eta_1 + \eta_3 + \omega^2\eta_5}{1 - \omega^2}, \tag{5.4}$$

where c is the gamma constant which appeared in (1.4).

(II) *Monodromy transformation.* Let $p_0 = (\lambda_0, \mu_0)$ be the point $(-\frac{1}{2}, \frac{1}{2})$ on Λ. We consider the following loops $\delta_1, \ldots, \delta_5$ in Λ, where we suppose that p_0 is the initial point of every δ_i $(i = 1, \ldots, 5)$:

δ_1 goes round the point $\lambda = 0$ in the positive sense on the hyperplane $\mu = \mu_0$,

δ_2 goes round the point $\lambda = 1$ in the positive sense on the same plane,

δ_3 goes round the point at infinity in the positive sense on the hyperplane $\lambda = 3\mu - 2$,

δ_4 goes round the point $\lambda=\mu_0$ in the positive sense on the hyperplane $\mu=\mu_0$,

δ_5 goes round the point $\mu=0$ in the positive sense on the hyperplane $\lambda=\lambda_0$.

We regard Λ as $\boldsymbol{P}^2$-{6 complex lines}. And let H be a general hyperplane in $\boldsymbol{P}^2$. Then the generators of the fundamental group of $H\cap\Lambda$ are also the generators of $\pi_1(\Lambda)$, this is the theorem of Lefschetz. Hence $(\delta_1, \dots, \delta_5)$ constitutes a generator system of $\pi_1(\Lambda)$. Every element δ of $\pi_1(\Lambda)$ induces a monodromy transformation $\tilde{\delta}$ of $H_2(\tilde{S}(\lambda_0, \mu_0), \boldsymbol{Z})$. And every divisor of $A(\tilde{S})$ is invariant under this transformation. Then we consider the monodromy transformation of $\Gamma_1, \dots, \Gamma_6$ in the following.

(i) Transformation $\tilde{\delta}_1$. When the point (λ, μ) moves along the loop δ_1, the critical points v_0, v_1, v_3 and v_∞ stay invariant and the point $v_2=(1-\lambda)/(\mu-\lambda)$ varies along a loop ω_1 which goes round v_3 in the positive sense. This loop defines a Jordan region $R(\omega_1)$ in the finite v-plane. Let $\tilde{S}(\lambda, \mu; v)$ be a fibre over v of the elliptic fibre surface $(\tilde{S}(\lambda, \mu), p, \boldsymbol{P})$. If v is a point on $\Delta'-\omega_1$, where $\Delta'=P-\{v_0, v_1, v_2, v_3, \infty\}$, then δ_1 induces a monodromy transformation $\delta_1'(v)$ of $H_1(\tilde{S}(\lambda_0, \mu_0; v), \boldsymbol{Z})$. At first we study this transformation.

A general fibre $\tilde{S}(\lambda, \mu; v)$ is realized as a covering Riemann surface by considering the representation (1.5). We denote this Riemann surface by $R(\lambda, \mu; v)$. There are three triply ramified points over $u=0$, $u=u_1=$ $=(1-\mu v)/\lambda$ and $u=u_2=1-v$ on $R(\lambda, \mu; v)$.

Let us make three cut arcs a_0, a_1 and a_2 on u-plane, they are line segments which connect $u=0$ and ∞, $u=u_1$ and ∞, $u=u_2$ and ∞, respectively. Then we can determine the i-th sheet w_i $(i=1, 2, 3)$ so that we have $w_3=\exp(2\pi i/3)w_2=\exp(4\pi i/3)w_1$. Here we consider an oriented arc on u-plane which connects two points α and β. We denote the lift of this arc into the i-th sheet w_i by $w_i(\alpha, \beta)$.

When a point (λ, μ) moves on Λ, the ramified points $u=0$ and $u=u_2$ are invariant and only $u=u_1(\lambda, \mu)$ varies. Let s_1 be a loop which is drawn by u_1 corresponding to δ_1. This loop negatively goes round the Jordan region $R(s_1)$ which is defined by s_1 in the finite u-plane. If u is an interior point of $R(s_1)$, then δ_1 does not induce a permutation of the branches of

$$w=w(\lambda, \mu; u, v)=\sqrt[3]{v^2\lambda u(u_1-u)(u_2-u)}\,.$$

And if u is an exterior point of $R(s_1)$, then the branch w_i of w changes according to the permutation $(1, 2, 3)$.

If v is an exterior point of $R(\omega_1)$, s_1 goes round 0 and u_2 in the negative sense. And the canonical basis system of $H_1(\tilde{S}(\lambda_0, \mu_0; v), \boldsymbol{Z})$ are given as the following:

$$\begin{cases} \gamma_1(v) = w_2(0, u_2) - w_3(0, u_2)\,, \\ \gamma_2(v) = w_1(0, u_2) - w_2(0, u_2)\,. \end{cases}$$

Then the projection of $\gamma_i(v)$ is contained in $R(s_1)$. Hence in this moment $\delta_1'(v)$ is the identity.

And if v is an interior point of $R(\omega_1)$, s_1 goes round only one ramified point $u = 0$ in the negative sense. We consider the oriented line segment connecting 0 and u_2 in this direction.
We denote this arc by ε. Then ε can be deformed to an arc passing through the point $u = v - 2$. Then we obtain

$$\begin{aligned} \delta_1'(\gamma_1(v)) &= \delta_1'(w_2(0, u_2) + w_3(u_2, 0)) \\ &= \big(w_1(0, u_1) + w_2(u_1, 0) + w_3(0, u_2)\big) + \big(w_1(u_2, 0) + w_3(0, u_1) + w_2(u_1, 0)\big) \\ &= 2w_2(u_1, 0) - w_3(u_1, 0) + w_3(0, u_2) - w_1(u_1, u_2)\,. \end{aligned}$$

By the direct observation we know

$$\delta_1'(\gamma_1)\gamma_1 = 1\,, \qquad \delta_1'(\gamma_1)\gamma_2 = 0\,.$$

Hence we know $\delta_1'(\gamma_1) = \gamma_2$. And by the relation $\gamma_2 = \varrho^{-1}\gamma_1$ we obtain

$$\delta_1'(\gamma_2) = \varrho^{-1}\delta_1'(\gamma_1) = -(\gamma_1 + \gamma_2)\,.$$

In the following we denote $-(\gamma_1 + \gamma_2)$ by γ_3, then we have

$$(5.5) \qquad \delta_1' = \begin{cases} \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_2 & \gamma_3 & \gamma_1 \end{pmatrix} & \text{if } v \text{ lies inside of } R(\omega_1)\,, \\ \text{identity} & \text{if } v \text{ lies outside of } R(\omega_1)\,. \end{cases}$$

Now we study the transformation $\tilde{\delta}_1$. The base arc α_1 of Γ_1, it is the line segment connecting $v = 0$ and $v = 1$, does not change as (λ, μ) moves along δ_1. And α_1 is contained in the exterior part of ω_1. Then $\delta_1'(v)$ is identity for the value v on α_1. Then it follows that $\tilde{\delta}_1\Gamma_1 = \Gamma_1$. Similarly we have $\tilde{\delta}_1\Gamma_2 = \Gamma_2$. The critical point v_2 goes round the critical point v_3 in the positive sense drawing the loop ω_1 as (λ, μ) moves along δ_1. Then the base arc α_2 of Γ_3 moves following the loop ω_1. Let v_j and v_k $(0 \leqq j, k \leqq 3)$ be two critical points, and we consider a Jordan arc α connecting v_j and v_k in $\Delta_0 = \Delta - \bigcup_{i=0}^{3} a_i$. We denote the 2-cycle $\bigcup_{v \in \alpha} \gamma_i(v)$ by $\Gamma_i(j, k)$.

Then the following holds:

$$\Gamma_1 = \Gamma_1(0,1)\,, \qquad \Gamma_3 = \Gamma_1(0,2)\,, \qquad \Gamma_5 = \Gamma_1(0,3)\,.$$

And also we have

$$\begin{aligned}\tilde{\delta}_1\Gamma_3 &= \Gamma_1(0,2) + \Gamma_2(2,3) + \Gamma_3(3,2)\\ &= \Gamma_1(0,2) + \big(\Gamma_2(0,3) - \Gamma_2(0,2)\big) - \big(\Gamma_3(0,3) - \Gamma_3(0,2)\big)\\ &= -2\Gamma_4 + \Gamma_5 + 2\Gamma_6\,.\end{aligned}$$

If we regard the base arc α_3 of Γ_5 as the composition of α_2 and the arc $\alpha_3 - \alpha_2$, we know that the former is invariant as (λ, μ) moves along δ_1 and that the latter is also invariant. But we must notice that $\alpha_3 - \alpha_2$ is contained in the interior part of ω_1. Hence we obtain the following, using the transformation (5.5):

$$\tilde{\delta}_1\Gamma_5 = \Gamma_1(0,2) + \Gamma_2(2,3) = \Gamma_1(0,2) + \Gamma_2(0,3) - \Gamma_2(0,2) = \Gamma_3 - \Gamma_4 + \Gamma_6\,.$$

Because of the relation (4.1) we obtain:

$$\begin{aligned}\tilde{\delta}_1\Gamma_4 &= 2\Gamma_3 + 2\Gamma_4 - 2\Gamma_5 - \Gamma_6\,,\\ \tilde{\delta}_1\Gamma_6 &= \Gamma_3 + 2\Gamma_4 - \Gamma_5 - \Gamma_6\,.\end{aligned}$$

Consequently we can describe the transformation $\tilde{\delta}_1$:

$$(5.6) \qquad \tilde{\delta}_1\begin{pmatrix}\Gamma_1\\ \Gamma_2\\ \Gamma_3\\ \Gamma_4\\ \Gamma_5\\ \Gamma_6\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -2 & 1 & 2\\ 0 & 0 & 2 & 2 & -2 & -1\\ 0 & 0 & 1 & -1 & 0 & 1\\ 0 & 0 & 1 & 2 & -1 & -1\end{pmatrix}\begin{pmatrix}\Gamma_1\\ \Gamma_2\\ \Gamma_3\\ \Gamma_4\\ \Gamma_5\\ \Gamma_6\end{pmatrix}.$$

This transformation is of order three.

(ii) Transformation $\tilde{\delta}_2$. When the point (λ, μ) moves along δ_2, the critical points v_0, v_1, v_3 and v_∞ are invariant. And v_2 moves along a loop ω_2 which goes round v_0 in the positive sense. This loop defines a Jordan region $R(\omega_2)$ in the finite v-plane. By the same procedure as (i) we obtain the tıansformation $\delta_2'(v)$ of $H_1(\tilde{S}(\lambda_0, \mu_0; v), \boldsymbol{Z})$ which is induced from δ_2, where v

is fixed on $\Delta'-\omega_2$:

$$(5.7)\qquad \delta_2'(v)=\begin{cases}\begin{pmatrix}\gamma_1 & \gamma_2 & \gamma_3\\ \gamma_2 & \gamma_3 & \gamma_1\end{pmatrix} & \text{if } v \text{ lies on } R(\omega_2)\,,\\ \text{identity} & \text{if } v \text{ is outside of } R(\omega_2)\,.\end{cases}$$

After the variation of the point (λ, μ) along δ_2 the base arc α_1 of Γ_1 is deformed to an arc which passes through v_2. This deformed arc is the composition of two subarcs: the one starts from v_0 and goes to v_2 and the other starts from v_2 and goes to v_1. The former lies in $R(\omega_2)$ and the latter lies outside of $R(\Gamma_2)$. Using (5.7) we have

$$\tilde{\delta}_2\Gamma_1=\Gamma_2(0,2)-\Gamma_1(0,2)+\Gamma_1(0,1)=\Gamma_1-\Gamma_3+\Gamma_4\,.$$

After the variation of the point (λ, μ) along δ_2 the base arc α_2 of Γ_3 does not change geometrically, but the argument increases by 2π and this arc is contained in $R(\omega_2)$. And by the same variation the arc α_3 is deformed to an arc which passes through the points v_0, v_2, a_0 in this order and ends at v_3, where a_0 is the line segment connecting v_0 and ∞.

We regard this deformed arc as a composition of two subarcs. The first starts from v_0 and goes to v_2, the second starts from v_2 and goes to v_3. The former lies in $R(\omega_2)$ and this part changes the argument by 2π. The latter lies outside of $R(\omega_2)$. According to (5.7) we have

$$\tilde{\delta}_2\Gamma_3=-\Gamma_3-\Gamma_4\,,$$

$$\tilde{\delta}_2\Gamma_5=-\Gamma_3-2\Gamma_4+\Gamma_5\,.$$

Because of the relation (4.1) we obtain

$$\tilde{\delta}_2\Gamma_4=\Gamma_3\,,$$

$$\tilde{\delta}_2\Gamma_6=2\Gamma_3+\Gamma_4+\Gamma_6\,.$$

Consequently we can describe the transformation $\tilde{\delta}_2$:

$$(5.8)\qquad \tilde{\delta}_2\begin{pmatrix}\Gamma_1\\ \Gamma_2\\ \Gamma_3\\ \Gamma_4\\ \Gamma_5\\ \Gamma_6\end{pmatrix}=\begin{pmatrix}1 & 0 & -1 & 1 & 0 & 0\\ 0 & 1 & -1 & -2 & 0 & 0\\ 0 & 0 & -1 & -1 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & -1 & -2 & 1 & 0\\ 0 & 0 & 2 & 1 & 0 & 1\end{pmatrix}\begin{pmatrix}\Gamma_1\\ \Gamma_2\\ \Gamma_3\\ \Gamma_4\\ \Gamma_5\\ \Gamma_6\end{pmatrix}.$$

This transformation is of order three.

(iii) Transformation $\tilde{\delta}_3$. When the point (λ, μ) moves along δ_3, the critical points v_0, v_1, v_2 and ∞ are invariant. The critical point v_3 varies along a loop ω_3 which goes round v_0 in the positive sense. This loop defines a Jordan region $R(\omega_3)$ in the finite v-plane. By the usual method we obtain the transformation $\delta_3'(v)$ of $H_1(\tilde{S}(\lambda_0, \mu_0; v), \boldsymbol{Z})$ induced from δ_3:

$$(5.9)\qquad \delta_3' = \begin{cases} \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_3 & \gamma_1 & \gamma_2 \end{pmatrix} & \text{if } v \text{ lies on } R(\omega_3)\,, \\ \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_2 & \gamma_3 & \gamma_1 \end{pmatrix} & \text{if } v \text{ lies outside of } R(\omega_3)\,. \end{cases}$$

If we use (5.9) we can describe the transformation $\tilde{\delta}_3$ by the same method as (i) and (ii):

$$(5.10)\qquad \tilde{\delta}_3\begin{pmatrix} \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \\ \Gamma_4 \\ \Gamma_5 \\ \Gamma_6 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & -1 & -2 \\ -1 & -1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & -1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \\ \Gamma_4 \\ \Gamma_5 \\ \Gamma_6 \end{pmatrix}.$$

This transformation is of order three.

(iv) Transformations $\tilde{\delta}_4$ and $\tilde{\delta}_5$. When the point (λ, μ) moves along δ_4, the critical points v_0, v_1, v_3 and ∞ are invariant. And v_2 goes around ∞ in the positive sense. And when the point (λ, μ) moves along δ_5, the critical points v_0, v_1 and ∞ are invariant. The critical point v_2 moves along a loop which is homotopic to zero in Δ'. The critical point v_3 goes around the point ∞ in the positive sense.

By the similar way we obtain the monodromy transformation $\tilde{\delta}_4$ and $\tilde{\delta}_5$ as follows:

$$(5.11)\qquad \tilde{\delta}_4\begin{pmatrix} \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \\ \Gamma_4 \\ \Gamma_5 \\ \Gamma_6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -2 & -1 & 1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \\ \Gamma_4 \\ \Gamma_5 \\ \Gamma_6 \end{pmatrix},$$

$$(5.12)\qquad \tilde{\delta}_5\begin{pmatrix} \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \\ \Gamma_4 \\ \Gamma_5 \\ \Gamma_6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & -2 & -1 & 1 & 0 \\ -2 & -1 & 1 & -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \\ \Gamma_4 \\ \Gamma_5 \\ \Gamma_6 \end{pmatrix}.$$

The transformation $\tilde{\delta}_4$ and $\tilde{\delta}_5$ are of order infinite.

(III) The monodromy transformation $\tilde{\delta}_i$ $(i = 1, 2, 3, 4, 5)$ induces a linear transformation δ_i^* of the periods $\eta_j(\lambda, \mu)$ $(j = 1, 3, 5)$. According to the results in (II) and (6.1′) we can describe them as follows:

$$\delta_1^* \begin{pmatrix} \eta_1 \\ \eta_3 \\ \eta_5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2\omega^2 & 1+2\omega^2 \\ 0 & 1-\omega^2 & \omega^2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_3 \\ \eta_5 \end{pmatrix},$$

$$\delta_2^* \begin{pmatrix} \eta_1 \\ \eta_3 \\ \eta_5 \end{pmatrix} = \begin{pmatrix} 1 & -1+\omega^2 & 1 \\ 0 & -1-\omega^2 & 0 \\ 0 & -1-2\omega^2 & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_3 \\ \eta_5 \end{pmatrix},$$

$$\delta_3^* \begin{pmatrix} \eta_1 \\ \eta_3 \\ \eta_5 \end{pmatrix} = \begin{pmatrix} \omega^2 & 0 & -1-2\omega^2 \\ 0 & \omega^2 & -1-2\omega^2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_3 \\ \eta_5 \end{pmatrix},$$

$$\delta_4^* \begin{pmatrix} \eta_1 \\ \eta_3 \\ \eta_5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2-\omega^2 & 1 & 1-\omega^2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_3 \\ \eta_5 \end{pmatrix},$$

$$\delta_5^* \begin{pmatrix} \eta_1 \\ \eta_3 \\ \eta_5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1+2\omega^2 & -2-\omega^2 & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_3 \\ \eta_5 \end{pmatrix}.$$

The transformed value $\delta_i^*(\eta_j(\lambda, \mu))$ is nothing but the analytic continuation of $\eta_j(\lambda, \mu)$ along the arc δ_i. According to the local Torelli type theorem of the period mapping for algebraic $K3$-surfaces (see [4]) $\eta_1(\lambda, \mu)$, $\eta_3(\lambda, \mu)$ and $\eta_5(\lambda, \mu)$ are linearly independent. And if we observe the transformations δ_i^*, it is easily shown that any $\eta_j(\lambda, \mu)$ $(j = 1, 3, 5)$ is obtained as a linear combination of one period $\eta(\lambda, \mu) = \int_\Gamma \psi$ for some 2-cycle Γ of $\tilde{S}$ and its analytic continuations. The function

$$f(\lambda, \mu) = \frac{A(\lambda, \mu)}{1-\omega^2}(\omega\eta_1 + \eta_3 + \omega^2\eta_5),$$

where $A(\lambda, \mu) = c\mu^{5/3}(\mu-\lambda)^{5/3}(\mu-1)^4$, of the right hand side in (5.4) is a solution for the differential equation (0.3) under the transformation of variables

$$\lambda = (y-x)/(x(y-1)), \quad \mu = 1/(1-y). \tag{5.13}$$

Then $A(\lambda, \mu)\eta_j(\lambda, \mu)$ $(j = 1, 3, 5)$ are the three independent solutions for the equation (0.3). Hence the ratios

$$\zeta_1(\lambda, \mu) = \frac{\eta_3(\lambda, \mu)}{\eta_1(\lambda, \mu)}, \quad \zeta_2(\lambda, \mu) = \frac{\eta_5(\lambda, \mu)}{\eta_1(\lambda, \mu)}$$

coincide with the Picard's original mapping. Consequently we obtain the following.

CONCLUSION 4. *The period mapping* $[\eta_1(\lambda, \mu), \eta_3(\lambda, \mu), \eta_5(\lambda, \mu)]$ *for the family of surfaces* $\tilde{S}(\lambda, \mu)$ *coincides with* $(\zeta_1(x, y), \zeta_2(x, y))$ *under the transformation* (5.13). *The inverse mapping* $(\lambda, \mu) = (\varphi_1(\zeta_1, \zeta_2), \varphi_2(\zeta_1, \zeta_2))$ *is equal to the Picard's modular functions* (*up to a projective linear transformation*). *The functions* φ_1 *and* φ_2 *are holomorphic on* Ω *which is determined by* (4.8), *and they are automorphic functions with respect to the transformation group generated by* $\delta_1^*, \ldots, \delta_5^*$.

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