FREDERICK J. ALMGREN
LEON SIMON

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*Annali della Scuola Normale Superiore di Pisa, Classe di Scienze* 4e série, tome 6, n° 3 (1979), p. 447-495

<http://www.numdam.org/item?id=ASNSP_1979_4_6_3_447_0>
Existence of Embedded Solutions of Plateau’s Problem (*)

FREDERICK J. ALMGREN Jr. (**) - LEON SIMON (***) (1)

Consider a uniformly convex open set $A \subset \mathbb{R}^3$ such that $\partial A$ is $C^2$, and let $\Gamma$ be a $C^3$ Jordan curve contained in $\partial A$. The classical approach to Plateau’s problem, as formulated by Douglas [D], Rado [R], and Courant [C], together with the work of Osserman [O], Alt [Al, 2] and Gulliver [G] concerning branch points (see also [GOR]), demonstrates that there is a minimal immersion $X$ from the disc $D_1 = \{x \in \mathbb{R}^2: |x| < 1\}$ into $\mathbb{R}^3$ such that $X|\partial D_1$ is a homeomorphism onto $\Gamma$. Furthermore $X$ is area minimizing relative to all immersions $Y$ from $D_1$ into $\mathbb{R}^3$ such that $Y|\partial D_1$ is a homeomorphism onto $\Gamma$.

Osserman conjectured that (for convex $A$) the immersion $X$ might be an embedding; that is, the solution surface might have no self intersections. This was proved subject to the (rather stringent) condition that $\Gamma$ has total curvature < $4\pi$ by Gulliver and Spruck [GS1] (1). Also, there is recent work of Tomi and Tromba [TT] which, based on an intriguing analysis of the pathwise connectivity of spaces of suitable minimal surfaces, asserts that $\Gamma$ bounds a (perhaps unstable) embedded minimal surface.

We here wish to demonstrate by a geometric measure theory argument (quite different than these other approaches) that one can always find an embedded minimal surface $M \subset \hat{A}$ which is diffeomorphic to $D_1$, satisfies $\partial M = \Gamma$, and minimizes area relative to all diffeomorphs $N$ of $D_1$ having $\partial N = \Gamma$. (See Theorem 6 in § 8 below.)

(*) This research was partly supported by NSF grants at Princeton and Stanford universities. The research of the second author was also partly supported by a Sloan Foundation grant.

(**) Princeton University.

(***) Stanford University, University of Adelaide.


(1) See also the correction [GS2] relating to Lemma 4.2 of [GS1].

Pervenuto alla Redazione il 13 Giugno 1978.
We here also discuss (in § 9) extensions of the result to the case when \( \Gamma \) consists of two or more contours, and (in § 10) we consider the case when the surfaces under consideration are allowed to have higher genus; Appendix B illustrates the fact that surfaces of higher genus will often have smaller area than the area minimizing discs obtained in § 8.

Concerning uniqueness, even in the single contour case, it is clear, by a slight modification of an argument of Nitsche [N, Section 2], that the embedded discs of § 8 are not necessarily unique. However, by using the methods of Morgan [M], it can be shown in a rather straightforward manner that there is a non-trivial (geometrically natural) measure \( \mu \) defined on \( C^{\infty} \) curves \( \Gamma \) contained in the boundaries of uniformly convex regions in \( \mathbb{R}^3 \), such that the set of curves which do not bound a unique area minimizing disc has \( \mu \)-measure zero.

We have also very recently received an announcement by Meeks and Yau [MY] dealing with similar questions on three dimensional manifolds from a somewhat different perspective and by different methods (viz. showing «Douglas-Morrey » immersed solutions to Plateau’s problem are in fact embedded). The techniques of our paper extend in a straightforward manner to yield embedded minimal surfaces in three dimensional manifolds (provided boundary regularity is guaranteed by convexity hypotheses, etc.). In this regard, see also [P].

1. - Terminology.

\( B(x_0, \rho), U(x_0, \rho) \) will denote respectively the closed and open balls, with radius \( \rho \) and centre \( x_0 \), in \( \mathbb{R}^3 \).

\( \mathcal{H}^k \), \( k > 0 \), denotes \( k \)-dimensional Hausdorff measure in \( \mathbb{R}^3 \).

\( D_1 = \{ x \in \mathbb{R}^2 : |x| < 1 \} \).

\( \mathcal{M} \) denotes the collection of all surfaces-with-boundary in \( \mathbb{R}^3 \) which are \( C^2 \) diffeomorphs of \( D_1 \).

If \( \gamma \) is a Jordan curve in \( \mathbb{R}^2 \), \( \text{int} \gamma \) denotes the bounded component of \( \mathbb{R}^2 \sim \gamma \). More generally, given \( M \in \mathcal{M} \) and a Jordan curve \( A \subset M \), we let \( \text{int}_M A \) denote the (unique) component \( M' \sim A \) such that \( M' \cap \partial M = 0 \).

\( \mathcal{M} \) will denote the collection of surfaces of the form \( \text{int}_M A \), where \( M \in \mathcal{M} \) and \( A \) is a piecewise \( C^2 \) Jordan curve in \( M \).

\( A \) denotes a uniformly convex open subset of \( \mathbb{R}^3 \) with \( C^2 \) boundary \( \partial A \).

\( \Gamma \) denotes a \( C^2 \) Jordan curve contained in \( \partial A \).

The remaining notation concerning varifolds and currents is the standard notation of [AW1, 2] and [FH].

Now let \( \{ M_k \} \) be a sequence in \( \mathcal{M} \) with \( \partial M_k = \Gamma \), \( k = 1, 2, \ldots \), and \( \lim_{k \to \infty} \mathcal{H}^2(M_k) = \inf \{ \mathcal{H}^2(M) : M \in \mathcal{M}, \partial M = \Gamma \} \).
We can assume, by taking a subsequence if necessary, that there is a varifold $V \in V_2(\mathbb{R}^3)$ such that $V = \lim_{k \to \infty} v(M_k)$. $V$ is of course stationary in $\mathbb{R}^3 \sim \Gamma$ (in the sense of [AW1, 4.2]) because

$$\|h_* V\|(U) > \|V\|(U)$$

whenever $U$ is a bounded open subset of $\mathbb{R}^3$ and $h$ is a diffeomorphism of $U$ leaving a neighbourhood of $\partial U \cup \Gamma$ fixed. Also, by the convex hull property for stationary varifolds (see the appendix), we have

$$\text{spt } \|V\| \subset A \cup \Gamma.$$ (1.1)

Our ultimate aim is to prove that there is an $M \in \mathcal{M}$ such that $\partial M = \Gamma$ and $V = v(M)$. (Of course it will then follow that $\mathcal{H}^2(M) = \inf \{\mathcal{H}^2(N) : N \in \mathcal{M} \text{ and } \partial N = \Gamma\}$ because of the manner in which $V$ was constructed.) An outline of the remainder of the paper is as follows:

In §§ 2-4 we prove some preliminary results (of a rather elementary nature). In §§ 5-6 these are used to prove interior regularity of $V$: it is proved that for each $x_0 \in \text{spt } \|V\| \sim \Gamma$ there is a positive integer $n_{x_0}$, a positive $\varepsilon_{x_0}$, and an analytic hypersurface $M_{x_0}$ such that

$$V \subset U(x_0, \varepsilon_0) \times G(3, 2) = n_{x_0} v(M_{x_0}).$$

In § 7 the corresponding boundary regularity results are given: we prove that for each $x_0 \in \Gamma$ there is a $\varepsilon_0 = \varepsilon(x_0) > 0$ and a $C^2$ surface-with-boundary $M_{x_0}$ such that $\partial M_{x_0} \cap U(x_0, \varepsilon_0) = \Gamma \cap U(x_0, \varepsilon_0)$ and $V \subset (\mathbb{R}^3 \sim \Gamma) \cap (U(x_0, \varepsilon_0) \times G(3, 2) = v(M_{x_0} \cap U(x_0, \varepsilon_0)).$

In § 8 it is shown that these regularity results imply the existence of the required minimizing $M \in \mathcal{M}$. In § 9 an analogous existence result is proved for the case when $\Gamma$ is a union of $k \geq 2$ smooth Jordan curves, and in § 10 we consider the possibility of finding area minimizing surfaces of higher genus.

2. Some preliminary lemmas.

**Lemma 1.** Suppose $F_1$, $F_2$ are oriented compact connected surfaces-with-boundary, and suppose $\partial F_1 = \partial F_2$. Suppose further that for each $j = 1, 2$, there is a diffeomorphism $\mu_j$ of $F_j$ onto a subset $E_j$ of $S^1$. Then there is a diffeomorphism $\chi$ of $F_1$ onto $F_2$ such that $\chi$ coincides with the identity map on $\partial F_1$. 

Proof. By replacing $\mu_j$ by $\theta \circ \mu_j$ if necessary, where $\theta$ is any orientation reversing diffeomorphism of $S^2$, we can assume that $\mu_j$ is an orientation preserving map from $F_j$ onto $E_j$, $j = 1, 2$. (We give $E_1$, $E_2$ the orientations induced by the inclusions $E_j \subset S^2$, $j = 1, 2$.) Since $\mu_2 \circ \mu_1^{-1} : \partial E_1$ is then an orientation preserving diffeomorphism of $\partial E_1$ onto $\partial E_2$, it is a standard fact that we can find an orientation preserving diffeomorphism $\mu$ of $E_1$ onto $E_2$ such that $\mu|\partial E_1 = \mu_2 \circ \mu_1^{-1}|\partial E_1$.

We then define $\chi$ by $\chi = \mu_2^{-1} \circ \mu \circ \mu_1$.

Before stating the next lemma, we need the following additional terminology.

We let $\Gamma_1, \ldots, \Gamma_n$ be pairwise disjoint $C^2$ Jordan curves in $\mathbb{R}^2$ and $D \subset \mathbb{R}^2$ we let be the union of a pairwise disjoint collection $D^{(1)}, \ldots, D^{(n)}$ of diffeomorphs of $D_1$. $\mathcal{Y} = \mathcal{Y}(\Gamma_1, \ldots, \Gamma_n)$ denotes the (possibly empty) collection of Lipschitz maps $Y : D \to \mathbb{R}^3$ satisfying the following conditions

\begin{equation}
\bigcup_{j=1}^n Y^{-1}(\Gamma_j) = \mu_1 \cup \cdots \cup \mu_m, \text{ where } \mu_1, \ldots, \mu_m \text{ are pairwise disjoint } C^2 \text{ Jordan curves contained in } D \sim \partial D;
\end{equation}

\begin{equation}
\text{for any component } U \text{ of } D \sim \bigcup_{i=1}^m \mu_i, \text{ } Y|\bar{U} \text{ is a } C^2 \text{ embedding of } \bar{U} \text{ into } \mathbb{R}^3;
\end{equation}

\begin{equation}
\text{for any pair of components } U, W \text{ of } D \sim \bigcup_{i=1}^m \mu_i, \text{ either } Y(U) \cap Y(W) = \emptyset, \text{ or else } Y(U) = Y(W).
\end{equation}

Notice that the above conditions imply that $Y(D) \sim \left( \bigcup_{j=1}^n \Gamma_j \cup Y(\partial D) \right)$ is a 2-dimensional submanifold of $\mathbb{R}^3$, each component of which has closure of the form $Y(\bar{U})$, where $U$ is a component of $D \sim \bigcup_{j=1}^n \mu_j$.

Subsequently, for $Y \in \mathcal{Y}$, we use the notation

\begin{equation}
\begin{align*}
U(Y) &= \bigcup_{i=1}^m \text{int}(\mu_i) \quad (\mu_i \text{ as in (2.1)}) \\
U_\eta(Y) &= \{ x \in D : \text{dist}(x, U(Y)) < \eta \}.
\end{align*}
\end{equation}

If $B$ is a Borel subset of $D$, then we let $A(Y|B)$ denote the area associated with $Y|B$; that is

\begin{equation}
A(Y|B) = \int_B \| A_2(DY(x)) \| \, dx.
\end{equation}

(In case $Y$ is 1-1 on $B$, $A(Y|B)$ is just $\mathcal{H}^2(Y(B))$.)
LEMMA 2. Suppose $X \in Y$. Then there is $\tilde{X} \in Y$ such that

\[ U(X) = U(\tilde{X}); \tilde{X}(x) = X(x), \quad x \in D \sim U(X); \tilde{X}(D) \subset X(D) \]

and such that for each $q > 0$ there is a $C^2$ embedding $\tilde{X}$ of $D$ into $\mathbb{R}^3$ with

\[ A(\tilde{X}|U') < A(X|U') \quad \text{for each component } U' \text{ of } U(X), \]

and such that for each $\eta > 0$ there is a $C^2$ embedding $\hat{X}$ of $D$ into $\mathbb{R}^3$ with

\[ \hat{X}(x) = X(x), \quad x \in D \sim U_\eta(X); \sup |\hat{X}(x) - \tilde{X}(x)| < \eta \]

\[ A(\hat{X}|U') < A(X|U') + \eta \quad \text{for each component } U' \text{ of } U_\eta(X). \]

Before giving a proof of this lemma, we note the following corollary.

COROLLARY 1. Suppose $M_1, \ldots, M_R \in \mathcal{M}$ with $\partial M_1, \ldots, \partial M_R$ pairwise disjoint and

\[ M_i \sim \partial M_i \subset B \sim \partial B, \quad \partial M_i \subset \partial B, \quad i = 1, \ldots, R, \]

where $B$ is a diffeomorph of the closed unit ball in $\mathbb{R}^3$.

Then for each $\eta > 0$ there exist pairwise disjoint $\tilde{M}_1, \ldots, \tilde{M}_R \in \mathcal{M}$ with

\[ \partial \tilde{M}_i = \partial M_i, \quad \tilde{M}_i \sim \partial M_i \subset B \sim \partial B, \quad \mathcal{H}^2(\tilde{M}_i) < \mathcal{H}^2(M_i) + \eta \]

for each $i = 1, \ldots, R$.

PROOF OF COROLLARY 1. The proof is by induction on $R$. The result is trivial if $R = 1$. Hence take $R > 2$ and assume that there exist pairwise disjoint $\tilde{M}_1, \ldots, \tilde{M}_{R-1}$ such that (2.8) holds with $\tilde{M}_i$ in place of $M_i$, $i = 1, \ldots, R - 1$. Next, let $\tilde{M}_R \in \mathcal{M}$ be such that $\partial \tilde{M}_R = \partial M_R$, $M_R \sim \partial \tilde{M}_R \subset B \sim \partial B$, $\mathcal{H}^2(\tilde{M}_R) < \mathcal{H}^2(M_R) + \eta$, and such that $\tilde{M}_R$ intersects each of $\tilde{M}_1, \ldots, \tilde{M}_{R-1}$ transversally. (Such $\tilde{M}_R$ can of course be obtained by making a slight perturbation of $M_R$.) We can then write $\tilde{M}_R \cap \left( \bigcup_{i=1}^{R-1} \tilde{M}_i \right) = \bigcup_{i=1}^n \Gamma_i$, where $\Gamma_1, \ldots, \Gamma_n$ are pairwise disjoint $C^2$ Jordan curves.

We now let $X$ be a $C^2$ immersion of $D$ into $\mathbb{R}^3$ such that the restriction of $X$ to each $D^{(i)}$ is a $C^2$ diffeomorphism of $D^{(i)}$ onto $\tilde{M}^{(i)}$. (Here $D = \bigcup_{i=1}^R D^{(i)}$ is as defined above.)

It is now easy to see that $X \in Y$ and hence by the lemma there is an embedding $\hat{X}$ of $D$ onto $\mathbb{R}^3$ satisfying $\hat{X}(\partial D^{(i)}) = \partial M_i$, $\hat{X}(D \sim \partial D) \subset B \sim \partial B$ and $\mathcal{H}^2(\hat{X}(D^{(i)})) < \mathcal{H}^2(\hat{M}_i) + \eta$, $i = 1, \ldots, R$. Then, defining $\tilde{M}_i = \hat{X}(D^{(i)})$, $i = 1, \ldots, R$, we see that (2.8) holds with $2\eta$ in place of $\eta$. This completes the proof.
PROOF OF LEMMA 2. For \( Y \in \mathcal{Y} \) we define \( Y^{(k)} : D \to \mathbb{R}^3 \) as follows:

(i) If \( Y^{-1}(\Gamma_k) = \emptyset \), define \( Y^{(k)} = Y \);

(ii) if \( Y^{-1}(\Gamma_k) \neq \emptyset \), then (since \( Y \in \mathcal{Y} \)) we have \( Y^{-1}(\Gamma_k) = \bigcup_{i=1}^{m} \mu_i \) for some pairwise disjoint \( C^2 \) Jordan curves \( \mu_1, \ldots, \mu_m \) in \( D \sim \partial D \). We let \( l \in \{1, \ldots, m\} \) be such that

\[
A(Y'|\text{int } \mu_i) < A(Y'|\text{int } \mu_j), \quad j = 1, \ldots, m,
\]

(notice that then \( (\text{int } \mu_i) \cap \mu_j = \emptyset \) for \( j \neq l \), and let \( U_1, \ldots, U_p \) be the components of \( \bigcup_{i=1}^{m} \text{int } \mu_i \). For \( j = 1, \ldots, p \) let \( \theta_j \) be any \( C^2 \) diffeomorphism of \( \overline{U}_j \) onto \( \text{int } \mu_i \) such that

\[
\theta_j|\partial U_j = (Y'|\text{int } \mu_i)^{-1}(Y'|\partial U_j).
\]

We then define \( Y^{(k)} \) by

\[
Y^{(k)} = \left\{ \begin{array}{ll}
Y & \text{on } D \sim \bigcup_{j=1}^{p} U_j \\
Y \circ \theta_j & \text{on } U_j, \quad j = 1, \ldots, p.
\end{array} \right.
\]

Of course \( Y^{(k)} \), so defined, depends on the choice of \( \theta_1, \ldots, \theta_p \) (and on the choice of \( l \) in case there is more than one integer \( l \) satisfying (2.9)).

For each \( j = 1, \ldots, n \) define

\[
(2.10) \quad \mathcal{Y}^{(j)} = \{ Y^{(k)} : Y \in \mathcal{Y} \}, \\
\mathcal{Y}_j = \bigcap_{k=1}^{\infty} \mathcal{Y}^{(k)}.
\]

We note that \( \mathcal{Y}^{(j)} \subset \mathcal{Y} \), and \( Y \in \mathcal{Y}^{(j)} \) if and only if \( Y \in \mathcal{Y} \) and \( Y^{-1}(\Gamma_j) = \bigcup_{i=1}^{m} \mu_i \), where \( \text{int } \mu_1, \ldots, \text{int } \mu_m \) are pairwise disjoint and such that there are \( C^2 \) diffeomorphisms \( \theta_j \) of \( \text{int } \mu_j \) onto \( \text{int } \mu_1 \) with \( Y'|\text{int } \mu_j = (Y'|\text{int } \mu_1) \circ \theta_j, \quad j = 1, \ldots, m \).

The following properties, except possibly the first, are straightforward consequences of the definition of \( Y^{(k)} \):

(i) \( Y \in \mathcal{Y}^{(j)} \Rightarrow Y^{(k)} \in \mathcal{Y}^{(j)}, \quad k, j = 1, \ldots, n; \)

(ii) \( U(Y) = U(Y^{(k)}) \) (notation as in (2.4));

(iii) \( Y^{(k)}(D) \subset Y(D), \quad Y^{(k)}((D \sim U(Y)) = Y((D \sim U(Y))). \)
(iv) $A(U^{(k)}|U') < A(Y|U')$ whenever $U'$ is a component of $U(Y)$.

To check (i), we first let $\tilde{\mu}_1, \ldots, \tilde{\mu}_r$ be pairwise disjoint Jordan curves in $D \sim \partial D$ such that $(Y^{(k)})^{-1}(I_i^j) = \bigcup_{i=1}^r \tilde{\mu}_i$. (Thus $\bigcup_{i=1}^r \text{int} \tilde{\mu}_i = \bigcup_{i=1}^r \text{int} \mu_i$, where $\mu_1, \ldots, \mu_m$ are as in the definition of $Y^{(k)}$.) Since $Y \in \mathcal{Y}^{(d)}$, one can now check that if a component $y$ of $(Y^{(k)})^{-1}(I_i^j)$ is contained in $\bigcup_{i=1}^r \text{int} \tilde{\mu}_i$, then any other component $y'$ is either likewise contained in $\bigcup_{i=1}^r \text{int} \tilde{\mu}_i$, or else $\text{int} y \cap \bigcup_{i=1}^r \text{int} \tilde{\mu}_i = \emptyset$. (Otherwise it is quite a straightforward matter to deduce that there is an $i \in \{1, \ldots, r\}$ with $\mu_i \subset \text{int} \mu_i$, thus contradicting (2.9).) Using the above fact, together with the definition of $Y^{(k)}$, it then quite easily follows that $Y^{(k)} \in \mathcal{Y}^{(d)}$ as required.

We are now ready to define $\hat{X}$; we define $\hat{X} = X_n$, where $X_n$ is the final element in the sequence $X_0, X_1, \ldots, X_n$, defined inductively as follows:

(a) $X_0 = X$.

(b) Assume $k \in \{1, \ldots, n\}$ and suppose $X_0, \ldots, X_{k-1} \in \mathcal{Y}$ have already been defined. Then define $X_k = (X_{k-1})^{(k)}$.

By properties (ii)-(iv) above, it is clear that $\hat{X}$ has the required properties (2.6). It remains to show that there exists a suitable embedding $\hat{X}$ for each $\eta > 0$.

First, note that by property (i) above we have the implication

$$Y \in \mathcal{Y}_{k-1} \Rightarrow Y^{(k)} \in \mathcal{Y}_k, \quad k = 1, \ldots, n.$$  

(Here we adopt the convention that $\mathcal{Y}_0 = \mathcal{Y}$ in case $k = 1$.) Hence from definition of $\hat{X}$ we have $\hat{X} \in \mathcal{Y}_n$.

The proof will now be completed by showing that for each $Y \in \mathcal{Y}_n$ and each $\eta > 0$, there is a $C^3$ embedding $\hat{Y}$ of $D_1$ into $\mathbb{R}^3$ with

$$Y(x) = \hat{Y}(x) \quad \text{for} \ x \in D \sim U_\eta(Y), \quad \sup |Y - \hat{Y}| < \eta$$

and

$$A(\hat{Y}|U') \leq A(Y|U') + \eta \quad \text{for each component } U' \text{ of } U_\eta(Y).$$

From now on we thus take $Y$ to be an arbitrary fixed element of $\mathcal{Y}_n$.

(*) In case $(Y^{(k)})^{-1}(I_i^j) = \emptyset$, then $Y^{(k)} = Y$ and hence $Y^{(k)} \in \mathcal{Y}^{(d)}$ trivially in this case.
We let 

\[ k = \max \{ \text{card} \left( Y^{-1}(p) : p \in \mathbb{R}^n \right) \}, \]

\[ K = \{ p \in \mathbb{R}^n : \text{card} \left( Y^{-1}(p) \right) = k \}. \]

We want to show that \( K \) is closed. Let \( y \) be any component of \( \bigcup_{j=1}^{n} Y^{-1}(\Gamma_j) \), and let \( U_y^+ \), \( U_y^- \) be the (unique) components of \( D \sim \bigcup_{j=1}^{n} Y^{-1}(\Gamma_j) \) such that 
\[ y \subset U_y^+ \cap U_y^- \setminus \text{int} y \neq \emptyset, \quad (D \sim \text{int} y) \cap U_y^+ \neq \emptyset. \]

By (2.3) we have either \( Y(U_y^+) \cap Y(U_y^-) = \emptyset \) or \( Y(U_y^+) = Y(U_y^-) \). We claim that since \( Y \in \mathcal{U}_n \) the latter possibility cannot occur. Indeed suppose \( Y(U_y^+) = Y(U_y^-) \), and let \( \mu_1 \) denote the outermost component of \( \partial U_y^- \). Note that then \( \mu_1 \cap \partial D = \emptyset \) by (2.1), hence \( \mu_1 \) is a component of \( \bigcup_{j=1}^{n} Y^{-1}(\Gamma_j) \). Since \( Y(U_y^+) = Y(U_y^-) \), it then follows that there is a \( \mu_2 \subset \partial U_y^- \) with \( Y(\mu_2) = Y(\mu_1) = \Gamma_j \) for some \( j \in \{ 1, \ldots, n \} \). On the other hand, because of our selection of \( \mu_1 \), we have \( \text{int} \mu_2 \subset \text{int} \mu_1 \). However, since \( Y(\mu_2) = Y(\mu_1) = \Gamma_j \), this contradicts the fact that \( Y \in \mathcal{U}_n \subset \mathcal{Y}^0 \) (in particular, see the characterization of \( \mathcal{Y}^0 \) following (2.10)). Thus we deduce that \( Y(U_y^+) \cap Y(U_y^-) = \emptyset \) for any component \( y \) of \( \bigcup_{j=1}^{n} Y^{-1}(\Gamma_j) \). Because of this, it is now quite a straightforward matter to show that \( K \) is closed. (One now needs only to use the fact that \( Y \in \mathcal{Y} \); no further use need be made of the fact that \( Y \in \mathcal{U}_n \).)

We let \( K_1, \ldots, K_R \) denote the (compact) components of \( K \), so that

\[ \text{card} \left( Y^{-1}(p) \right) = k, \quad p \in K_r, \]

and we define \( L_r = Y^{-1}(K_r), \ r = 1, \ldots, R \).

If \( k = 1 \), we simply set \( \hat{Y} = Y \) and we are finished. Hence we assume \( k > 1 \). Using (2.1)-(2.3) we can see that \( \text{card} \left( Y^{-1}(p) \right) \) is constant on each component of \( Y(D) \sim \bigcup_{j=1}^{n} \Gamma_j \), and

\[ \partial L_r \subset \bigcup_{j=1}^{n} Y^{-1}(\Gamma_j). \]

Also, if \( \Gamma_j \cap K_r \neq \emptyset \), then, by (2.11) and the fact that \( Y \in \mathcal{U}_n \subset \mathcal{Y}^0 \subset \mathcal{Y} \), we have

\[ Y^{-1}(\Gamma_j) = \bigcup_{i=1}^{k} \gamma_i^j \]

for some pairwise disjoint smooth Jordan curves \( \gamma_1^j, \ldots, \gamma_k^j \) with

\[ \text{int} \gamma_i^j \cap \text{int} \gamma_l^j = \emptyset, \quad i \neq l, \quad \bigcup_{i=1}^{k} \text{int} \gamma_i^j \subset L_r. \]
By combining (2.11), (2.12), (2.13) with the fact that $K_r$ is closed, we deduce that

\[ L_r = \bigcup_{i=1}^{k} E_i', \quad r = 1, ..., R, \]

where $E_1', ..., E_k'$ are pairwise disjoint diffeomorphs of $D_l$ with $\partial E_i' \subset \bigcup_{i=1}^{n} Y^{-1}(I_i')$ and

\[ Y(E_i') = ... = Y(E_k') = K_r. \]

By (2.11) and (2.15) it is clear that $Y[E_i']$ is 1-1 for $i = 1, ..., k$. Notice also that $Y(B - E_i') \cap Y(E_i') = \emptyset$ whenever $B \cap \bigcup_{l=1, l \neq i}^{k} E_i' = \emptyset$, because otherwise by (2.15) we would have a contradiction to the definition of $k$. Using this fact together with the definition of $B$, it then follows that we can find pairwise disjoint subsets $F_1', ..., F_k'$ of $D - \partial D$ having the properties

\[ F_i' \in \mathcal{M}, \quad F_i' \sim \partial F_i' \cap E_i', \quad Y[F_i'] \text{ is 1-1, dist } \{E_i', \partial F_i'\} < \eta. \]

Since $Y(\partial E_i')$ is one of the $I_i'$, we can also clearly arrange

\[ \left( \bigcup_{i=1}^{k} Y(F_i' \sim E_i') \right) \cap \left( \bigcup_{i=1}^{n} I_i' \right) = \emptyset. \]

Furthermore by (2.3) we can arrange to choose the $F_i'$ so that, for each $i, j = 1, ..., k$, either

\[ Y(F_i' \sim E_i') \cap Y(F_i' \sim E_j') = \emptyset \]

or

\[ Y(F_i') = Y(F_j'). \]

Now let $\tilde{Y}$ be a Lipschitz map of $D$ into $\mathbb{R}^3$ such that

(i) $\tilde{Y}|F_i'$ is a $C^2$ embedding into $\mathbb{R}^3$, $i = 1, ..., k$;

(ii) $\tilde{Y}(F_i') = \tilde{Y}(F_j')$ for each $i, j$ such that (2.18)' holds

\[ \tilde{Y}(F_i') \cap \tilde{Y}(F_j') = \emptyset \text{ for each } i, j \text{ such that (2.18) holds}; \]

(iii) $\tilde{Y}\left( \bigcup_{r=1}^{R} \bigcup_{i=1}^{k} F_i' \right) \cap \left( \bigcup_{i=1}^{n} I_i' \right) = \emptyset$, $\tilde{Y}\left( \bigcup_{r=1}^{R} \bigcup_{i=1}^{k} F_i' \right) \cap Y(D \sim \bigcup_{r=1}^{R} \bigcup_{i=1}^{k} F_i') = \emptyset$;

(iv) $\tilde{Y}(x) = Y(x)$ for $x$ in some neighbourhood of $D \sim \bigcup_{r=1}^{R} \bigcup_{i=1}^{k}$ interior $(F_i')$.

(v) $\sup |\tilde{Y} - Y| < \eta$. 

Because of (2.16), (2.17), (2.18), (2.18)', it is clear that such a \( Y \) can be constructed by a perturbation and smoothing of \( Y \) inside the sets \( F_i \) (holding \( Y \) fixed in some neighbourhood of \( D \) - \( \bigcup \bigcup_{r=1}^{k} F_r \)).

We now claim \( Y \in \mathcal{Y}_n \). One can readily check this on the basis of (2.16), (2.17) and (i)-(iv) above. It of course follows from the construction of \( Y \) that \( \max \{ \text{card } (Y^{-1}(p)): p \in \mathbb{R}^1 \} < k \).

Then in view of (iv)-(vi) and the arbitrariness of \( \eta \), the existence of the desired embedding \( \tilde{Y} \) now follows by induction on \( k \).

3. - A replacement theorem.

The following theorem guarantees that if \( U \) is an open convex subset of \( \mathbb{R}^3 \) and if \( M \in \mathcal{M} \) intersects \( \partial U \) in a sufficiently nice manner (to be made precise below), then we can always replace \( M \) by \( \tilde{M} \in \mathcal{M} \) in such a way that \( \partial \tilde{M} = \partial M \), \( \tilde{M} \sim \{ x \in U: \text{dist } (x, \partial U) < \theta \} \subset M \sim \{ x \in U: \text{dist } (x, \partial U) < \theta \} \) \( (\theta \) any preassigned positive constant), \( \mathcal{H}^2(\tilde{M}) < \mathcal{H}^2(M) \), and such that \( \tilde{M} \cap \bar{U} \) is a disjoint union of elements of \( \mathcal{M} \). The replacement procedure also has other nice properties. The precise result is given in the following theorem. In this theorem we use the notation that

\[
U_\theta = \{ x \in U: \text{dist } (x, \partial U) > \theta \}.
\]

**Theorem 1** (Replacement Theorem). Suppose \( \theta > 0 \) is given and

(i) \( U \) is a \( C^2 \) convex open set in \( \mathbb{R}^3 \);

(ii) \( M \in \mathcal{M}, \partial M \subset \mathbb{R}^3 \sim U, \) and \( \partial M \sim \partial U \) is contained in the union of the unbounded components of \( \mathbb{R}^3 \sim (\bar{U} \cup (M \sim \partial M)) \);

(iii) \( M \) intersects \( \partial U \) transversally; in case \( \partial M \cap \partial U \neq \emptyset \), we will always take this to mean that there is a \( C^2 \) (open) surface \( N \) with \( M \subset N \), with \( (N \sim M) \cap \partial U = \emptyset \) and with \( N \) intersecting \( \partial U \) transversally.

Then there is \( \tilde{M} \in \mathcal{M} \) such that

(iv) \( \partial M = \partial \tilde{M}, \tilde{M} \sim U \subset M \sim U, \tilde{M} \cap U_\theta \subset M \cap U_\theta \);

(v) \( \tilde{M} \) intersects \( \partial U \) transversally (in the same sense as in (iii));

(vi) \( \mathcal{H}^2(\tilde{M}) + \mathcal{H}^2((M \sim \tilde{M}) \cap U_\theta) < \mathcal{H}^2(M) \);

(vii) \( \tilde{M} \cap \bar{U} \) is a disjoint union \( \bigcup_{i=1}^{k} N_i \) of elements \( N_i \in \mathcal{M} \) (and consequently \( \tilde{M} \subset \bar{U} \) in case \( \partial M \subset \partial U \)).
If in addition to the hypotheses (i)-(iii) we have
(viii) $\mathcal{K}^2(M) < \mathcal{K}^2(P) + 0$ for every $P \in \mathcal{M}$ with $\partial P = \partial M$, then there are non-negative numbers $\theta_1, \ldots, \theta_k$ with $\sum_{j=1}^k \theta_j < 0$ and
(ix) $\mathcal{K}^2(N_j) < \mathcal{K}^2(P) + 0$, for every $P \in \mathcal{M}$ with $\partial P = \partial N_j$, $j = 1, \ldots, k$.

**Proof.** In the proof we will need the following area minimizing property of subsets $F$ of $\partial U$:

Suppose $F \subset \partial U$ and suppose each component of $F$ is a $C^2$ surface-with-boundary; suppose that $E \subset \mathbb{R}^3 \sim U$ is a $C^2$ surface-with-boundary satisfying $\partial E = E \cap \partial U = \partial F$ and $E \cup F = \partial G$ for some open $G \subset \mathbb{R}^3 \sim \bar{U}$. Then

$$\mathcal{K}^2(F) < \mathcal{K}^2(E). \quad (3.1)$$

The proof of (3.1) consists of an application of the divergence theorem on $\bar{G}$. We note first that since $U$ is convex, then the outward unit normal $\nu$ of $\partial U$ extends, by defining it to be constant on rays normal to $\partial U$, to a $C^1(\mathbb{R}^3 \sim U)$ function $\nu^*$ with $|\nu^*| = 1$ and $\text{div} \, \nu^* > 0$. Applying the divergence theorem on $\bar{G}$ we then deduce that

$$\int_F \nu \cdot \nu^* d\mathcal{K}^2 < \int_E \eta \cdot \nu^* d\mathcal{K}^2$$

where $\eta$ denotes the outward unit normal of $G$ at points of $E$. Since $\nu = \nu^*$ on $F$ and $\eta \cdot \nu^* < 1$ on $E$, with equality if and only if $\eta = \nu^*$, the required result (3.1) then easily follows.

We now proceed with the proof of the theorem. Let $E$ be any component of $M \sim \bar{U}$ such that $E \cap \partial M = \emptyset$ (1). Since $E$ is connected and $\bar{E} \sim E$ is a union of smooth Jordan curves contained in $\partial U$, a straightforward modification of a standard topological result enables us to assert that there is a unique bounded component $U_\varepsilon$ of $\mathbb{R}^3 \sim (\bar{U} \cup E)$. Let $F$ be defined by

$$F = \partial U_\varepsilon \cap \partial U. \quad (3.2)$$

Then $\bar{E}$, $F$ are $C^2$ surfaces-with-boundary (although of course $F$ is not necessarily connected) with $\partial \bar{E} = \partial F$. Furthermore, given any component $E'$ of $M \sim \bar{U}$ with $E' \subset U_\varepsilon$, it is clear that $\partial M \cap E' = \emptyset$ (by condition (ii) of the theorem) and $U_{E'} \subset U_\varepsilon$. From this last inclusion it follows

(1) Notice of course that $(E \sim E) \cap \partial M = \emptyset$, by (iii).
that we can select a component $E$ of $M \sim \overline{U}$ such that $\partial M \cap E = \emptyset$ and

\begin{equation}
U \cap M = \emptyset.
\end{equation}

Henceforth it is assumed that $E$ has been so selected, and $F$ is defined by (3.2).

Now let $\chi$ be a $C^2$ diffeomorphism of $D_1$ onto $M$, and consider the set $\chi^{-1}(E)$; this is a compact connected subset $H$ of $D_1$ bounded by a finite collection $\gamma_1, \ldots, \gamma_k$ of pairwise disjoint $C^2$ Jordan curves. Suppose without loss of generality that $\gamma_1$ is the outermost of these curves. That is, suppose $\gamma_2, \ldots, \gamma_k$ are all contained in $\text{int} \, \gamma_1$. Notice that, by the connectedness of $H$, we can then write

$$H = \overline{\text{int} \, \gamma_1} \sim \bigcup_{i=2}^k \text{int} \, \gamma_i.$$ 

We now define $F^* \subset F$ to be the component of $F$ which contains the Jordan curve $\chi(\gamma_1)$, and we define $I = \cup \text{int} \, \gamma_i$, where the union is over those $j$ such that $\chi(\gamma_i) \subset \partial F \sim \partial F^*$. Also, define $E^* = \chi(I)$. Then $E \cup E^*$ is diffeomorphic to the connected subset $H \cup I$ of $D_1$, and $\partial (E \cup E^*) = \partial F^*$. Hence by Lemma 1 we can construct a diffeomorphism $\mu$ of $E \cup E^*$ onto $F^*$ such that $\mu$ coincides with the identity on $\partial (E \cup E^*)$ ($\sim \partial F^*$). We now define a Lipschitz mapping $\hat{x}: D_1 \to \mathbb{R}^3$ by

$$\hat{x}(x) = \begin{cases} 
(\mu \circ \chi)(x) & \text{if } x \in H \cup I, \\
\chi(x) & \text{otherwise},
\end{cases}$$

and we consider the following two cases:

**Case I.** $\partial M \cap (F^* \sim \partial F^*) = \emptyset$. In this case (by (3.3)) it is clear that $\hat{x}$ is 1-1 on $D_1$, and we define

$$\hat{M} = \hat{x}(D_1) \quad (= (M \sim (E \cup E^*)) \cup F^*).$$

**Case II.** $\partial M \cap (F^* \sim \partial F^*) \neq \emptyset$. In this case we know by (3.3) and the hypotheses (ii) and (iii) of the theorem that $\partial M \subset F^* \sim \partial F^*$, and hence $\hat{x}^{-1}(\partial M) = \partial D_1 \cup \gamma$, where $\gamma$ is a $C^2$ Jordan curve in $D_1 \sim \partial D_1$. In this case we define $\hat{M} = \hat{x}(\text{int} \, \gamma)$ and we note that

$$\hat{M} \subset \hat{x}(D_1) = (M \sim (E \cup E^*)) \cup F^*.$$ 

In either of the above two cases we thus obtain $\hat{M}$ which is homeomorphic to $D_1$ (via a bilipschitz homeomorphism). Also, writing $E^* = E^*$
in Case I and $E^* = E^* \cup \dot{\gamma}(D_1 \sim \text{int } \gamma)$ in Case II, we have

(3.4)  
$$\mathcal{K}^*(\hat{M}) + \mathcal{K}^*(E^*) < \mathcal{K}^*(M).$$

To check this we first note that, by (3.1),

(3.5)  
$$\mathcal{K}^*(F^*) < \mathcal{K}^*(F) < \mathcal{K}^*(E).$$

Next we note that, by definition of $\hat{M}$,

(3.6)  
$$\mathcal{K}^*(\hat{M}) = \mathcal{K}^*(M) - \mathcal{K}^*(E) - \mathcal{K}^*(E^*) + \mathcal{K}^*(F^*).$$

By adding (3.5) and (3.6) we then have (3.4) as required.

The next step in the argument involves a smoothing and slight perturbation of $\hat{M}$ near $F^* \cap \hat{M}$, holding the sets $\partial M \cap \partial U$, $\hat{M} \sim \bar{U}$ and $\hat{M} \cap U_0$ fixed, and taking points of $(F^* \cap \hat{M}) \sim \partial M$ into $U \sim \bar{U}_0$. In this way we can obtain an $M_1 \in \mathcal{M}$ with $M_1$ intersecting $\partial U$ transversally, with

(3.7)  
$$\begin{align*}  
\partial M_1 &= \partial M, \quad M_1 \sim \bar{U} \subset \hat{M} \sim \bar{U} \subset M \sim \bar{U}, \\
M_1 \cap U_0 &= \hat{M} \cap U_0 = (M \sim E^*) \cap U_0, 
\end{align*}$$

and with

(3.8)  
$$\mathcal{K}^*(M_1) < \mathcal{K}^*(\hat{M}) + \delta$$

for any preassigned $\delta > 0$. Then, if $\delta$ is taken small enough, (3.8) together with (3.4) implies

$$\mathcal{K}^*(M_1) + \mathcal{K}^*(E^*) < \mathcal{K}^*(M).$$

By (3.7) this of course gives

(3.9)  
$$\mathcal{K}^*(M_1) + \mathcal{K}^*((M \sim M_1) \cap U_0) < \mathcal{K}^*(M).$$

Next we note that $M_1 \sim \bar{U}$ has fewer components than $M \sim \bar{U}$ (by construction). Thus, by induction on the number of components of $M \sim \bar{U}$, we obtain a sequence $M_0 = M$, $M_1$, $M_2$, ..., $M_k = \hat{M}$, where $\partial M_j = \partial M$, where $\hat{M} \sim \bar{U}$ has no components $E$ such that $E \cap \partial M = \emptyset$ (notice this gives conclusion (vii)), and where, for $j = 1, ..., k$, $M_j$ intersects $\partial U$ transversally, $\partial M_j = \partial M$, $M_j \sim \bar{U} \subset M_{j-1} \sim \bar{U}$, $M_j \cap U_0 \subset M_{j-1} \cap U_0$, and

$$\mathcal{K}^*(M_j) + \mathcal{K}^*((M_{j-1} \sim M_j) \cap U_0) < \mathcal{K}^*(M_{j-1}).$$
Summing over \( j \) in this last inequality, we then obtain (vi). The remaining conclusions are now evident.

It thus remains to prove the last part of the theorem subject to the additional hypothesis (viii). Let \( N_1, \ldots, N_k \) be as in (vii) and suppose that (ix) fails; then we must have
\[
\inf \{ \mathcal{K}^j(P_j): P_j \in \mathcal{M}, \quad \partial P_j = \partial N_j \} = \mathcal{K}^j(N_j) - \alpha_j,
\]
for \( j = 1, \ldots, k \), where \( \sum_{j=1}^{k} \alpha_j > \theta \). Then choose \( \bar{P}_1, \ldots, \bar{P}_k \in \mathcal{M} \) with \( \partial \bar{P}_j = \partial N_j \) and
\[
\mathcal{K}^j(\bar{P}_j) + \beta_j < \mathcal{K}^j(N_j),
\]
\[
\sum_{j=1}^{k} \beta_j > \theta.
\]

By applying the first part of the theorem with \( \bar{P}_j \) in place of \( M \), we see by conclusion (vii) that we may take \( \bar{P}_j \) such that \( \bar{P}_j \approx \partial \bar{P}_j \subset U \). By Corollary 1, we can also choose the \( \bar{P}_j \) to be pairwise disjoint. We can then define \( M^* = \left( \bar{M} \cup \bigcup_{j=1}^{k} N_j \right) \cup \left( \bigcup_{j=1}^{k} \bar{P}_j \right) \). Clearly \( M^* \) is homeomorphic to \( D_t \) and \( \partial M^* = \partial \bar{M} = \partial M \). Also by (3.10) and (3.11) we have
\[
\mathcal{K}^j(M^*) = \mathcal{K}^j(M) - \sum_{j=1}^{k} \mathcal{K}^j(N_j) + \sum_{j=1}^{k} \mathcal{K}^j(\bar{P}_j) < \mathcal{K}^j(\bar{M}) - \theta < \mathcal{K}^j(M) - \theta.
\]
That is
\[
\mathcal{K}^j(M^*) + \theta < \mathcal{K}^j(M);
\]
in view of the fact that \( M^* \) may be smoothed to give \( M' \in \mathcal{M} \) having the same boundary as \( M^* \) and area arbitrarily close to \( M^* \), this last inequality contradicts the assumption (viii).

(3.12) Remark. By applying the above theorem with \( A \) in place of \( U \) and with \( M_k \) (as in § 1) in place of \( M \) (4), and noting again that \( \text{spt} \| V \| \subset A \cup \Gamma \), it is not difficult to see that we can replace the sequence \{\( M_k \)\} by a sequence \{\( \bar{M}_k \)\} such that \( \partial \bar{M}_k = \Gamma \), \( \bar{M}_k \subset A \cup \Gamma \), \( \lim_{k \to \infty} \mathcal{K}^j(\bar{M}_k) = \lim_{k \to \infty} \mathcal{K}^j(M_k) \)
\[
\left(= \inf \{ \mathcal{K}^j(M): M \in \mathcal{M}, \partial M = \Gamma \right) \) and \( \left(\lim_{k \to \infty} \nu(M_k)\right) \perp (R^2 \sim \Gamma) \times G(3, 2) = V \perp (R^2 \sim \Gamma) \times G(3, 2). \) That is, we may assume that the sequence \{\( M_k \)\} of § 1 is such that \( M_k \subset A \cup \Gamma \), \( k = 1, 2, \ldots \), without changing the varifold limit \( V \) in \( R^2 \sim \Gamma \). We will therefore henceforth make this assumption.

(4) If \( M_k \) does not intersect \( \partial A \) transversally, it is necessary to apply Theorem 1 to a slightly perturbed version of \( M_k \). That a transversally intersection surface can be obtained by arbitrarily slight perturbations of \( M_k \) holding \( \Gamma \) fixed follows from Sard’s Theorem together with the fact that if \( \Gamma' \subset M_k \sim \Gamma \) is a \( C^4 \) Jordan curve, then \( \Gamma, \Gamma' \) are not linked. (This last fact is a consequence of the orientability of \( M_k \), and does not depend on the fact that \( M_k \) has genus zero).
4. - Filigree lemma.

Here the word «filigree» means (very roughly) «a collection of thread-like protrusions from a surface». For example, if \( M \) is a surface in \( \mathbb{R}^3 \) with \((\bar{M} \sim M) \cap U(x_0, \epsilon) = \emptyset\) and \( \mathcal{K}^2(M \cap U(x_0, \epsilon)) = \epsilon \), where \( \epsilon \) is small, then \( M \cap U(x_0, \epsilon/2) \) would be classed as filigree.

The following lemma will enable us to «prune off» such sets under appropriate circumstances.

**Lemma 3 (Filigree Lemma).** Suppose \( \{ Y_t \}_{t \in [0, 1]} \) is an increasing family of convex sets with \( Y_t = \{ x \in \mathbb{R}^3 : f(x) < t \}, \ t > 0, \) where \( f \) is a non-negative function on \( \mathbb{R}^3 \) which is \( C^2 \) on \( \mathbb{R}^3 \sim \bar{Y}_0 \), \( Df \neq 0 \) on \( Y_1 \sim \bar{Y}_0 \) and

\[
\sup_{Y_t \sim \bar{Y}_t} |Df| < c_1
\]

for some constant \( c_1 > 0 \). Suppose also that there is a constant \( c_2 < \infty \) such that, whenever \( \Gamma \) is a \( C^2 \) Jordan curve which is nullhomotopic in \( \partial Y_t \), then there is a region \( E \subset \partial Y_t \) with \( \partial E = \Gamma \) and

\begin{equation}
\mathcal{K}^2(E) < c_2 (\mathcal{K}^1(\Gamma))^2. \quad (*)
\end{equation}

Finally, suppose \( M \in \mathcal{M} \) and \( \epsilon > 0 \) are such that \( \partial M \) is contained in the union of the unbounded components of \( \mathbb{R}^3 \sim (\bar{Y}_t \cup (M \sim \partial M)) \) for all \( t \in (0, 1) \), and

\begin{equation}
\mathcal{K}^2(M) < \mathcal{K}^2(N) + \epsilon, \quad \forall N \in \mathcal{M} \text{ with } \partial N = \partial M.
\end{equation}

Then

\begin{equation}
\mathcal{K}^2(M \cap Y_t) < 2\epsilon \text{ whenever } t < 1 - 2c_1 \sqrt{c_2 \sqrt{\mathcal{K}^2(M \cap \bar{Y}_t)}}.
\end{equation}

**Proof.** By Sard’s Theorem, we know \( M \) intersects \( Y_t \) transversally for almost all \( t \in (0, 1) \). Then for almost all \( t \in (0, 1) \) we can apply Theorem 1, with \( Y_t \) in place of \( U \), to give \( \bar{M} \) such that \( \mathcal{K}^2(\bar{M}) < \mathcal{K}^2(M) \), \( \bar{M} \cap \partial Y_t \subset \)

\(^(*)\) There exists such a constant \( c_2 \) in the cases when all the \( \partial Y_t \) are spheres, or planes, or cylinders with circular cross-section; these are the only cases considered subsequently.
\( c \mathcal{M} \cap \partial Y_t, \text{ and } \overline{\mathcal{M}} \cap \overline{Y}_t = \bigcup_{j=1}^k N_j \text{ for some } N_j \in \mathcal{M} \text{ satisfying} \)

\[
\sum_{j=1}^k N_j = 0 \quad \forall N \in \mathcal{M} \text{ with } \partial N = \partial N_j,
\]

where \( \sum_{j=1}^k \varepsilon_j < \varepsilon \).

Let \( j \in \{1, \ldots, k\} \) and let \( I_j = \partial N_j \). By hypothesis we have an \( E \subset \partial Y_t \) with \( \partial E = I_j \) and with (4.1) holding. However, by (4.4) we have

\[
\mathcal{H}^2(N_j) < \mathcal{H}^2(E) + \varepsilon_j,
\]

hence

\[
\mathcal{H}^2(N_j) < \varepsilon_j.
\]

Summing over \( j \), we then have

\[
\mathcal{H}^2(\overline{\mathcal{M}} \cap Y_t) < \varepsilon \mathcal{H}^1(\overline{\mathcal{M}} \cap \partial Y_t)^2 + \varepsilon.
\]

Since \( \overline{\mathcal{M}} \cap \partial Y_t \subset \mathcal{M} \cap \partial Y_t \), this gives

\[
\mathcal{H}^2(\overline{\mathcal{M}} \cap Y_t) < c_2(\mathcal{H}^1(\overline{\mathcal{M}} \cap \partial Y_t))^2 + \varepsilon.
\]

However \( \mathcal{H}^2(\mathcal{M} \cap Y_t) < \mathcal{H}^2(\overline{\mathcal{M}} \cap Y_t) + \varepsilon \) (by (4.2) with \( N = \overline{\mathcal{M}} \) together with the fact that \( \overline{\mathcal{M}} \sim Y_t \subset \mathcal{M} \sim Y_t \)). Hence (4.5) gives

\[
\mathcal{H}^2(\mathcal{M} \cap Y_t) < c_2(\mathcal{H}^1(\mathcal{M} \cap \partial Y_t))^2 + 2\varepsilon.
\]

We can now suppose \( \mathcal{H}^2(\mathcal{M} \cap Y_t) > 2\varepsilon \), otherwise the required conclusion is trivial. Then let \( t_0 = \inf \{ t : \mathcal{H}^2(\mathcal{M} \cap Y_t) > 2\varepsilon \} \) and define \( f(t) = \mathcal{H}^2(\mathcal{M} \cap Y_t) - 2\varepsilon, \ t \in [t_0, 1] \). By the co-area formula and the definition of \( c_2 \) we see that (4.6) implies

\[
f(t) < c_1^2 e_1 (f'(t))^2, \quad \text{a.e. } t \in [t_0, 1].
\]

Integrating this inequality (using the fact that \( f(t) \) is an increasing function of \( t \)), we obtain

\[
\sqrt{f(t)} < \sqrt{f(1)} - \frac{1 - t_0}{2 \sqrt{c_1 e_1}}.
\]

However \( f(1) = \mathcal{H}^2(\mathcal{M} \cap Y_t) - 2\varepsilon < \mathcal{H}^2(\mathcal{M} \cap Y_t) \); hence we deduce \( 1 - t_0 < 2 \sqrt{c_1 e_1} \frac{\sqrt{\mathcal{H}^2(\mathcal{M} \cap Y_t)}}{2} \). That is, \( t_0 > 1 - 2 \sqrt{c_1 e_1} \sqrt{\mathcal{H}^2(\mathcal{M} \cap Y_t)} \), and the required result is proved.
5. Interior regularity.

Theorem 2. Suppose \( \{A_k\} \) is an increasing sequence of open sets such that \( \mathbb{R}^3 \sim A_k \) has no bounded components, \( \{N_k\} \) is a sequence in \( \mathcal{M} \) with \( \partial N_k \subset \partial A_k \), \( N_k \sim \partial N_k \subset A_k \), and

\[
\mathcal{K}^0(N_k) < \mathcal{K}^0(N) + \varepsilon_k, \quad \forall N \in \mathcal{M} \text{ with } \partial N = \partial N_k,
\]

where \( \varepsilon_k \to 0 \) as \( k \to \infty \). Suppose further that \( W = \lim_{k \to \infty} \mathbf{v}(N_k) \) exists in \( V_g(\mathbb{R}^3) \).

Then \( W \) is a stationary integral varifold in \( \bigcup A_k \), with the property that if \( x_0 \in \text{spt} \| W \| \cap \left( \bigcup_{k=1}^{\infty} A_k \right) \) and if \( W \) has a varifold tangent \( C \) at \( x_0 \) with \( \text{spt} \| C \| \subset H \), where \( H \) is a plane, then there is a \( \varrho > 0 \) such that

\[
\| W \| \subset U(x_0, \varrho) \times G(3, 2) = n\mathbf{v}(M),
\]

where \( n \) is a positive integer and \( M \) is an analytic oriented connected minimal surface containing \( x_0 \).

Notice that (in view of (3.12) and the fact that \( \text{spt} \| V \| \subset A \cup \Gamma \)) the above theorem (in case \( A_k = A_k \), \( N_k = M_k \)) implies that the varifold \( V \) of § 1 is a stationary integral varifold in \( \mathbb{R}^3 \sim \Gamma \), and \( V \) is regular in a neighbourhood of any point of \( \mathbb{R}^3 \sim \Gamma \) where there is a varifold tangent with support contained in a plane. (By rectifiability there is such a tangent plane at \( \| V \| \)-almost all points of \( \mathbb{R}^2 \).)

Proof of Theorem 2. \( W \subset \bigcup_{k=1}^{\infty} A_k \times G(3, 2) \) is stationary in \( \bigcup_{k=1}^{\infty} A_k \) because \( \| h_\ast W \| (U) > \| W \| (U) \) whenever \( U \) is a bounded open subset of \( \bigcup_{k=1}^{\infty} A_k \) and \( h \) is a diffeomorphism of \( U \) leaving a neighbourhood of \( \partial U \) fixed.

We first want to show that there is a constant \( c > 0 \) such that

\[
\Theta^2_0(\| W \|, x_0) > c
\]

whenever \( x_0 \in \text{spt} \| W \| \cap \left( \bigcup_{k=1}^{\infty} A_k \right) \). (It follows from this that \( W \subset \bigcup_{k=1}^{\infty} A_k \times G(3, 2) \) is rectifiable by virtue of [AW1, 5.5].) Suppose \( x_1 \in \text{spt} \| W \| \) and \( U(x_1, \varrho) \subset \bigcup_{k=1}^{\infty} A_k \), and let \( c_\ast \) be a constant such that if \( \Gamma_1 \) is a Jordan curve in the unit sphere \( S^2 \), and if \( E_1, E_2 \) are the two components of \( S^2 \sim \Gamma_1 \),
then

$$\min \{\mathcal{H}^s(E_1), \mathcal{H}^s(E_2)\} < c_2(\mathcal{H}^s(I_1))^2.$$ 

By the filigree lemma (which can be applied with \(f(x) = |x|^{1/4}, c_1 = 1/q, \ Y_i = U(x_1, t_i)\), we know that if \(\mathcal{H}^s(N_k \cap U(x_1, g)) < (1/4c_2) g^2\), then \(\mathcal{H}^s(N_k \cap U(x_1, g/2)) < 2\varepsilon_k\). Hence if there is a subsequence \(\{k'\} \subset \{k\}\) with \(\mathcal{H}^s(N_{k'} \cap U(x_1, g)) < (1/4c_2) g^2\), then we would have \((\text{spt} \|W\|) \cap U(x_1, g/2) = \emptyset\), thus contradicting the fact that \(x_1 \in \text{spt} \|W\|\). Hence for all sufficiently large \(k\) we have

$$\mathcal{H}^s(N_k \cap U(x_1, g)) > \frac{1}{4c^2} g^2,$$

from which we deduce

$$\|W\| (U(x_1, g)) > \frac{1}{4c^2} g^2.$$

Thus we obtain (5.2) with \(c = 1/(4c^2)\). In particular, \(W \cup \bigcup_{k=1}^{\infty} A_k \times G(3, 2)\) is rectifiable by [AW1, 5.5].

Now let \(x_0, C, H\) be as in the statement of the theorem. For convenience of notation we will suppose that \(x_0 = 0\) and that \((0, 0, 1)\) is normal to the plane \(H\). We write

$$D_\sigma = \{x \in \mathbb{R}^3: |x| < \sigma\}, \quad K_{0,\sigma} = (D_\sigma \sim \partial D_\sigma) \times (-\sigma, \sigma),$$

and we let \(\mu_s\) denote the transformation of \(\mathbb{R}^3\) defined by \(\mu_s(x) = tx\). By definition of \(C\), we know there is a sequence \(\{r_k\} \to \infty\) such that

$$\mu_{r_k} W \to C \quad \text{as} \quad k \to \infty.$$

By (5.4) it is then clear that for any \(\sigma \in (0, 1)\) we can find \(r\) such that

$$K_{1,1} \cap \text{spt} \|\mu_{r_k} W\| \subset K_{1,\alpha/2}.$$

We can of course also choose \(r\) such that

$$\mathcal{H}^s(\partial \text{spt} \|\mu_{r_k} W\| \cap (\partial D_1 \times \mathbb{R})) = 0.$$

Henceforth we will suppose that \(r\) has been thus chosen.

Now by hypothesis

$$\mathcal{H}^s(\mu_{r_k}(N_0)) < \mathcal{H}^s(N) + r^2 \varepsilon.$$
for every \( N \in \mathcal{M} \) with \( \partial N = \tilde{\partial} \mu_\alpha(N, \alpha) \). From (5.6) and the co-area formula we know that for almost all \( \sigma \in (\sigma_0/2, 1) \)

\[
\mathcal{H}^1(\mu_\alpha(N) \cap (D_\sigma \times (\{- \sigma \} \cup \{\sigma\})) \to 0 \quad \text{as } k \to \infty.
\]

Thus for any given \( \eta > 0 \) we can assert that, for sufficiently large \( k \), there is a \( \alpha_\eta \in (3/4, \sigma_0) \) such that

\[
\mathcal{H}^1(\mu_\alpha(N, \alpha_\eta) \cap (D_\sigma \times (\{- \sigma_\eta \} \cup \{\sigma_\eta\})) < \eta.
\]

Hence, assuming \( \eta \) is taken small enough, we can deduce

\[
\mu_\alpha(N_k) \cap (\partial D_{\alpha_k} \times (\{- \sigma_k \} \cup \{\sigma_k\})) = \emptyset,
\]

for some \( \alpha_k \in (3/4, 1) \). We can also arrange, by Sard's theorem, that \( \mu_\alpha(N_k) \) intersects both \( D_{\alpha_k} \times (\{- \sigma_k \} \cup \{\sigma_k\}) \) and \( \partial D_{\alpha_k} \times [-1, 1] \) transversally.

We now apply Theorem 1 with \( \mu_\alpha(N_k) \) in place of \( N \) and with \( K_{\alpha_k, \alpha_k} \) in place of \( U \) (\( \star \)). Then we find \( P^1_k, \ldots, P^i_k, \ldots, P^{i_k+1}_k \), \( i_k \in \mathbb{M} \), where \( R_k, R_k \) are integers depending on \( k \), such that

\[
\partial P^1_k, \ldots, \partial P^{i_k}_k \subset D_{\alpha_k} \times (\{- \sigma_i \} \cup \{\sigma_i\})
\]

and such that (by virtue of Theorem 1)

\[
\mathcal{H}^2(P^i_k) \leq \mathcal{H}^2(P) + \varepsilon_k, \quad \forall P \in \mathbb{M} \text{ with } \partial P = P^i_k,
\]

\( i = 1, \ldots, R_k \), where \( \sum_{i=1}^{R_k} \varepsilon_{k,i} < r^2 \varepsilon_k \), and (by (5.7) and [AW1, 2.6(2) (d)])

\[
(\mu_{\epsilon_k} W) \perp K_{1,1} \times G(3, 2) = \lim_{k \to \infty} \sum_{i=1}^{R_k} v(P^i_k \cap K_{1,1}).
\]

Next we claim that \( P^i_k, i = R_k + 1, \ldots, R_k \) can be discarded without changing the varifold limit in (5.12); that is, we claim

\[
(\mu_{\epsilon_k} W) \perp K_{1,1} \times G(3, 2) = \lim_{k \to \infty} \sum_{i=1}^{R_k'} v(P^i_k \cap K_{1,1}).
\]

\( \star \) To be strictly precise, we first apply Theorem 1 with the * edges * \( \partial D_{\alpha_k} \times (\{- \sigma_i \} \cup \{\sigma_i\}) \) smoothed out; because of (5.10), no modifications are needed in subsequent parts of the argument.
To justify this we notice that, by (5.11) and the isoperimetric inequality, 
each of the curves $\partial P^i_k$, $i = R'_k + 1, \ldots, R_k$ encloses an area $A_k \subset \partial D \times \times (\{-\sigma\} \cup \{\sigma\})$ which is such that $\mathcal{H}^n(A_k^{n}) < (4\pi)^{-1} \eta^2$; hence by (5.11) $\mathcal{H}^n(P^i_k) \eta^2 > \eta^2$ for all sufficiently large $k$. Thus, applying the filigree lemma with $f(x) = \left| x_j / \sigma \right|$, $Y_i = \{x : \left| x_j / \sigma \right| < t\}$, $a_i = 2\sigma^{-1}$ and $e_i = (4\pi)^{-1}$, we deduce that, provided $\eta$ is sufficiently small ($\eta < \sigma / 4$ will suffice),

$$\mathcal{H}^2(P^i_k \cap K_{1, c_{1/2}}) < 2\epsilon_i k.$$  

(5.13) now follows from this, (5.6) and (5.12).

Suppose now that $i \in \{1, \ldots, R'_k\}$ and $\Gamma_k^i = \partial P^i_k$ is nullhomotopic in $\partial D \times \{-\sigma, \sigma\}$. Then letting $A_k^i$ be that part of the cylinder $\partial D \times \mathbb{R}$ interior to $\Gamma_k$, we clearly have

$$\mathcal{H}^2(A_k^i) < \mathcal{H}^n(\partial D \times \{-\sigma, \sigma\}) = 4\pi \sigma \sigma.$$  

By using (5.11) with $N = A_k^i$, we deduce

$$\mathcal{H}^2(P^i_k) < 4\pi \sigma + \epsilon_{k,i} < 5\pi \sigma$$

for sufficiently large $k$. Hence we can again use the filigree lemma, this time with $Y_i = \{x = (x_1, x_2, x_3) : x_1^2 + x_2^2 < t \theta_0, f(x) = \theta_k^{-1} \sqrt{x_1^2 + x_2^2}, a_i = \theta_k^{-1}$, and a suitable $e_i$. (By homogeneity we can take $e_i$ to be such that (4.1) holds for any null-homotopic $\Gamma_i$ in $Y_1$.) We thus obtain

$$\mathcal{H}^2(P^i_k \cap K_{1, c_1}) < 2\epsilon_{i,k},$$

provided $\sigma$ is sufficiently small. Hence by (5.13) we have

$$\sum_{i \in \mathcal{S}_k} \mathcal{H}^2(P^i_k \cap K_{1, c_1}) \rightarrow (\mu, W) \sqsubseteq K_{1, c_1} \times G(3, 2),$$

where $\mathcal{S}_k$ denotes the collection of those $P^i_k (i \in \{1, \ldots, R'_k\})$ which are such that $\partial P^i_k$ is not null-homotopic in $\partial D \times \{-\sigma, \sigma\}$. For each $i \in \mathcal{S}_k$ we clearly have

$$\pi \epsilon^2 < \mathcal{H}^2(P^i_k \cap K_{1, c_1}), \quad \epsilon \in (0, \theta_0].$$

Also, by letting $A^i_k$ denote the component of $(\partial D \times \{-\sigma, \sigma\}) \sim P^i_k$ which contains the circle $\partial D \times \{\sigma\}$, we define $N = A^i_k \cup (D \times \{\sigma\})$. Clearly $N$ is homeomorphic to $D_1$ (via a bilipschitz homeomorphism) and $\mathcal{H}^n(N) < \pi \epsilon^2 + 4\pi \sigma \sigma$. However we can now use (5.11) (with $N$ is place of $P$)
to deduce
\begin{equation}
\mathcal{K}^2(P_k^i) < \pi\sigma_0^2 + 5\pi\sigma_0
\end{equation}
for all sufficiently large $k$.

Using (5.15) together with the fact that $\sum_{i \in I_k} v(P_k^i \cap K_{i,1})$ is bounded (e.g. by (5.14)), we then deduce that $\text{card } I_k$ is bounded independent of $k$. Thus we can find a positive integer $n$ and a subsequence $\{k'\} \subset \{k\}$ such that, after a suitable relabelling of $P_k^i$,

(i) $q_{k'} \to q_0 \in [\frac{2}{3}, 1]$ as $k' \to \infty$;

(ii) for $i = 1, \ldots, n$, $v(P_k^i)$ converges to a varifold $W_i$, where (by (5.15), (5.16) and (5.6)), for each $q \in (0, 1]$, \begin{equation}
\pi q^2 < \|W_i\|(K_{0,1}) - \|W_i\|(K_{1,1}) \leq \pi + 5\pi\sigma_0 q_0^{-2} < \pi + 32\sigma_0,
\end{equation}
and
\begin{equation}
\text{spt } W_i \subset K_{1,s},
\end{equation}
and
\begin{equation}
(\mu_{q_0^{-1}} \ast W) \cap K_{1,s} \times G(3, 2) = \sum_{i=1}^n W_i \cap K_{1,1} \times G(3, 2).
\end{equation}

Now by (5.17), (5.18) and the fact that $K_{1-s,1} \subset U(0, 1) \subset K_{1,1}$ we have
\begin{align*}
\|W_i\|(U(0, 1)) &> \|W_i\|(K_{1-s,1}) = \|W_i\|(K_{1-s,1}) \\
&> \pi(1 - \sigma_0)^2
\end{align*}
and
\begin{align*}
\|W_i\|(U(0, 1)) &< \|W_i\|(K_{1,1}) < \pi + 32\sigma_0.
\end{align*}

Then by using the arbitrariness of $\sigma_0$, together with (5.18), (5.19) and the monotonicity of $q^{-2}\|W\|(U(x_0, q))$ ([AW1, 5.1(3)]), we conclude that $\Theta(\|W\|, x_0) = n$ for some positive integer $n$. Thus we have proved that $\Theta(\|W\|, x_0)$ is a positive integer whenever $x \in \text{spt } W \cap \bigcup_{k=1}^\infty A_k$ and $W$ has a varifold tangent at $x_0$ supported in a plane. Since $W \subset \bigcup_{k=1}^\infty A_k \times G(3, 2)$ is rectifiable, we know that such a varifold tangent exists for $\mathcal{K}^2$-almost all $x_0 \in \text{spt } W \cap \left(\bigcup_{k=1}^\infty A_k\right)$. Hence we finally deduce that $W$ is a stationary
integral varifold in $\bigcup_{k=1}^{\infty} A_k$, thus completing the proof of the first part of the theorem.

We can now apply this first part of the theorem to the sequences $\{v_{\varepsilon k}(P_i^k)\}$ (in place of $\{N_k\}$), and hence deduce that each $W_i$ in (5.17)-(5.19) is a stationary integral varifold in $K_{i,1}$. But then by (5.17) we can apply the regularity theorem of W. Allard [AW1, §8]; for small enough $\sigma_0$ we then deduce that

$$W_i \bot K_{i,1} \times G(3, 2) = \nu(M_i), \quad i = 1, \ldots, n,$$

where $M_i$ is an analytic minimal surface which can be represented in the form

$$M_i = \{(x, z): z = u_i(x), x \in D_i\},$$

where $u_i$ is a solution of the minimal surface equation on $D_i$ satisfying $\sup_{D_i} |Du_i| < 1$. Further, since $\Theta(\|W\|, x_0) = n$, we easily see from (5.19) that $u_i(0) = 0, \; i = 1, \ldots, n$ (7). On the other hand, from the construction of the $W_i$, it is clear that for each $i, j = 1, \ldots, n$ we have either $u_i(x) > u_j(x)$ for all $x \in D_i$ or $u_i(x) < u_j(x)$ for all $x \in D_i$. But $u_i(0) = u_j(0) = 0$, and hence, since the difference of two solutions of the minimal surface equation satisfies the strong maximum principle, we deduce $u_i = u_j$ on $D_i$. Thus by (5.19) $(\mu_{\sigma, \Theta} W \bot K_{i,1} \times G(3, 2) = n\nu(M_i))$. This completes the proof of Theorem 1.

(5.20) REMARK. For later reference we wish to make a note concerning another consequence of the above theorem and its proof. Suppose instead of the sequence $\{v(N_k)\}$ of varifolds we were to consider the sequence $\{\tau(N_k)\}$ of currents, where $\tau(N)$ denotes the rectifiable current associated (in the usual way) with an oriented surface $N$. With respect to the flat norm topology ([FH: 4.1.12]) we get convergence of some subsequence $\{\tau(N_{k'})\}$ to a rectifiable current $T$, with $\partial T \subseteq \bigcup_{k=1}^{\infty} A_k = 0$. Now if $x_0$ is as in the statement of the above theorem, then an examination of the proof shows clearly that we must have

$$\partial T \subseteq U(x_0, \varepsilon) = n'\nu(M),$$

where $M$ is as in (5.1) and $n'$ is an integer such that $|n'| < n$ and $n - n'$ is even. ($n$ also as in (5.1).) In particular, if $n = 1$ then $n' = \pm 1$.

(*) Of course for all sufficiently large $r$ it follows from (5.17), (5.18) that the integer $n$ in (5.19) is precisely $\Theta(\|W\|, x_0)$. 


6. – Interior regularity (continued).

**Theorem 3.** Suppose \( \{A_k\}, \{N_k\}, W \) are as in Theorem 2, suppose \( B_1, \ldots, B_R \) are distinct open discs of radius \( e \) which intersect in a common diameter \( L \) of a ball \( U(x_0, e) \), where \( B(x_0, e) \subset \bigcup A_k \). Suppose also that \( \text{spt} \| W \| \cap \bigcup_{k=1}^{\infty} U(x_k, e) \subset \bigcup B_j \) and that \( \text{spt} \| W \| \cap (B_1 \sim L) \neq \emptyset \). Then

\[
W \subseteq U(x_0, e) \times G(3, 2) = n\nu(B_1)
\]

for some positive integer \( n \).

**Proof.** For each \( i = 1, \ldots, R \), let \( B_i^+, B_i^- \) be \( \frac{1}{2} \)-discs with common diameter \( L \) such that \( B_i^+ \cup B_i^- = B_i \). According to Theorem 2

\[
(6.1) \quad W \subseteq U(x_0, e) \times G(3, 2) = \sum_{j=1}^{R} (n_j^+ \nu(B_j^+) + n_j^- \nu(B_j^-)),
\]

where \( n_j^+, n_j^- \) are non-negative integers.

We now consider 3 cases:

**Case I.** \( \sum_{j=1}^{R} (n_j^+ + n_j^-) < 2 \).

Using the fact that \( W \) has first variation zero in \( U(x_0, e) \), the case

\[
\sum_{j=1}^{R} (n_j^+ + n_j^-) = 1
\]

is easily shown to be impossible, hence we need only consider the possibility \( \sum_{j=1}^{R} (n_j^+ + n_j^-) = 2 \). However, using the facts that \( W \) is stationary in \( U(x_0, e) \) and that \( \text{spt} \| W \| \cap (B_1 \sim L) \neq \emptyset \), it is easy to show in this case that we must have \( n_i^+ = n_i^- = 1 \) (and \( n_j^+ = n_j^- = 0 \), \( j \neq 1 \)). Hence the required result, with \( n = 1 \), is established in Case I.

**Case II.** \( \sum_{j=1}^{R} (n_j^+ + n_j^-) = 3 \).

In this case we can find 3 half-discs \( H_1, H_2, H_3 \) with common diameter \( L \) such that

\[
W \subseteq U(x_0, e) \times G(3, 2) = \sum_{i=1}^{3} \nu(H_i);
\]

furthermore, using the fact that \( W \) has first variation zero in \( U(x_0, e) \), one easily shows that \( H_1, H_2, H_3 \) are distinct. (In fact they must meet along \( L \) at angles of 120°.) Now by the remark (5.20) we know that (provided the \( H_i \) are appropriately oriented and provided we use the notation of (5.20))

\[
T \subseteq U(x_0, e) = \sum_{i=1}^{3} \tau(H_i).
\]
However we then clearly have $\partial T \subseteq U(x_0, q) \neq 0$, thus contradicting the fact that $\partial T \subseteq \bigcup_{k=1}^{\infty} A_k = 0$.

**Case III.** \[ \sum_{j=1}^{R} (n_j^+ + n_j^-) > 4. \]

We will prove in this case that for each sufficiently small $\theta \in (0, q)$, there is a positive integer $n$ such that

\[ W \subseteq U_0 \times G(3, 2) = n\nu(B_i \cap U_0), \]

where, here and subsequently, $U_0 = U(x_0, q - \theta)$. This clearly suffices to establish the desired result.

Choose a $C^2$ convex open set $G$ such that $\bigcup_{j=1}^{R} B_j \subseteq G \subseteq U(x_0, q)$, and

\[ \mathcal{H}^2(\partial G) = 4\pi q^2 - \alpha \quad \text{for some } \alpha > 0. \]

We can suppose $N_k$ intersects $\partial G$ transversally (otherwise modify $N_k$ slightly without upsetting the hypotheses concerning $N_k$), $k = 1, 2, \ldots$. By Theorem 1 (with $G$ in place of $U$ and $N_k$ in place of $M$) we can then find the pairwise disjoint $P_1^k, \ldots, P_j^k \in \mathcal{H}(N_k)$ (for a positive integer) such that $P_k^j \sim \partial P_k^j \subseteq G$, $\partial P_k^j \subseteq \partial G$,

\[ \mathcal{H}^2(P_k^j) < \mathcal{H}^2(P) + \epsilon_{k,j}, \quad \forall P \in \mathcal{H} \text{ with } \partial P = \partial P_k^j, \]

where $\sum_{j=1}^{R} \epsilon_{k,j} < \epsilon_k$, and such that (by conclusions (iv) and (vi) of Theorem 1 together with [AW1, 2.6(2) (d)])

\[ W \subseteq U_0 \times G(3, 2) = \lim_{k \to \infty} \sum_{j=1}^{R} \nu(P_k^j \cap U_0) \]

for any preassigned $\theta > 0$. (Notice that $\text{spt } ||W|| \cap \overline{U_0}$ is a compact subset of $G$ because $\text{spt } ||W|| \cap \overline{U_0} \subseteq \bigcup_{j=1}^{R} B_j \cap \overline{U_0}$, hence (6.5) does follow from Theorem 1.)

By (6.4) and the filigree lemma, there is an $\alpha > 0$ (independent of $k$, but depending on $\theta$) such that if $\mathcal{H}^2(P_k^j \cap U_0) < \alpha$, then $\mathcal{H}^2(P_k^j \cap U_{2\theta}) < 2\epsilon_{k,j}$; hence we have by (6.5) and [AW1, 2.6(2) (d)] that

\[ \lim_{k \to \infty} \sum_{j \in \mathcal{H}} \nu(P_k^j \cap U_{2\theta}) = W \subseteq U_{2\theta} \times G(3, 2), \]
where $\mathcal{T}_k$ is the set of $j$ such that $\mathcal{H}^2(P_k \cap U_0) > \alpha$. In view of the fact that 
$$\limsup_{k \to \infty} \sum_{i \in J_k} \mathcal{H}^2(P_k \cap U_i) < \infty \quad \text{by (6.5)}$$
we then deduce that $\text{card } (\mathcal{T}_k)$ is bounded independent of $k$. Thus we can select a subsequence \{$k'$\} \subseteq \{k\} and assert that (after a suitable relabelling of the $P_k$)
$$\lim_{k' \to \infty} \sum_{i=1}^n \mathcal{v}(P_{k'} \cap U_{2\theta}) = W \cup U_0 \times G(3, 2).$$

We can choose the subsequence \{$k'$\} such that each individual sequence \{v($P_{k'}$)\}, $j = 1, \ldots, n$, has a varifold limit $W_j$; thus

\begin{equation}
W_j = \lim_{k' \to \infty} \mathcal{v}(P_{k'}), \quad j = 1, \ldots, n,
\end{equation}

$$W \cup U_0 \times G(3, 2) = \sum_{i=1}^n W_i \cup U_0 \times G(3, 2).$$

By (6.4) and Theorem 2 (with $P_{k'}$ in place of $N_k$), we know that $W_i$ is a stationary integral varifold in $G_i$ and since 

spt $W_i \cap U_0 \subseteq \bigcup_{j=1}^R U_j$, Theorem 2 also implies that

\begin{equation}
W_i \cup U_0 \times G(3, 2) = \sum_{i=1}^R \left( m_i^+ \mathcal{v}(B_i^+ \cap U_0) + m_i^- \mathcal{v}(B_i^- \cap U_0) \right),
\end{equation}

where $m_i^+$, $m_i^-$ are non-negative integers.

Now by (6.4),

$$\mathcal{H}^2(P_{k_i}) \leq \min \{\mathcal{H}^2(S^+), \mathcal{H}^2(S^-)\} + \epsilon_{k,i},$$

where $S^+$, $S^-$ are the components of $\partial G \sim \partial P_{k_i}$. In view of (6.3), this last inequality gives

$$\mathcal{H}^2(P_{k_i}) \leq 2\pi \theta^2 - \frac{\alpha}{2} + \epsilon_{k,i},$$

whereupon we must have

$$\|W\|_{(U_0)} < 2\pi(q - \theta)^2$$

for sufficiently small $\theta \in (0, q)$. But then the $m_i^+$, $m_i^-$ in (6.6) must satisfy

$$\sum_{i=1}^R (m_i^+ + m_i^-) < 3,$$

and hence we deduce by Cases I and II above that

\begin{equation}
W_i \cup U_0 \times G(3, 2) = \mathcal{v}(B_i \cap U_0),
\end{equation}
for any \( j \in \{1, \ldots, n\} \) and \( k \in \{1, \ldots, R\} \) with \( \text{spt} \|W_j\| \cap B_k \cap U_\theta \neq \emptyset \). Because \( \text{spt} \|W\| \cap (B_1 \sim L) \neq \emptyset \), we see that (for any \( \theta \in (0, \varrho) \) such that \( \text{spt} \|W\| \cap (B_i \sim L) \neq \emptyset \)) there is a \( j \), say \( j_1 \), such that (6.7) holds with \( j = j_1 \) and \( k = 1 \). If on the other hand there is an \( i \neq 1 \) such that \( \text{spt} \|W\| \cap (B_i \sim L) = \emptyset \) then (for \( \theta \in (0, \varrho) \) sufficiently small) there is a \( j_2 \) such that (6.7) holds with \( j = j_2 \) and \( k = i \).

Thus we would have

\[(6.8) \quad v(P_{ki} \cap U_\theta) \to v(B_1 \cap U_\theta), \quad v(P_{ki} \cap U_\theta) \to v(B_i \cap U_\theta), \quad i \neq 1.\]

This is clearly impossible in view of the fact that \( P_{ki} \), \( P_{ki} \) are disjoint elements of \( \mathcal{M} \) for each \( k' \). (For example we can argue as follows: since \( P_{ki} \cap P_{ki} = \emptyset \), we have disjoint open sets \( U_{ki} \), \( U_{ki} \) in \( U(x_0, \varrho) \) with \( \partial U_{ki} = P_{ki} \), \( r = 1, 2 \). But by (6.8) and (5.20) we see that the characteristic functions of \( U_{ki} \), \( U_{ki} \) converge in the \( L^1 \) sense to the characteristic functions of hemispheres \( U^1 \), \( U^2 \) of \( U(x_0, \varrho) \) with \( \partial U^1 \cap U(x_0, \varrho) = B_1 \) and \( \partial U^2 \cap \cap U(x_0, \varrho) = B_i \). Such convergence is clearly impossible in view of the disjointness of \( U_{ki} \) and \( U_{ki} \).) Thus we deduce that \( W_j \cap U_\theta \cap G(3, 2) \neq \emptyset \) whenever \( W_j \cap U_\theta \cap G(3, 2) \neq \emptyset \), \( j = 1, \ldots, n \). The desired conclusion (6.2) now follows from (6.5).

**Corollary 2.** Let \( V \) be as in § 1. Then at each point \( x_0 \in \text{spt} \|V\| \sim \Gamma \) there is a varifold tangent \( C \) of \( V \) having the form \( C = n v(H) \), where \( H \) is a plane in \( \mathbb{R}^3 \) and \( n \) is a positive integer.

**Proof.** For convenience of notation we suppose \( x_0 = 0 \). Let \( \{a_k\} \to \infty \) be a sequence such that \( C = \lim_{k \to \infty} \mu_{a_k} V \). Clearly we can find a sequence \( \{N_k\} \subset \mathcal{M} \) such that

\[
N_k = \mu_{a_k}(M_k) \quad (r_k \to \infty \text{ as } k \to \infty)
\]

\[
v(N_k) \to C \quad \text{as } k \to \infty,
\]

and

\[
\mathcal{C}^2(N_k) \leq \mathcal{C}^2(N) + \varepsilon'_k, \quad \forall N \in \mathcal{M} \text{ with } \partial N = \partial N_k,
\]

where \( \varepsilon'_k \to 0 \) as \( k \to \infty \).

If \( \text{spt} \|C\| \) is contained in a plane, Theorem 2 applies and the required result is immediately obtained. If \( \text{spt} \|C\| \) is not contained in a plane, then it follows (see [AW1, 6.5], [AA]) that we can find \( x_1 \neq 0 \) and \( \varrho > 0 \) such that

\[
\text{spt} \|C\| \cap B(x_1, \varrho) = \bigcup_{i=1}^{B} H_i,
\]
where \( R > 2 \) and \( H_1, \ldots, H_R \) are distinct closed half-discs having common
diameter \( L \), \( L \) also being a diameter of \( B(x_1, \rho) \). But then we can apply
Theorem 3 with \( C \) in place of \( W \), whereupon we deduce

\[
C \subset U(x_1, \rho) \times G(3, 2) = n , v(B)
\]

for some disc \( B \), where \( n \) is a positive integer. This completes the proof.

By combining Theorems 2 and 3 the following interior regularity the-
orem is now evident.

**Theorem 4 (Interior regularity of \( V \)).** Let the varifold \( V \) be as in § 1.
At each point \( x_0 \in \text{spt } V \) there is a positive integer \( n_{x_0} \), a \( \theta_{x_0} > 0 \), and
an analytic minimal surface \( M_{x_0} \) such that

\[
V \subset U(x_0, \rho) \times G(3, 2) = n_{x_0} v(M_{x_0}) .
\]

7. – Boundary regularity.

In this section \( v_0 \) denotes a fixed point of the boundary curve \( \Gamma \subset \partial A \).
Since \( \Gamma \) is \( C^2 \) and \( A \) is uniformly convex, we know there are planes

\[
\pi_{x_0}^+ = \{ x \in \mathbb{R}^3 : (x - x_0)^+ \cdot v_0^+ = 0 \}, \quad \pi_{x_0}^- = \{ x \in \mathbb{R}^3 : (x - x_0)^- \cdot v_0^- = 0 \},
\]

where \( v_0^+, v_0^- \in S^2 \) satisfy \( |v_0^+ \cdot v_0^-| < 1 \), such that \( \pi_{x_0}^+ \cap \Gamma = \pi_{x_0}^- \cap \Gamma = \{ x_0 \} \) and

(7.1) \( \Gamma \sim \{ x_0 \} \subset W = \{ x \in \mathbb{R}^3 : (x - x_0)^+ \cdot v_0^+ > 0 , \quad (x - x_0)^- \cdot v_0^- > 0 \} .
\]

We assume that the minimizing sequence \( \{ M_k \} \) of § 1 has been chosen
so that \( M_k \sim \Gamma \subset A , k = 1, 2, \ldots \). (By Remark (3.12) this can always be
arranged without any change in \( V \subset (\mathbb{R}^3 \sim \Gamma) \times G(3, 2) .\))

We let \( V = V \subset (\mathbb{R}^3 \sim \Gamma) \times G(3, 2) \), and we claim that

(7.2) \( \Gamma \subset \text{spt } V' \).

In fact some subsequence of \( \tau(M_k) \) (notation as in (5.20)) converges with
respect to the flat norm to an integral current \( T \) such that \( \partial T = \tau(\Gamma) \),
where \( \tau(\Gamma) \) denotes the one-dimensional current associated with a suitably
oriented version of \( \Gamma \). Thus \( \Gamma \subset \text{spt } T \). On the other hand (by (5.20))
\( \text{spt } T(= \text{spt } T \subset (\mathbb{R}^3 \sim \Gamma)) \subset \text{spt } V' \), and hence we deduce (7.2) as required.

Now define \( L = \pi_{x_0}^+ \cap \pi_{x_0}^- \), and henceforth (for convenience of notation
only) assume \( x_0 = 0 \in \Gamma \). Letting \( C_+ = \lim_{k \to \infty} \mu_{r_k} V' \) (\( r_k \to \infty \) as \( k \to \infty \)) be any varifold tangent of \( V' \) at 0 ([AW1, 3.4(1)]) we then deduce that the line \( L \) defined above is contained in \( \text{spt} \|C_+\| \) (by applying [AW2, 3.4(1)] to each \( \mu_{r_k} V' \)). Furthermore (by applying [AW2, 3.4(2)] to each \( \mu_{r_k} V' \)) we deduce

\[
(7.3) \quad \|C_+\|(L) = 0
\]

\[
(7.4) \quad r^{-2}\|C_+\|(U(x_1, r)) \text{ is non-decreasing in } r \text{ for } x_1 \in L.
\]

By using (7.4) with \( x_1 = 0 \), together with the definition \( C_+ \), we then see that

\[
(7.5) \quad r^{-2}\|C_+\|(U(0, r)) = \Theta(\|V'\|, 0), \quad \forall r > 0.
\]

Also, by definition of \( C_+ \), we can use the sequence \( \{M_k\} \) to construct a sequence \( \{N_k\} \in \mathcal{M} \) with the following properties:

\[
(7.6) \quad N_k \subset W \cup L(W \text{ as in (7.1)}), \quad \partial N_k \cap U(0, 1) \subset L \cap U(0, 1);
\]

\[
(7.7) \quad k^2(N_k) + 1/k, \quad \forall N \in \mathcal{M} \text{ with } \partial N = \partial N_k;
\]

\[
(7.8) \quad \lim_{k \to \infty} v(N_k) \sqsubseteq (\overline{W} \sim L) \times G(3, 2) = C_+ \sqsubseteq (\overline{\mathbb{R}^3} \sim L) \times G(3, 2).
\]

Thus by Theorem 2 we deduce that \( C_+ \) is a stationary integral varifold in \( \mathbb{R}^3 \sim L \), and

\[
(7.9) \quad \text{spt} \|C_+\| \subset \overline{W}.
\]

Next, define \( C = C_+ + q_+ C_+ \), where \( q \) is the reflection of \( \mathbb{R}^3 \) through the line \( L \). By (7.3) and the reflection principle [AW2, 3.2], \( C \) is a stationary integral varifold in \( \mathbb{R}^3 \). Then by (7.5) and [AW1, 5.1(2)] it is clear that \( \mu_{r_k} C = C, \forall r > 0 \), hence

\[
(7.10) \quad \mu_{r_k} C_+ = C_+, \quad \forall r > 0.
\]

By Theorem 3 together with the classification of 1-dimensional stationary varifold tangents, given in [AA], we then know that (since \( C_+ \) is integral and \( L \subset \text{spt} \|C_+\| \subset \overline{W} \))

\[
(7.11) \quad C_+ = \sum_{i=1}^{R} v(H_i),
\]

where \( H_1, ..., H_R \) are half-planes with common boundary \( L \) and \( \bigcup_{i=1}^{R} H_i \subset \overline{W} \).
Our aim now is to show that \( R = 1 \). We let

\[
U^+(0, q) = \{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 : |x| < q, x_3 > 0 \}.
\]

and suppose (for notational convenience) that \( W \subset \{ x \in \mathbb{R}^3 : x_3 > 0 \} \).

By Sard's theorem we can find \( q_k \in (\frac{1}{k}, 1) \) such that \( N_k \) intersects \( \partial U^+(0, q_k) \) transversally. (Since \( \partial U^+(0, q_k) \) is not smooth at the edge \( E = \{ x = (x_1, x_2, x_3) : x_3 = 0, |x| = q_k \} \) this last statement has to be appropriately interpreted; at points in \( E \) we require the tangent plane of \( N_k \) to differ from both the tangent plane of \( \partial U(0, q_k) \) and the plane \( x_3 = 0 \).) Then \( N_k \cap \partial U^+(0, q_k) = \bigcup_{j=1}^{n_k} \Gamma^j_k \), where \( \Gamma^j_k \) are pairwise disjoint Jordan curves such that \( \Gamma^j_k \) is a union of \( L \cap U(0, q_k) \) and a \( C^2 \) Jordan arc \( \gamma_k \subset \partial U(0, q_k) \cap \bigcap W \), and where \( \Gamma^1_k, ..., \Gamma^{n_k}_k \) are \( C^2 \) Jordan curves contained in \( \partial U(0, q_k) \cap W \). Let \( \varepsilon > 0 \) be given. Because of (7.8), (7.11), it is not difficult to show (e.g. by using the co-area formula) that we can also arrange for \( q_k \) to be such that

\[
\sum_{j=1}^{n_k} \mathcal{H}(\{ x : \text{dist} \left( x, \bigcup_{i=1}^{n_k} H_i \right) > \varepsilon \}) < \varepsilon,
\]

provided \( k \) is sufficiently large (depending on \( \varepsilon \)). From this it clearly follows that (for \( k \) sufficiently large) each of the curves \( \Gamma^j_k, j = 2, ..., n_k \), encloses an area \( A^j_k \subset \partial U(0, q_k) \) with

\[
\mathcal{H}(A^j_k) < \varepsilon k,
\]

where \( k \) is an absolute constant.

Now recall that Theorem 1 was stated for the case when \( U \) was a \( C^2 \) convex open set. The theorem is easily seen to be valid (with only minor modifications in the proof) in case \( U \) is the intersection of two \( C^2 \) convex open sets (like \( U^+(0, q_k) \) for example), although in this case we can only require the \( N_j \) of (vii) to be elements of \( \mathcal{M} \) (rather than \( \mathcal{M} \)) and (ix) holds for all \( P \in \mathcal{M} \) with \( \partial P_j = \partial N_j \). (Here \( \mathcal{M} \) is as in §1.) Using such a modified version of Theorem 1, with \( U^+(0, q_k) \) in place of \( U \), we deduce that there is an \( \mathcal{N}_k \in \mathcal{M} \) with \( \partial \mathcal{N}_k = \partial N_k \), \( \mathcal{N}_k \cap U^+(0, q_k) = \bigcup_{i=1}^{m_k} P^i_k, \mathcal{N}_k \subset \mathcal{M}, \partial P^i_k = \Gamma^i_k, \) \( P^1_k, ..., P^{m_k}_k \in \mathcal{M}, \partial P^1_k \subset \partial U(0, q_k) \cap \bigcap W \)

\[
C_+ \subset U(0, \frac{1}{2}) \times G(3, 2) = \lim_{k \to \infty} \sum_{j=1}^{m_k} v(P^j_k \cap U(0, \frac{1}{2}))
\]

(using conclusions (iv) and (vi) of Theorem 1 together with [AW1, 2.6(2) (d)]),
and where

\[(7.16) \quad \mathcal{K}^2(P^j_k) < \mathcal{K}^2(P) + \varepsilon_{k,j}, \quad \forall P \in \mathcal{M}, \text{ with } \partial P = \partial P^j_k, \]

\[j = 1, \ldots, m_k. \] Here \( \varepsilon_{k,j} \) are constants such that \( \sum_{j=1}^{m_k} \varepsilon_{k,j} < 1/k \) (which Theorem 1 guarantees by virtue of (7.7)).

By (7.13) and (7.16) we then have \( \mathcal{K}^2(P^j_k) < (e + 1)\varepsilon, \quad j = 2, \ldots, m_k, \) for all sufficiently large \( k. \) Hence by the filigree lemma (for \( \varepsilon \) small enough) we can discard \( P^k_2, \ldots, P^k_{m_k} \) without changing the limit in (7.18). That is,

\[(7.15') C^+ \subseteq U(0, \frac{1}{2}) \times G(3, 2) = \lim_{k \to \infty} \nu(P^k_k \cap U(0, \frac{1}{2})). \]

Also, by (7.16) and Theorem 2 we have a subsequence \( \{k'\} \subset \{k\} \) with

\[(7.17) \quad g_{k'} \to \sigma \in [\frac{2}{3}, 1], \quad v(P^1_{k'}) \to Y, \]

where \( Y \) is a stationary integral varifold in \( U^+(0, \sigma) \) with \( \text{spt} \|Y\| \subset \overline{W}. \) Clearly by (7.15') and (7.17)

\[(7.18) \quad Y \subseteq U^+(0, \frac{1}{2}) \times G(3, 2) = C^+ \subseteq U^+(0, \frac{1}{2}) \times G(3, 2). \]

We next let \( S^+, S^- \) be the two components of \((\partial U(0, \sigma_k) \cap W) \sim I^1_k \)

and let \( D^+, D^- \) be the half discs having common diameter \( L \cap U(0, \sigma_k) \)

such that \( \partial(U(0, \sigma_k) \cap W) = \tilde{S}^+ \cup \tilde{S}^- \cup \tilde{D}^+ \cup \tilde{D}^-, \) and suppose the labelling

is such that \( (\tilde{D}^+ \cap \tilde{S}^-) \sim L \neq 0. \) Then \( E^+ = \tilde{D}^+ \cup \tilde{S}^+ \) and \( E^- = \tilde{D}^- \cup \tilde{S}^- \)

are both bilipschitz homeomorphs of \( D_1, \) and \( \partial E^+ = \partial E^- = I^2_k. \) Furthermore \n
\[\min \{\mathcal{K}^2(E^+), \mathcal{K}^2(E^-)\} = \mathcal{K}^2(D^+) + \min \{\mathcal{K}^2(S^+), \mathcal{K}^2(S^-)\} \]

\[\leq \left( \frac{\pi}{2} + \frac{2\pi - 2\alpha}{2} \right) \varepsilon^2 = \left( \frac{3\pi}{2} - \alpha \right) \varepsilon^2 \]

for some \( \alpha > 0. \) \( (2\alpha = \frac{1}{2} \mathcal{K}^2(\partial U(0, \sigma_k)) \) \) \( - \mathcal{K}^2(\partial U(0, \sigma_k) \cap W)). \) Hence by

(7.16) and (7.17) we conclude

\[(7.19) \quad \|Y\| (U(0, \sigma)) < \frac{3\pi}{2} \varepsilon^2. \]

Also, writing \( Y = Y \subseteq U^+(0, \sigma) \times G(3, 2), \) we then deduce from (7.19) and the monotonicity of \( r^{-2} \|Y\| (U(0, r)) \) \( ([WW2, 3.2; WW1, 5.1(3)]) \) that

\[(7.20) \quad \Theta(\|C^+\|, 0) = \Theta(\|Y\|, 0) < \frac{\pi}{2}. \]
Thus we can deduce that the integer \( R \) in (7.11) satisfies \( R < 2 \). On the other hand we know that by (5.20) there is a rectifiable current \( T \) with
\[
\partial T \setminus U(0, \varrho) = \tau(L \cap U(0, \varrho)), \quad T \setminus U(0, \varrho) = \sum_{i=1}^{R} \chi_{i} \tau(H_{i} \cap U(0, \varrho)),
\]
where each \( \chi_{i} = \pm 1 \), \( i = 1, \ldots, R \). (Here we take account of the fact that the half-spaces \( H_{1}, \ldots, H_{R} \) are not necessarily distinct.) In (7.21) \( \tau(L \cap U(0, \varrho)) \) denotes the one-dimensional current associated with a suitably oriented version of \( L \cap U(0, \varrho) \), and the \( H_{i} \) are also assumed to have been given some definite orientation. Now (7.21) clearly implies that \( R \) is odd, hence we deduce finally that \( R = 1 \). That is we have
\[
\Theta(\|v'\|, 0) = \Theta(\|C_{+}\|, 0) = \frac{1}{2},
\]
and by the regularity theorem of Allard [AW2, § 4] we then deduce the following result concerning \( V' \) in a neighbourhood of \( x_{0} \in \Gamma \).

**Theorem 5 (Boundary regularity of \( V \)).** Suppose \( V \) is as in § 1, \( x_{0} \in \Gamma \) and \( V' = V \setminus (\mathbb{R}^{2} \setminus \Gamma) \times G(3, 2) \). Then there is a \( \varrho > 0 \) and a \( C^{2} \) surface-with-boundary \( M \) such that \( \partial M \cap U(x_{0}, \varrho) = \Gamma \cap U(x_{0}, \varrho) \) and \( V' \setminus U(x_{0}, \varrho) \times \times G(3, 2) = v(M \cap U(x_{0}, \varrho)) \).

**Remark.** The regularity theorem of [AW2, § 4] guarantees \( M \) is \( C^{1,\gamma} \) for each \( \gamma \in (0, 1) \); the fact that \( M \) is \( C^{2} \) then follows from the fact that \( \Gamma \) is \( C^{3} \) together with the fact that \( M \) can be locally represented as the graph of a solution of the minimal surface equation. (The Schauder theory for elliptic equations with Hölder continuous coefficients is thus applicable.)

8. – Main result.

From Theorems 4 and 5 (together with the fact that there exist no compact minimal surfaces without boundary) we now immediately deduce that \( V' = v(M) \), where \( M \) is a compact connected \( C^{0} \) surface-with-boundary such that \( \partial M = \Gamma \). It thus only remains to show \( M \in \mathcal{M} \).

In order to do this we first show that there is a \( C^{2} \) map \( \chi \) of \( \partial D_{1} \) into \( M \) such that \( \chi|_{\partial D_{1}} \) is a diffeomorphism of \( \partial D_{1} \) onto \( \Gamma \). Notice that the existence of such a map \( \chi \) immediately implies orientability of \( M \) (because, for example, we can use the map \( \chi \) to demonstrate that the 2-dimensional singular integral homology group \( H_{2}(M, \partial M) \) is non-zero). It then follows from standard classification theory for 2-dimensional surfaces that either \( M \in \mathcal{M} \) or else \( M \) is a disc with \( h > 1 \) handles [AS 41G, 46]. However the latter
case cannot occur because the existence of the map \( \chi \) clearly implies that
the homeomorphism \( \phi: [0, 1] \to \Gamma \) defined by \( \phi(t) = \chi(e^{it}) \), is null-homotopic
in \( M \). (A homotopy is given by \( F(s, t) = \chi(se^{it}), \ (s, t) \in [0, 1] \times [0, 1] \).) However, if \( M \) has \( h > 1 \), then one can express the homotopy group of \( \phi \)
explicitly in terms of generators of the free group \( \pi(M) \) ([AS, 43B]); certainly
one can conclude in this case that \( \phi \) is not null-homotopic.

Thus to prove \( M \in \mathcal{M} \) it merely remains to prove the existence of a map \( \chi \)
as described above. To do this, first note that there is a \( \delta > 0 \) such that
there is a \( C^2 \) retract \( \mu \) of \( S_\delta = \{ x \in \mathbb{R}^3; \text{dist} (x, M) < \delta \} \) onto \( M \). We can
write \( \mathbb{R}^3 = \bigcup_{i=1}^{\infty} K_i \), where \( K_1, K_2, \ldots \) are closed cubes of edge length \( \delta/4 \),
with pairwise disjoint interiors and with faces contained in the planes
\( x_i = k\delta/4, \ i = 1, 2, 3, \ k = 0, \pm 1, \pm 2, \ldots \). Suppose the labelling is such
that for some \( R \)
\[
(R^3 \sim S_\delta) \cap \overline{A} \subset \bigcup_{i=1}^{R} K_i
\]
and
\[
M \cap \bigcup_{i=1}^{R} K_i = \emptyset.
\]

Now let \( e(K_i) \) denote the union of the edges of \( K_i \) (thus \( e(K_i) \) is a union
of 12 closed line segments, each of length \( \delta/4 \)). Since \( \mathcal{C}^2 \left( M \cap \left( \bigcup_{i=1}^{R} K_i \right) \right) \to 0 \)
as \( k \to \infty \) (by (8.2)), it follows that, for \( k \) sufficiently large, we can slightly
perturb \( M_k \) to give \( N \in \mathcal{M} \), with \( N \sim \partial N \subset A, \partial N = \Gamma \), \( N \cap \left( \bigcup_{i=1}^{R} e(K_i) \right) = \emptyset \),
and such that \( N \) meets \( \bigcup_{i=1}^{R} (\partial K_i \sim e(K_i)) \) transversally.

Having thus chosen \( N \), apply Theorem 1 with \( N \) in place of \( M \) and interior \( (A \cap K_R) \) in place of \( U \) (*) . Then we obtain \( \tilde{N} \in \mathcal{M} \) with \( \partial \tilde{N} = \Gamma \),
\( \tilde{N} \sim \partial \tilde{N} \subset A, \tilde{N} \cap K_R = \bigcup_{i=1}^{R} P_i \), where \( P_i \in \mathcal{M} \), and with \( \tilde{N} \cap \left( \bigcup_{i=1}^{R} \partial K_i \right) \subset \bigcap_i \cap \left( \bigcup_{i=1}^{R} \partial K_i \right) \). It is then clear, since \( \partial P_i \cap (\partial K_R \sim e(K_R)) \cap N \), that we can
modify \( \tilde{N} \) to give \( N^* \in \mathcal{M} \), with \( \partial N^* = \Gamma \), \( N^* \sim \partial N^* \subset A, \ N^* \cap \left( \bigcup_{i=1}^{R} e(K_i) \right) = \emptyset \),
\( N^* \) intersecting \( \bigcup_{i=1}^{R} \partial K_i \) transversally, \( N^* \cap \left( \bigcup_{i=1}^{R} \partial K_i \right) \subset \cap_i \cap \left( \bigcup_{i=1}^{R} \partial K_i \right) \), and

(*) To be strictly precise, we should here replace \( A \cap K_R \) by a \( C^2 \) convex set \( U \)
with \( U \subset A \cap K_R \) and \( \overline{U} \cap N = K_R \cap N \). The subsequent conclusions are then still valid.
$N^* \cap \partial K_R = \emptyset$. Thus, by induction on $R$, we can obtain $N \in \mathcal{M}$ with 
$\partial N = \Gamma, N \sim \partial N \subset A$, and $N \cap \left( \bigcup_{i=1}^{R} \partial K_i \right) = \emptyset$ (and hence $N \cap \left( \bigcup_{i=1}^{\infty} \partial K_i \right) = \emptyset$).

Thus by (8.1) we deduce that $N \subset S_3$. We now let $\chi_1$ be a $C^3$ diffeomorphism of $D_1$ onto $N$ and let $\mu$ be the retract of $S_3$ onto $M$ described above. We finally define $\chi = \mu \circ \chi_1$; this gives the desired mapping $\chi$.

Thus we have finally proved the following.

**Theorem 6.** If $A, \Gamma$ are as in § 1, then there is an $M \in \mathcal{M}$ such that 
\[ \partial M = \Gamma \text{ and } \mathcal{H}^2(M) = \inf \{ \mathcal{H}^2(N) : N \in \mathcal{M}, \partial N = \Gamma \} . \]

9. - The $k$-contour case, $k > 2$.

In this section we wish to point out that the techniques of the previous sections imply analogous results for the case when $\Gamma$ is a disjoint union $\Gamma_1 \cup \Gamma_2 \cup \ldots \cup \Gamma_k$ of $k > 2$ smooth Jordan curves contained in $\partial A$. For simplicity we will here only explicitly consider the case $k = 2$. We let $\Gamma = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1, \Gamma_2$ are disjoint $C^3$ Jordan curves in $\partial A$, $\mathcal{K}_2$ denotes the set of surfaces $M \subset \mathbb{R}^3$ which are $C^2$ diffeomorphs of the annulus \{$x : \frac{1}{2} < |x| < 1$\}, and we let 
\[ \alpha_1 = \inf \{ \mathcal{H}^2(M) : M \in \mathcal{M}_1, \partial M = \Gamma_1 \} \]
\[ \alpha_2 = \inf \{ \mathcal{H}^2(M) : M \in \mathcal{M}_2, \partial M = \Gamma_2 \} \]
\[ \alpha = \inf \{ \mathcal{H}^2(M) : M \in \mathcal{M}_2, \partial M = \Gamma \} . \]

**Theorem 7.** Suppose $\alpha < \alpha_1 + \alpha_2$.

Then there is an $M \in \mathcal{M}_2$ with $\partial M = \Gamma$ and $\mathcal{H}^2(M) = \alpha$.

The general approach to the proof is as for the single contour case. We take a sequence \{$M_k$\} $\subset \mathcal{M}_2$ with $\partial M_k = \Gamma$, $\mathcal{H}^2(M_k) \to \alpha$, and such that $V = \lim_{k \to \infty} \nu(M_k) \in V_{\alpha}(\mathbb{R}^3)$ exists. We need to show that there is a compact $C^2$ surface-with-boundary $M \in \mathcal{M}_2$ with $\partial M = \Gamma$ and $\nu(M) = V \setminus (\mathbb{R}^3 \sim \Gamma) \times G(3, 2)$. (The fact that $\mathcal{H}^2(M) = \alpha$ then of course easily follows from the construction of $V$.) In order to show the existence of such an $M$ we need the following variant of Theorem 1.

**Theorem 1'.** Suppose the hypotheses are as in Theorem 1, except that $M \in \mathcal{M}_2$.

Then either

(a) there exist $M_1, M_2 \in \mathcal{M}$ with $\partial M = \partial M_1 \cup \partial M_2$ and $\mathcal{H}^2(M_1) + \mathcal{H}^2(M_2) < \mathcal{H}^2(M)$, or
(b) there is an \( \tilde{M} \in \mathcal{M}_2 \) such that the conclusions (iv), (v), (vi) of Theorem 1 hold, and in place of conclusion (vii) we have:

(vii)' \( \tilde{M} \sim \overline{U} \) has no components \( E \) with \( \partial M \cap E = \emptyset \); in case \( \partial M \subset \partial U \), we deduce \( \tilde{M} \subset \overline{U} \) as in (vii).

If in addition neither of the two components \( \Gamma_1, \Gamma_2 \) of \( \partial M \) is contained entirely in \( \partial U \), and if there is a single component \( F \) of \( M \sim \overline{U} \) containing \( \partial M \sim \partial U \), then conclusion (vii) of Theorem 1 holds as before; if hypothesis (viii) holds for every \( P \in \mathcal{M}_2 \) with \( \partial P = \partial M \), then conclusion (ix) of Theorem 1 also holds.

**Proof.** Let \( X \) denote a diffeomorphism of \( G \) onto \( M_1 \), where \( G = S^2 \sim (D(1) \cup D(2)), \) with \( D(1), D(2) \subset S^2 \) disjoint diffeomorphs of \( D_1 \), and let \( \Gamma_i = X^{-1}(D(i)) \), \( i = 1, 2 \).

Now let \( U_1, E, F \) be constructed as in the proof of Theorem 1 (with \( \partial M \cap E = \emptyset \) and \( U_1 \cap \overline{M} = \emptyset \)). For \( i = 1, 2 \) let \( \delta_i \subset G \) be the component of \( \chi^{-1}(\partial E) \) such that some component \( K \) of \( G \sim \chi^{-1}(E) \) contains both \( \delta_i \) and \( \partial D(0) \). Let \( \hat{F} \) denote the component of \( F \) which contains \( \chi(\delta_i) \), let \( I_i \) denote the union of those components \( W \) of \( G \sim \chi^{-1}(E) \) such that

\[
\chi(\partial W \sim \partial G) \cap \partial \hat{F} = \emptyset,
\]

and let \( E_i = \chi(I_i), i = 1, 2 \). (cf. the definition of \( F_i^*, I_i, E_i^* \) in the proof of Theorem 1.)

We now consider two cases:

- **Case I.** \( \Gamma_2 \notin \hat{E}_1 \) or \( \Gamma_1 \notin \hat{E}_2 \).
- **Case II.** \( \Gamma_2 \subset \hat{E}_1 \) and \( \Gamma_1 \subset \hat{E}_2 \).

In Case I we only explicitly consider the alternative \( \Gamma_2 \notin \hat{E}_1 \); the procedure in case of the other alternative is almost identical.

We define a Lipschitz mapping \( \tilde{\chi} \) by setting \( \tilde{\chi}(x) = \chi(x) \) for \( x \in G \sim \chi^{-1}(E \cup \hat{E}) \) and \( \tilde{\chi}(x) = (\mu_1 \circ \chi)(x) \) for \( x \in \chi^{-1}(E \cup \hat{E}) \), where \( \mu_1 \) denotes a diffeomorphism of \( E \cup \hat{E} \) onto \( F \) with \( \mu_1 = \text{identity on } \partial F \). (Notice that such a \( \mu_1 \) exists by Lemma 1, because \( \Gamma_2 \notin \hat{E}_1 \).)

If \( (\Gamma_1 \cup \Gamma_2) \cap F_1^* \sim \partial F_1^* = \emptyset \), then \( \tilde{\chi} \) is 1-1 on the whole of \( G \) and we can define a Lipschitz surface \( \tilde{M} = \tilde{\chi}(G) \). Then by (3.1) \( \mathcal{H}(\tilde{M}) + \mathcal{H}(F_1^*) < \mathcal{H}(M) \) and, by making a slight perturbation of \( \tilde{M} \) in a neighbourhood of \( F_1^* \) (cf. the procedure of Theorem 1), we obtain \( \tilde{M} \) satisfying

\[
\left\{ \begin{array}{l}
\tilde{M} \in \mathcal{M}_2, \\
\partial \tilde{M} = \partial M, \\
\tilde{M}_1 \sim \overline{U} \subset M \sim \overline{U}, \\
\tilde{M}_1 \cap U_0 \subset M \cap U_0 \\
\mathcal{H}(\tilde{M}) + \mathcal{H}(M \sim \tilde{M}_1) < \mathcal{H}(M).
\end{array} \right. \tag{9.1}
\]

If \( (\Gamma_1 \cup \Gamma_2) \cap F_1^* \sim \partial F_1^* = \emptyset \) (cf. Case II of Theorem 1), then we have a curve \( \gamma_i \subset G \) with \( \tilde{\gamma}(\gamma_i) = \Gamma_i \) for either \( i = 1 \) or \( 2 \) (or both). Suppose for
example that such a curve \( \gamma_1 \) exists. We consider the possibilities:

(i) \( \gamma_1 \) is not null-homotopic in \( G \);

(ii) \( \gamma_1 \) is null-homotopic in \( G \).

In case (i) holds, we define \( \tilde{\gamma} = \tilde{\gamma}|_{G'} \), where \( G' \) denotes the component of \( G \sim \gamma_1 \) with closure containing \( \partial D^{01} \). In case (ii) holds, we first let \( \mu \) be a diffeomorphism of \( D^{01} \) onto \( \text{int} \gamma_1 \subset G \) with \( \mu(\partial D^{01}) = (\tilde{\gamma}|_{\gamma_1})^{-1}(\tilde{\gamma}|_{\gamma_1}(\partial D^{01})) \), and we define \( G' \) and \( \tilde{\gamma} \) by \( G' = (G \sim \text{int} \gamma_1) \cup D^{01} \), \( \tilde{\gamma} = \tilde{\gamma} \) on \( \tilde{G} \cap G' \), \( \tilde{\gamma} = \chi \circ \mu \) on \( D^{01} \). Then (regardless of whether (i) or (ii) holds), if there is no curve \( \gamma_2 \subset G' \) with \( \tilde{\gamma}(\gamma_2) = \Gamma_2 \), we can slightly perturb \( \tilde{\gamma} \) to give a smooth embedding \( \tilde{\gamma}^*: G' \to \mathbb{R}^3 \) such that \( \tilde{M}_1 = \tilde{\gamma}^*(G') \) satisfies (9.1). If on the other hand there is a curve \( \gamma_2 \subset G' \) with \( \tilde{\gamma}(\gamma_2) = \Gamma_2 \), then a repetition of the above argument, starting with \( \tilde{\gamma}, G', \gamma_1 \) in place of \( \tilde{\gamma}, G, \gamma_0 \) respectively, again yields \( \tilde{M}_1 \) as in (9.1). Thus Case I always leads to a surface \( \tilde{M}_1 \) as in (9.1).

We now turn to Case II. In this case, since \( \partial D^{01} \notin \tilde{I}_1 \), we must have

\[
F_1 \neq F_2.
\]

Defining \( \hat{M}_i = (M \sim (E \cup \tilde{E}_i)) \cup F_i \), we then see by (9.2) and (3.1) that

\[
\mathcal{K}(\hat{M}_1) + \mathcal{K}(\hat{M}_2) < \mathcal{K}(M),
\]

and, with the aid of Lemma 1, we also see that \( \hat{M}_i = Y_i(D_1) \), where \( Y_i: D_1 \to \mathbb{R}^2 \) and \( Y_i|_{\text{int} \gamma_1} \) is a bilipschitz homeomorphism for some Jordan curve \( \gamma_1 \subset D_1 \) with \( Y_i(\gamma_1) = \Gamma_i \). It follows that we can find \( M_1, M_2 \in \mathcal{M} \) such that alternative (a) of Theorem 1' holds. Thus Case II implies that alternative (a) holds.

We can now complete the proof that there is \( \hat{M} \) as in (vii)' by using induction on the number of components \( E \) of \( M \sim \tilde{U} \) such that \( \partial M \cap \tilde{E} = \emptyset \). (cf. the proof of (vii) in Theorem 1; the only essential difference here is that at the inductive step of the argument the occurrence of Case II will always yield alternative (a) and the argument can therefore be immediately terminated.)

To prove the second part of Theorem 1', we simply note that (vii) implies (vii) in case neither component of \( \partial M \) is contained entirely in \( \partial U \) and there is a single component \( E \) of \( M \sim \tilde{U} \) containing \( \partial E \sim \partial U \). (The remaining part is then also proved in an almost identical fashion to the corresponding part of Theorem 1.)

To prove (vii)' implies (vii) under the above hypotheses, first note that we can take \( N \) in condition (iii) of Theorem 1 to be such that \( \tilde{N} \in \mathcal{M} \) and
\((\tilde{N} \sim M) \cap \tilde{U} = 0\). It then easily follows that there is a \(\tilde{N} \in \mathcal{M}_2\) with \(\tilde{M} \subset \tilde{N} \sim \partial \tilde{N}\) and \((\tilde{N} \sim \tilde{M}) \cap \tilde{U} = 0\). Furthermore (by conclusion (iv)) \(F \subset \tilde{M}\); \(F\) is thus a component of \(\tilde{M} \sim \tilde{U}\) containing all of \(\partial \tilde{M} \sim \partial \tilde{U}\). It then follows that \(\tilde{N} \sim \tilde{U}\) has a component containing all of \(\partial \tilde{N}\), and hence (by (vii)) \(\tilde{N} \sim \tilde{U}\) is connected. Thus if we let \(\chi\) be a \(C^2\) diffeomorphism of \(\{x: \frac{1}{2} < |x| < 1\}\) onto \(\tilde{N}\), so that \(\chi^{-1}(\tilde{N} \cap \partial \tilde{U}) = \bigcup_{i=1}^{m} \gamma_i\), where \(\gamma_1, \ldots, \gamma_m\) are pairwise disjoint \(C^2\) Jordan curves in \(\{x: \frac{1}{2} < |x| < 1\}\), then we must have \(\text{int} \gamma_1, \ldots, \text{int} \gamma_m\) pairwise disjoint and contained in \(\{x: \frac{1}{2} < |x| < 1\}\). (Otherwise we contradict connectedness of \(\tilde{N} \sim \tilde{U}\).) It is then clear that \(\tilde{N} \cap \tilde{U}\) is a union \(\bigcup_{i=1}^{m} \chi(\text{int} \gamma_i)\) of elements of \(\mathcal{M}\) as required.

In conjunction with the above modification of Theorem 1, we point out the following property of the minimizing sequence \(\{M_k\}\).

**Lemma 4.** Suppose \(x < x_1 + x_2\) as in Theorem 7, and suppose \(\{M_k\} \subset \mathcal{M}_2\) satisfies \(\partial M_k = \Gamma_k\), \(k = 1, 2, \ldots\), and \(\lim_{k \to \infty} \mathcal{H}^2(M_k) = \alpha\).

Then we have the following conclusion: there is a constant \(\rho > 0\) such that if \(U\) is any \(C^2\) open convex subset of \(\Delta\) with diameter \(U < \rho\) if \(M_k\) intersects \(\partial U\) transversally for \(k = 1, 2, \ldots\), and if \(\Gamma_1 \sim \partial U, \Gamma_2 \sim \partial U\) are both non-empty and connected, then for all sufficiently large \(k\) both \(\Gamma_1 \sim \partial U\) and \(\Gamma_2 \sim \partial U\) are contained in the same component of \(M_k \sim \tilde{U}\).

**Proof.** Suppose no such \(\rho\) exists. Then for each \(\varepsilon > 0\) we can find a convex set \(U \subset \Delta\) with diameter \(U < \varepsilon\), with the sets \(\Gamma_1 \sim \partial U, \Gamma_2 \sim \partial U\) non-empty, connected, and contained in different components of \(M_k \sim \tilde{U}\) for all \(k\) in some subsequence \(\{k'\} \subset \{k\}\), and with \(M_k\) intersecting \(\partial U\) transversally. We can suppose \(\Gamma_1 \cap \partial U = \partial U = 0\) and \(\Gamma_2 \cap \partial U = \partial U = 0\); otherwise this can be achieved, without upsetting the above hypotheses, by replacing \(U\) by a slightly smaller set. We then let \(X_{\kappa}\) be a \(C^2\) diffeomorphism of \(\{x \in \mathbb{R}^2: \frac{1}{2} < |x| < 1\}\) onto \(M_k\), such that \(X_{\kappa}(\partial D_1) = \Gamma_1\) and \(X_{\kappa}(\partial D_2) = \Gamma_2\).

Since \(\Gamma_1, \Gamma_2\) are in different components of \(M_k \sim \tilde{U}\), we can find a Jordan curve \(\gamma \subset \{x \in \mathbb{R}^2: \frac{1}{2} < |x| < 1\}\) such that \(D_1 \subset \text{int} \gamma\) and \(X_{\kappa}(\gamma) \subset \partial U\). Now let \(P_{\kappa}\) be one of the two components of \(\partial U \sim X_{\kappa}(\gamma)\) and let \(Y_{\kappa}\) be a bilipschitz immersion of \(D_1\) into \(\mathbb{R}^2\) such that \(Y_{\kappa}(D_1) = P_{\kappa} \cup X_{\kappa}(D_1 \sim \text{int} \gamma)\).

Then by Lemma 2 it follows that there is an \(M_{k'} \in \mathcal{M}\) such that

\[
\begin{align*}
1. & \quad \partial M_{k'} = \Gamma_1, \quad \mathcal{H}^2(M_{k'}) < \mathcal{H}^2(P_{\kappa}) + \mathcal{H}^2(X_{\kappa}(D_1 \sim \text{int} \gamma)) + \varepsilon^2. \\
2. & \quad \partial M_{k'} = \Gamma_2, \quad \mathcal{H}^2(M_{k'}) < \mathcal{H}^2(P_{\kappa}) + \mathcal{H}^2(X_{\kappa}(\text{int} \gamma \sim D_1)) + \varepsilon^2.
\end{align*}
\]
Now $\mathcal{K}^1(P_k) + \mathcal{K}^2(M_k) > \alpha_1 + \alpha_2$, hence adding (i) and (ii), we see that

$$\alpha_1 + \alpha_2 < 2\mathcal{K}^2(P_k) + \mathcal{K}^2(M_k) + 2\varepsilon^2.$$ 

Since $\mathcal{K}^2(P_k) < 4\pi\varepsilon^2$ and $\mathcal{K}^2(M_k) \to \alpha$, we then deduce, because $\varepsilon > 0$ is arbitrary, that $\alpha_1 + \alpha_2 < \alpha$, contrary to the hypothesis of the lemma. This completes the proof of Lemma 4.

Because of Theorem 1' and Lemma 4 it should now be clear that the techniques of § 5-8 can be modified in a very straightforward manner to yield a proof of Theorem 7.

10. – Minimizing surfaces of higher genus.

In this section we wish to show that there is an analogue of Theorem 6 for surfaces of higher genus. The discussion of Appendix B illustrates the wide range of applicability of the results obtained here.

We let $A, \Gamma, \mathcal{K}$ be as in §§ 1-8, and we also introduce the following further notation:

$\mathcal{M}(g, \Gamma)$ ($g > 0$ an integer) denotes the collection of connected oriented $C^2$ surfaces-with-boundary $M$ with $\partial M = \Gamma$, and genus $M = g$;

$$\alpha_g = \inf \{ \mathcal{K}^2(M) : M \in \mathcal{M}(g, \Gamma) \} (= \inf \{ \mathcal{K}^2(M) : M \in \mathcal{M}(h, \Gamma) \text{ for some } h < g \});$$

$\{M_k\}$ will denote a sequence in $\mathcal{M}(g, \Gamma)$ such that

$$\mathcal{K}^2(M_k) \to \alpha_g$$

and such that there is a varifold $V_g$ with

$$V_g = \lim \nu(M_k).$$

Of course by the convex hull property of Appendix A we have

$$\text{spt } V_g \subset A \cup \Gamma$$

(cf. (1.1)).

The main result we want to prove is the following.

**Theorem 8.** Suppose $\alpha_g < \alpha_{g-1}$.

Then there is an $M \in \mathcal{M}(g, \Gamma)$ with $\mathcal{K}^2(M) = \alpha_g$.

We will show that, like the main result of § 9, this theorem can be proved by rather straightforward modifications of the techniques of §§ 5-8.
The key point is to prove interior and boundary regularity of $V$, in the sense (cf. Theorems 2, 3) that there is a smooth surface-with-boundary $M$ satisfying $\partial M = \Gamma$ and

$$\nu(M) = V_s \cap (\mathbb{R}^3 \sim \Gamma);$$

it then remains to prove genus $\chi(M) = g$ (cf. § 8).

We first need the following generalization of Theorem 1.

**Theorem 9.** Suppose the hypotheses are as in Theorem 1, except that $M \in \mathcal{M}(g, \Gamma)$ rather than $M \in \mathcal{M}$.

Then there is an integer $0 \leq h \leq g$ and $M \in \mathcal{M}(h, \Gamma)$ such that the conclusions (iv), (v), (vi) of Theorem 1 hold and such that $\mathbb{R}^3 \sim (M \cup \bar{U})$ is connected. (Consequently $M \subset U$ in case $\Gamma \subset \partial U$.)

**Proof.** Let $E, U, \Gamma$ (satisfying (3.2), (3.3)) be as in the proof of Theorem 1 and consider the cases $F \subset F - \partial F$, $\Gamma \not\subset F - \partial F$.

**Case I.** $\Gamma \not\subset F - \partial F$. In this case we let $M^*$ be the compact (not necessarily connected) oriented Lipschitz surface with boundary $\Gamma$ defined by $M^* = (M - E) \cup F$, and we let $M^{(1)}$ be the component of $M^*$ containing $\Gamma$. We want to show genus $\chi(M^{(1)}) \leq \chi(M)$.

To show this we are going to use the *Euler characteristic* $\chi(Y)$ of a compact (not necessarily connected) oriented surface $Y$; $\chi(Y)$ has the properties that it is an integer, and if $Y$ is connected then

$$\chi(Y) = 2 - R - 2g.$$  

Here $R = 0$ if $\partial Y = 0$ and $R$ is the number of components of $\partial Y$ otherwise; $g$ denotes the genus (number of handles) of $Y$ and is zero if and only if $Y$ is homeomorphic to a compact surface-with-boundary in $S^2$. Thus, in case $Y$ is connected we always have

$$\chi(Y) \leq 2,$$

with equality if and only if $Y$ is homeomorphic to $S^2$. The Euler characteristic has the additional property that if $Z \subset Y$ with $Z$ also a compact surface with boundary, then

$$\chi(Y) = \chi(Y \sim (Z \sim \partial Z)) + \chi(Z).$$

Now let $k$ be the number of components of $F$ and let $M^{(2)}, \ldots, M^{(l)}$ be the components of $M^* \sim M^{(1)}$. Using the fact that $l \leq k$ together with (10.5),
(10.6) and (10.7), we have

\[
2(k - 1) + \chi(M^{(1)}) > \sum_{i=1}^{k} \chi(M^{(i)}) = \chi(M^*) \\
= \chi(M) + \chi(F) - \chi(E) \\
= \chi(M) + 2k - 2 + 2g_E,
\]

where \(g_E\) denotes, the genus of \(E\). Thus we have \(\chi(M^{(1)}) > \chi(M)\), and hence genus \(M^{(1)} < \text{genus } M\) as required. Of course by (3.1) and the construction of \(M^{(1)}\) we have

\[
(10.8) \quad M^{1} \sim \partial U \subset M, \quad \mathcal{K}^2((M \sim M^{(1)}) \cap U) + \mathcal{K}^2(M^{(1)}) < \mathcal{K}^2(M).
\]

**Case II.** \(I' \subset F \sim \partial F\). Let \(I'' \subset M \sim \Gamma\) be a \(C^2\) Jordan curve homotopic to \(I\) in \(M\), and let \(N \subset M\) be the compact surface with boundary \(I''\). Since \(M \cap U \subset = 0\), we can arrange to choose \(I'' \) sufficiently close to \(I\) to ensure that \(I'' \subset U\). We now write \(N^* = (N \sim E) \cup F\) and let \(N^{(1)}\) be the component of \(N^*\) containing \(I''\). Exactly as in **Case I** we show that genus \(N^{(1)} < \text{genus } N\) (= genus \(M\)). If \(I' \notin N^{(1)}\), we write \(M^{(1)} = N^{(1)} \cup (M \sim N)\) and we note that genus \(M^{(1)} < \text{genus } M\) and (10.8) again holds. If \(I' \subset N^{(1)}\) we first want to argue that \(N^{(1)} \sim I'\) is not connected. Indeed \(U \sim M\) consists of open components, each lying on one side of the oriented surface \(M\). Let \(W\) be any connected component of \(U \sim M\) with \(I' \subset W\). If \(N^{(1)} \sim I'\) is connected we could, after letting \(F^*\) be the component of \(F\) which contains \(I'\), construct a curve in \(N^{(1)} \sim I'\) joining any two given points on \(F^* \sim I'\). Since \(N^{(1)}\) is oriented we could then construct a curve in \((U \cup F^*) \sim M\) joining any two given points on \(F^* \sim I'\). Thus it would follow that \(F^* \subset W\), and this in turn clearly implies that there are points of \(M \sim I'\) (close to \(I'\)) which lie in the interior of \(W\), thus contradicting the fact that \(W\) lies on one side of \(M\). Thus \(N^{(1)} \sim I'\) is not connected. We now let \(M^{(1)}\) be the closure of the component of \(N^{(1)} \sim I'\) which does not contain \(I''\). We then note (by using (10.5), (10.6), (10.7), together with the established fact that genus \(N^{(1)} < \text{genus } M\)) that genus \(M^{(1)} < \text{genus } M\) and that (10.8) again holds.

Thus in either Case I or Case II we obtain a Lipschitz surface \(M^{(1)}\) with genus \(M^{(1)} < \text{genus } M\) and with (10.8) holding. By making an arbitrarily slight perturbation of \(M^{(1)}\) in a neighborhood of \(F\) (and leaving the rest of \(M^{(1)}\) fixed) we obtain a smooth surface \(\tilde{M}^{(1)}\) with \((\tilde{M}^{(1)} \sim U) \cup (\tilde{M}^{(1)} \cap U_\delta) \subset M\) and with \(\mathcal{K}^2((M \sim \tilde{M}^{(1)}) \cap U_\delta) + \mathcal{K}^2(\tilde{M}^{(1)}) < \mathcal{K}^2(M)\). (Cf. the relevant part of the proof of Theorem 1.) By induction on the number of components of \(M \sim \tilde{U}\) (as in Theorem 1), we thus obtain the desired surface \(\tilde{M}\).
We next define $\mathcal{M}(g, \Gamma)$ (for $q > 0$) to be the collection of surfaces $M \in \mathcal{M}(g, \Gamma)$ with the following property:

If $U$ is an open $C^2$ convex set with diameter $< q$, if $\partial M \subset R^3 \sim U$, and if $M$ and $\partial U$ intersect transversally (in the sense of Theorem 1 (iii)), then for each component $A$ of $M \cap \partial U$ there is an $N \in \mathcal{M}$ with $N \subset M$ and $\partial N = A$.

We have the following lemma.

**Lemma 5.** Suppose $g > 1$ and $M \in \mathcal{M}(g, \Gamma)$ is such that $\mathcal{K}(M) < \alpha_{q-1}$. Then $M \in \mathcal{M}(g, \Gamma)$ for any $q < \left(\frac{\alpha_{q-1} - \mathcal{K}(M)}{(8\pi + 3)}\right)^4$.

Before proving this, we note that the following corollary can be obtained with the aid of Theorem 9 and Corollary 1.

**Corollary 3.** If the hypotheses are as in Theorem 9, with $\mathcal{K}(M) < \alpha_{q-1} - \beta$ and diameter $U < \left(\beta(8\pi + 3)\right)^4$ for some $\beta > 0$, then the surface $\bar{M}$ of Theorem 9 is such that

$$\bar{M} \cap U = \bigcup_{i=1}^{k} N_i,$$

where $N_1, \ldots, N_k$ are pairwise disjoint elements of $\mathcal{M}$.

If it is in addition hypothesized that $\mathcal{K}(M) < \mathcal{K}(P) + \theta$ for each $P \in \mathcal{M}(g, \Gamma)$, then there are positive $\theta_1, \ldots, \theta_k$ so that $\sum_{i=1}^{k} \theta_i < \theta$ and

$$\mathcal{K}(N_i) < \mathcal{K}(P) + \theta_i, \quad \forall P \in \mathcal{M} \text{ with } \partial P = \partial N_j, \quad j = 1, \ldots, k.$$

**Proof of Lemma 5.** Let $U$ have diameter $< q$, where for the moment $q > 0$ is arbitrary, and suppose that $M$ intersects $\partial U$ transversally. We can also suppose $\partial M \cap \partial U = \emptyset$, otherwise initially replace $U$ by a slightly smaller set.

Let $\Gamma_1$ be a component of $M \cap \partial U$ which is not null-homotopic in $M$ such that one of the two components (let us call it $P$) of $\partial U \sim \Gamma_1$ satisfies either $M \cap P = \emptyset$ or each component of $M \cap P$ is null-homotopic in $M$.

(Of course such a component $\Gamma_1$ of $M \cap \partial U$ exists unless every component of $M \cap \partial U$ is null-homotopic in $M$.) Then (even if $M \cap P$ is empty) we can find pairwise disjoint diffeomorphs $M_1, \ldots, M_n$ of $D_4$, with $\bigcup_{i=1}^{n} M_i \subset M$ and $\left(\bigcup_{i=1}^{n} (M_i \sim \partial M_i)\right) \cap P = M \cap P$. By Lemma 2 we then have pairwise disjoint $\bar{M}_1, \ldots, \bar{M}_n$, $\bar{P} \in \mathcal{M}$ with $\partial \bar{M}_i = \partial M_i$, $\partial \bar{P} = \partial P$.

(10.9) $$\mathcal{K}(\bar{M}_i) < \mathcal{K}(M_i) + \theta_i, \quad \mathcal{K}(\bar{P}) < \mathcal{K}(P) + \theta_i$$
and
\[(10.10) \quad \tilde{M}_i \cap (M \sim \bigcup_{i=1}^{R} M_i) = \emptyset, \quad i = 1, \ldots, R.\]

(Notice that this last conclusion follows from the first two assertions in (2.6)' of Lemma 2.) Then, defining \(\tilde{M} = \left( M \sim \bigcup_{i=1}^{R} M_i \right) \cup \left( \bigcup_{i=1}^{R} \tilde{M}_i \right)\), we see that \(\tilde{M}\) is homeomorphic (via a bilipschitz homeomorphism) to \(M\), and also \(\tilde{M} \cap \tilde{P} = \emptyset, \tilde{M} \cap (\partial U \sim P) = M \cap (\partial U \sim P)\).

Since \(U\) is convex with diameter \(\leq g\), we have \(\mathcal{K}^2(P) < 4\pi g^2\), and hence by (10.9) we deduce
\[(10.11) \quad \mathcal{K}^2(\tilde{P}) < (4\pi + 1)g^2, \quad \mathcal{K}^2(\tilde{M}) < \mathcal{K}^2(M) + g^2.\]

We now construct a surface \(N\) as follows (the construction depends on consideration of two cases):

**Case I.** If \(M \sim I_1\) is not connected, we let \(Q\) be the component of \(\tilde{M} \sim I_1\) which does not contain \(I_1\), and we define \(N = (\tilde{M} \sim Q) \cup \tilde{P}\).

**Case II.** If \(M \sim I_1\) is connected, we let \(S = \{x \in \tilde{M} : \text{dist}(x, I_1) < \varepsilon\}\), where \(\varepsilon\) is chosen small enough to ensure that \(\tilde{S}\) is diffeomorphic via a diffeomorphism \(X\), to the annulus \(\{x : \frac{1}{2} < |x| < 1\}\). For \(\varepsilon\) sufficiently small, we can construct \(E_1, E_2 \in \mathcal{M}\) (each being constructed by a slight perturbation of \(\tilde{P}\)) such that
\[(10.12) \quad E_1 \cap E_2 = \emptyset, \quad \partial E_1 \cup \partial E_2 = \partial S, \quad (E_1 \cup E_2) \cap \tilde{M} = \partial S, \quad \mathcal{K}^2(E_1) + \mathcal{K}^2(E_2) < (2(4\pi + 1)g^2).\]

We then define \(N = (\tilde{M} \sim S) \cup E_1 \cup E_2\).

(That is, \(N\) is constructed in this case by cutting out the annulus \(S\) and replacing it by two discs.)

By (10.11), (10.12) we now have (in either Case I or Case II)
\[(10.13) \quad \mathcal{K}^2(N) < \mathcal{K}^2(M) + (2(4\pi + 1) + 1)g^2 = \mathcal{K}^2(M) + (8\pi + 3)g^2.\]

We now claim that (again in either Case I or Case II)
\[(10.14) \quad \text{genus } N < g(= \text{genus } \tilde{M} = \text{genus } M).\]
Of course once (10.14) is established we immediately conclude

\[ \varphi^\beta \geq \frac{\alpha_{g-1} - \mathcal{K}^g(M)}{8\pi + 3} \]

from (10.13) because otherwise (10.13) would imply \( \mathcal{K}^g(N) < \alpha_{g-1} \).

However (10.14) is an immediate consequence of (10.5)-(10.7) and hence the proof is complete.

Using Theorem 9 and Corollary 3, we can now very directly modify the method of §§5-7 in order to prove that there is a \( C^2 \) surface \( M \) with boundary \( \Gamma \) as in (10.4). (Also, analogously to Remark 3.12, we note that Theorem 9 and (10.3) can be used to construct a minimizing sequence \( \{M_k\} \subset M(\varphi, \Gamma) \) with \( M_k \sim \Gamma \subset A \) and \( \lim_{k \to \infty} \varphi(M_k) \subset (R^3 \sim \Gamma) \times G(3, 2) \), that is, we may suppose without loss of generality that the minimizing sequence \( \{M_k\} \) is such that \( M_k \sim \Gamma \subset A \).

It thus remains only to show that \( M \) is orientable and genus \( M = g \). By virtue of Theorem 9 and Corollary 3, for each \( \delta > 0 \) we can use the argument of §8 in order to construct a surface \( N^\delta \in M(\varphi, \Gamma) \) with \( N^\delta \sim \Gamma \subset A \) and such that there is a smooth mapping \( \varphi^\delta: N^\delta \to M \) with \( \varphi^\delta|\Gamma \) a diffeomorphism of \( \Gamma \), and with \( \operatorname{dist}(x, \varphi^\delta(x)) < \delta \) for each \( x \in N^\delta \). As in §8 this immediately implies the orientability of \( M \). Letting \( K_1, K_2 \) denote the closures of the two components of \( A \sim M \), we note that the exactness of the Mayer-Vietoris sequence for the couple \( K_1, K_2 \) (together with the facts that \( H_1(A) = 0, H_2(A) = 0 \)) implies that there is an isomorphism

\[ H_1(M) = H_1(K_1 \cap K_2) \approx H_1(K_1) \oplus H_1(K_2) \]

Likewise

\[ H_1(N^\delta) \approx H_1(K_1^\delta) \oplus H_1(K_2^\delta) \],

where \( K_1^\delta, K_2^\delta \) are the closures of the two components of \( A \sim N \), labelled so that \( \operatorname{dist}(x, K_i^\delta) < \delta \) for each \( x \in K_i, i = 1, 2 \). By smoothness of \( M \) there exists a compact \( C_i \subset \text{interior } K_i \) and a retract \( r_i: K_i \to C_i \) which induces an isomorphism \( H_1(C_i) \approx H_1(K_i) \). Taking \( \delta \) small enough to ensure \( K_i^\delta \supset C_i \) it then follows that the inclusion map \( C_i \subset K_i^\delta \) induces a monomorphism of \( H_1(C_i) \) into \( H_1(K_i^\delta) \). Thus we conclude that \( \operatorname{rank} H_1(K_i^\delta) \leq \operatorname{rank} H_1(K_i) \), and hence that \( \operatorname{rank} H_1(M) \leq \operatorname{rank} H_1(N^\delta) \) by (10.15) and (10.16). This of course gives genus \( M \leq \text{genus } N^\delta \). Since \( \mathcal{K}^g(M) = \alpha_g < \alpha_{g-1} \), we then have genus \( M = g \) as required, and hence Theorem 8 is completely proved.
Appendix A. The convex hull property.

**Theorem.** Suppose $I'$ is any compact subset of $\mathbb{R}^n$, suppose $V \in V_k(\mathbb{R}^n)$ \((k < n)\) is stationary in $\mathbb{R}^n \sim I'$, and suppose $\|V\|((\mathbb{R}^n) < \infty$. Then $\text{spt } V$ is contained in the convex hull of $I'$.

**Proof.** It suffices to prove $\text{spt } V \subset H_+$ for any half-space $H_+$ such that $I' \subset H_+$. For convenience of notation we will prove this only for the case when $H_+ = \{x = (x_1, \ldots, x_n): x_n < 0\}$ and $I' = \{x = (x_1, \ldots, x_n): x_n < 0\}$. (The general case of course follows by considering $\phi^* V$, where $\phi$ is a suitable isometry of $\mathbb{R}^n$.)

Since $V$ is stationary in $\mathbb{R}^n \sim I'$ we can write (in the notation of [AW1])

\[
\int_{\text{int}(\mathbb{R}^n)} S \cdot Dg(x) \, dV(x, S) = 0
\]

whenever $g$ is smooth and $\text{spt } g$ is a compact subset of $\mathbb{R}^n \sim H_+$. Since $\|V\|((\mathbb{R}^n) < \infty$, one can then easily verify that (A.1) holds whenever $g$ is smooth and $\text{spt } g \subset \mathbb{R}^n \sim H_+$ (even if $\text{spt } g$ is not compact.)

Then we may choose $g(x)$ of the form $(0, \ldots, 0, \gamma(x_n))$ in (A.1), where $\gamma$ is any $C^1(\mathbb{R})$ function with $\text{spt } \gamma$ contained in the positive real numbers. This choice of $g$ immediately yields

\[
\int_{\text{int}(\mathbb{R}^n)} \gamma'(x_n) s^{nn} \, dV(x, S) = 0,
\]

where $(s^{ij})$ denotes the matrix of the orthogonal projection of $\mathbb{R}^n$ onto $S$, so that $s^{nn} > 0$. In view of the arbitraryness of $\gamma$ (for any $\epsilon > 0$ we can choose $\gamma$ so that $\gamma'(t) > 0$ for all $t \in \mathbb{R}$ and $\gamma'(t) > 0$ for $t > \epsilon$), this of course implies that $V[(\mathbb{R}^n \sim H_+) \times \{S \in G(n, k): s^{nn} \neq 0\}] = 0$. However, since $(s^{ij})$ is the matrix of an orthogonal projection (so that $\sum_{j=1}^n (s^{nj})^2 = s^{nn}$), $s^{nn} = 0$ implies $s^{nj} = s^{i\ast} = 0$, $j = 1, \ldots, n$. Then choosing $g(x) = \gamma(x_n) x$ in (A.1), and noting that $S \cdot Dx = k$ for every $S \in G(n, k)$, we obtain

\[
\int_{\text{int}(\mathbb{R}^n)} \gamma(x_n) \, dV(x, S) = 0.
\]

Again using the arbitraryness of $\gamma$, this gives $\text{spt } V \subset \{x: x_n < 0\}$ as required.
Appendix B. Comparisons with surfaces of higher topological type.

Our reason for including the discussion of § 10 above is that the discs obtained (in § 8) bounding simple closed curves \( \Gamma \) on the boundaries of uniformly convex sets \( A \) frequently have area which is larger than that of manifolds of higher topological type such as arise as two dimensional mass minimizing integral currents having \( \Gamma \) as boundary. For example, for suitable \( 1 < m < n < p < \infty \) let

\[
K = \{(x, y, z): x^2 + y^2 < 1\},
\]

\[
A = \{(x, y, z): x^2 + y^2 + (nz)^2 < m^2\},
\]

and \( \Gamma \subset \partial A \) be the simple closed curve

\[
[\partial A \cap \partial K \sim \{(x, y, z): x > 0, (py)^2 < 1\}] \cup [\partial A \cap \{(x, y, z): (py)^2 = 1\}]
\]

with smoothing near the four corners

\[
\partial A \cap \partial K \cap \{(x, y, z): (py)^2 = 1\}
\]

and with the orientation illustrated in Figure 1. For proper choices of \( m, n, p \) it is intuitively clear and readily checkable with the use [FH1, 5.4.3., 5.4.5., 5.4.15.] and [AW2] that the support of any two-dimensional mass minimizing integral current \( T \) having \( \Gamma \) as boundary must be an embedded minimal submanifold of \( \mathbf{R}^3 \) having \( \Gamma \) as boundary and having genus at least 1. In particular, when \( n \) is large one checks by area comparison that

\[
\text{spt} \ T \cap \{(x, y, z): x = y = 0\} = \emptyset
\]

and, also, since for large values of \( p, \Gamma \) is close in the flat metric topology to \( \partial A \cap \partial K \), it follows that \( T \) very nearly must lie within \( K \) and hence for most \( 1 < r < m \),

\[
\text{spt} \ T \cap \{(x, y, z): x = r, z = 0\} = \emptyset.
\]

More generally the references cited above imply that if \( T, T_1, T_2, T_3, \ldots \in \mathcal{I}_c(\mathbf{R}^3) \) are mass minimizing, if \( \lim_i T_i = T \) in the flat topology, if \( L \) is a compact subset of \( \mathbf{R}^3 \), and if

\[
\lim_i \text{Hausdorff distance (spt} \ T_i, L) = 0,
\]
then
\[ \lim r \text{ Hausdorff distance (spt } T_r, \text{ spt } T \cup L) = 0. \]

For such \( r \) as above one notes that the homotopy group
\[ \pi_1(\mathbb{R}^3 \sim [(x, y, z): x = y = 0] \cup [(x, y, z): x = r, z = 0]); (0, 0, 0) \]
is the free group \( \mathbb{Z} \ast \mathbb{Z} \) on two generators and that the homotopy class of \( \Gamma' \)
is the commutator of two such generators, which commutator is of course not equal to the identity. In particular then, spt \( T \) cannot topologically
be a disc. The support of $T$ presumably resembles the surface indicated in Figure 1.

For each $g = 1, 2, 3, \ldots$ our construction readily generalizes to produce smooth simple closed curves $\Gamma$ on the boundary of any uniformly convex subset of $\mathbb{R}^3$ such that the genus of the support of any mass minimizing integral current $T$ having such $\Gamma$ as boundary is at least $g$. Such a curve $\Gamma$

---

Figure 2

on $S^2$ corresponding to $g = 3$ is illustrated in Figure 2. If one wishes, such curves $\Gamma$, corresponding to any $g$, can be required to have arbitrarily short prescribed length; the curve if Figure 2 for example can be slid on $S^2$ to a small neighbourhood of any point.

Curves like that illustrated in Figure 2 have increasingly large curvature as $g$ increases. Indeed, it follows from [AW1] [AW2] [FH1, 4.2.17, 5.4.5, 5.4.15] that if $\Gamma$ is any collection of twice continuously differentiable oriented simple closed curves $\Gamma$ lying in, say, $S^2$ which is compact in the $C^2$ topology, then there is a finite number $g_0 = g_0(\Gamma)$ such that whenever $T$ is a mass minimizing integral current with $\partial T \in \Gamma$ then the genus of $\text{spt} \ T$ does not exceed $g_0$.

On the other hand William P. Thurston has pointed out that for each $\varepsilon > 0$ and $g_0 < \infty$ there is a $C^\infty$ simple closed oriented curve $\Gamma$ lying on $S^2$
such that the length of $\Gamma$ does not exceed $\varepsilon$, the total absolute curvature of $\Gamma$ does not exceed $4\pi + \varepsilon$, and the genus of the support of any mass minimizing integral current having $\Gamma$ as boundary is not less than $g_0$. Such a curve $\Gamma$ can be obtained as follows for $g_0 = 3$. We consider one dimensional integral cycles shaped like $\Gamma_1$ in Figure 3a and note that in case $\Gamma_1$ lies in a plane and $T_1$ is the unique mass minimizing integral current with $\partial T = \Gamma_1$, then spt $T_1$ does not intersect the open region $A \cup B \cup C$. It readily follows that if an integral cycle like $\Gamma_1$ lies on a sphere of sufficiently large radius, then any mass minimizing integral current $T_1$ having $\Gamma_1$ as boundary largely will not cover the region $A \cup B \cup C$. We now fix such $\Gamma_1$ on such a sphere of large radius and observe that the points on $\Sigma$ at positions $a$, $b$, $c$ as indicated in Figure 3a lie somewhat outside the convex hull of $\Gamma_1$. If $\Gamma_1$ is now modified to become the $C^\infty$ simple closed curve $\Gamma_2$ lying on $\Sigma$ of Figure 3b by adding extremely thin bridges as indicated (so that $\Gamma_2$ is very close in the flat topology to $\Gamma_1$) one checks, by arguments similar to those above, that, whenever $T_2$ is a mass minimizing integral current having $\Gamma_2$ as
boundary, the region $A \cup B \cup C$ still will largely be uncovered and there will exist straight lines $L_a$, $L_b$, $L_c$ as indicated entirely missing spt $T_2$ while passing just beneath the three bridges and above the main part of spt $T_2$. One checks, for example by computing the rank of the intersection matrix of appropriately chosen elements of the homology group $H_1(\text{spt } T_2; \mathbb{Z}_2)$ as in [AT], that the genus of spt $T_2$ is at least 3. It is clear from the illustration that such $\Gamma_3$ can be constructed with total curvature as close to $4\pi$ as desired, and, since the radius of $\Sigma$ is large, the curve $\Gamma$ on our standard two dimensional sphere $S^2$ (which is our desired curve) corresponding to $\Gamma_3$ on $\Sigma$ will be short.

REFERENCES


