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Holomorphic Mapping of Annuli in C^n and the Associated Extremal Function.

ERIC BEDFORD (*) - DAN BURNS (*)

Introduction.

For a complex manifold Ω, Chern, Levine and Nirenberg [6] have defined a seminorm on the homology groups $H_*(\Omega, \boldsymbol{R})$ that has the property of decreasing under holomorphic maps. Thus a holomorphic mapping $f: \Omega_1 \to \Omega_2$ is restricted by the norm-decreasing property of its induced action on homology $f_*: H_*(\Omega_1, \boldsymbol{R}) \to H_*(\Omega_2, \boldsymbol{R})$. In this paper we will show that for certain domains $\Omega_1, \Omega_2 \subseteq \boldsymbol{C}^n$, f_* is an isometry of the homology groups if and only if f is a biholomorphism. (In fact we will use only the top-dimensional homology group H_{2n-1}.) In the case where $\Omega_1, \Omega_2 \subset \boldsymbol{C}$ are annuli, this result was established by Schiffer [22] and was extended to the case of d-to-1 mappings by Huber [12].

We will consider domains Ω of the following special form:

$$\Omega = D_1 \setminus \bar{D}_0 , \qquad D_0 \Subset D_1 \Subset \boldsymbol{C}^n, \ n \geqslant 2$$

(*) D_0 and D_1 are strictly pseudoconvex with smooth boundary, $\bar{D}_0$ is connected and holomorphically convex in D_1.

(If Ω is multicircular and $0 \notin \Omega$, then Ω is actually a topological annulus.) The norm of the homology class $\Gamma = [\partial D_0] = [\partial D_1]$ is defined as

$$(**) \qquad N\{\Gamma\} = \sup_{v \in \mathcal{F}} \int_\Gamma d^c v \wedge (dd^c v)^{n-1} = \sup_{v \in \mathcal{F}} \int_\Gamma d^c v \wedge dd^c v \wedge \ldots \wedge dd^c v$$

where the family $\mathcal{F}$ consists of $v \in C^2(\Omega)$, $0 < v < 1$ which are plurisubharmonic and satisfy $(dd^c v)^n = dd^c v \wedge \ldots \wedge dd^c v = 0$. This is a higher-dimen-

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sional analogue of harmonic measure, and it may be shown that this supremum is achieved by the solution u of (1.1) if $u \in C^2(\Omega)$. We will refer to $N\{\Gamma\}$ as the *norm* $N\{\Omega\}$ of Ω.

THEOREM 1. *Let $f\colon \Omega_1 \to \Omega_2$ be a holomorphic mapping, with Ω_1 and Ω_2 as above, and let Ω_1, Ω_2 be Reinhardt* (i.e. *multicircular*) *domains. If $N\{\Omega_1\} = N\{\Omega_2\}$ and the mapping $f_*\colon H_{2n-1}(\Omega_1, \boldsymbol{R}) \to H_{2n-1}(\Omega_2, \boldsymbol{R})$ is not zero, then f is a biholomorphism.*

This theorem is proved by showing that the solution of (1.1) is the unique function of $\mathcal{F}$ which attains the supremum in $N\{\Gamma\}$. This technique was used for Riemann surfaces by Landau and Osserman [17]. In fact the proof (see Theorem 2.1) yields

THEOREM 1′. *Let Ω_1, Ω_2 be simply connected domains of the form* (∗), *and assume that the solutions u_1 and u_2 of* (1.1) *on Ω_1 and Ω_2 are of class $C^4(\bar{\Omega}_j)$ and satisfy* (1.4). *If $N\{\Omega_1\} = N\{\Omega_2\}$, and $f\colon \Omega_1 \to \Omega_2$ is a holomorphic mapping with $f_* \neq 0$, then f is a biholomorphism.*

The restriction to the class of Reinhardt domains in Theorem 1 arises because it is not known whether the solution of (1.1) is smooth and satisfies (1.4) on more general domains. Another approach to the study of holomorphic mappings via a related Dirichlet problem has been described by Kerzman [16].

If u satisfies $(dd^c u)^{n-1} \neq 0$, $(dd^c u)^n = 0$ on a domain $D \subseteq \boldsymbol{C}^n$, there is an associated foliation $\mathcal{F} = \mathcal{F}(u)$ on D: each leaf M of $\mathcal{F}$ is a Riemann surface, and $u|_M$ is harmonic on M. The foliation $\mathcal{F}$ is an important part of the proof of Theorem 2.1 and seems to be the main feature which distinguishes the cases $n = 1$ and $n \geqslant 2$. In Section 4, several remarks are made about $\mathcal{F}$. They center around the observation that if the normal bundle $\mathcal{N}$ of $\mathcal{F}$ is given the fiber metric $dd^c u$, then the Ricci curvature of this metric measures the antiholomorphic twist of $\mathcal{F}$. The reason for studying $\mathcal{F}$ in more detail is Theorem 4.6, which yields

THEOREM 1″. *Theorem 1′ remains valid if the hypothesis* (1.4) *is replaced by the assumption that u_j is real analytic on Ω_j, $j = 1, 2$.*

In the beginning of Section 4, it is shown that there is a topological obstruction which prevents the solution of (1.1) from satisfying (1.4) if D_0 has more than one component. Some information is also obtained concerning the real Monge-Ampère equation $\det(\partial^2 u/\partial x_i\, \partial x_j) = 0$.

In order to discuss self maps of domains $\Omega \subseteq \boldsymbol{C}^n$, one may use a continuous (generalized) solution of (1.1), which is known to exist. Thus it is possible to show that certain bounded domains $\Omega \subseteq \boldsymbol{C}^n$ are holomorphically « rigid » in the sense of H. Cartan [5]. For this purpose one may define several seminorms $\tilde{N}$ on $H_{2n-1}(\Omega, \boldsymbol{R})$ which are similar to that of Chern, Levine and Nirenberg. Let us suppose that the bounded components of $\boldsymbol{C}^n \setminus \Omega$ consist of $K_1 \cup \ldots \cup K_J \cup E$ where the K_j and E are closed and pairwise disjoint.

THEOREM 2. *Let Ω be a bounded domain in $\boldsymbol{C}^n$, and assume that $\tilde{N}$ is not identically zero on $H_{2n-1}(\Omega, \boldsymbol{R})$ but that $\tilde{N}$ is identically zero on*

$$H_{2n-1}(\Omega \cup K_1 \cup \ldots \cup K_J, \boldsymbol{R}) .$$

The following are equivalent for every holomorphic mapping $f\colon \Omega \to \Omega$:

(i) f *is an automorphism*

(ii) f_* *is injective*

(iii) f_* *is an isometry.*

This theorem applies, for instance, to the case where $n \geqslant 2$ and at least one of K_j has nonempty interior (Corollary 3.2). Section 3 contains examples where various norms $\tilde{N}$ are zero (or nonzero) and examples of singularities which are removable (or not) for plurisubharmonic functions. Rigidity theorems for plane domains have been proved by Huber [13] and Landau and Osserman [17]; these results are more general than Theorem 2 in the case $n = 1$. The reader should also compare these results with related results of Eisenman [8, p. 72].

A different measure on homology classes, which we discuss only briefly, may be defined in terms of the Carathéodory metric. If γ is a k-dimensional homology class on Ω, then $C\{\gamma\}$ is the infimum of the k-dimensional Hausdorff measure (with respect to the Carathéodory distance) taken over all chains representing γ. Let $\boldsymbol{B}^n$ be the unit ball in $\boldsymbol{C}^n$, and let K_1, K_2 be compact subsets that are convex in the Carathéodory metric of $\boldsymbol{B}^n$.

THEOREM 3. *Let $f\colon \boldsymbol{B}^n \setminus K_1 \to \boldsymbol{B}^n \setminus K_2$ be holomorphic, and assume that $C\{\partial K_1\} = C\{\partial K_2\}$. If $f_* \neq 0$, then f is an automorphism of $\boldsymbol{B}^n$.*

Examples are given to show that $N\{\gamma\}$ and $C\{\gamma\}$ are not functionally related.

1. – The Cauchy problem for the complex Monge-Ampère equation.

We use the notation $d^c = i(\bar{\partial} - \partial)$ so that $dd^c = 2i\partial\bar{\partial}$ and $(dd^c)^n = dd^c \wedge \ldots \wedge dd^c$. Let us summarize some known results (see [4] for details). The operator $(dd^c)^n$ has a continuous extension to the space of continuous, plurisubharmonic functions, with $(dd^c v)^n$ being defined as a measure on Ω. If we consider the class of functions

$$\mathcal{F}' = \{v \in C(\Omega),\ 0 < v < 1,\ v \text{ plurisubharmonic and } (dd^c v)^n = 0\}$$

then we may define a seminorm $\tilde{N}\{\gamma\}$ using (**) with $\mathcal{F}$ replaced by the larger class $\mathcal{F}'$ and the integration being taken with respect to a smooth current representing γ. In connection with this, one is led to consider the extremal function $u = \sup v$, where the supremum is taken over the functions $v \leqslant 1$ which are plurisubharmonic on Ω_1 and satisfy $v \leqslant 0$ on Ω_0: (This function has been studied by Siciak [23] and Zaharjuta [26] in other contexts.) The function u thus obtained is Lipschitz continuous and is a generalized solution of the Dirichlet problem

$$(1.1)\qquad \begin{cases} u & \text{plurisubharmonic} \\ u = 1 & \text{on } \partial\Omega_1 \\ u = 0 & \text{on } \partial\Omega_0 \\ (dd^c u)^n = 0\,. \end{cases}$$

The following estimate holds:

$$(1.2)\qquad N\{\gamma\} \leqslant \tilde{N}\{\gamma\} = \int_\Omega du \wedge d^c u \wedge (dd^c u)^{n-1}$$

where the integral is interpreted in a generalized sense. The problem (1.1) has been solved explicitly in several examples and the solution is regular and satisfies $(dd^c u)^{n-1} \neq 0$ in each case (but cf. § 4 on this point). If $u \in C^2(\Omega)$, it follows that equality holds in (1.2) and

$$(1.3)\qquad N\{\Gamma\} = \int_\gamma d^c u \wedge (dd^c u)^{n-1}\,.$$

If $u \in C^2(\Omega)$ satisfies

$$(1.4)\qquad (dd^c u)^n = 0\,, \qquad (dd^c u)^{n-1} \neq 0$$

there is further structure associated with u. The $(n-1, n-1)$-form $(dd^c u)^{n-1}$ may be integrated by the Frobenius theorem to yield a foliation $\mathcal{F} = \mathcal{F}(u)$ of Ω by complex manifolds of dimension one (Riemann surfaces). This foliation has the property that $u|_L$ is harmonic on L for each leaf L of $\mathcal{F}$ (see [2] for details). We note that the solution u of (1.1), if it is smooth, always satisfies (1.4) in a neighborhood of $\partial\Omega$ since it is constant on each (strictly pseudoconvex) boundary component.

Let $S = \{z \in C^n : r(z) = 0\}$, $dr \neq 0$ on S, $r \in C^2(C^n)$ be a real hypersurface. We will say that S is *non-characteristic* for a solution u of (1.4) if

$$(dd^c u)^{n-1} \wedge dr \wedge d^c r \neq 0 . \tag{1.5}$$

This is equivalent to the condition that each leaf $L \in \mathcal{F}$ intersects S transversally. This leads to uniqueness in the Cauchy problem.

PROPOSITION 1.1. *Let us suppose that u, v are C^3 and satisfy* (1.4) *in a neighborhood of S. Suppose that $u(z) = v(z)$ and $du(z) = dv(z)$ for $z \in S$, and let S be non-characteristic for u at $z_0 \in S$. If u, v satisfy* (1.4) *in a neighborhood of z_0, $u = v$ in a neighborhood of z_0.*

PROOF. We will show that there is an open set W containing z_0 such that the foliations $\mathcal{F}(u)$ and $\mathcal{F}(v)$ concide on W in the following sense: if $z \in W$ and L, L' are the leaves containing z, then the connected components of $L \cap W$ and $L' \cap W$ containing z are the same. By the non-characteristic condition, the leaves are transverse to S, and it suffices to show this for $z \in S \cap W$.

Thus we suppose that $z \in S \cap W$, and that L, L' are the integral curves of $(dd^c u)^{n-1}$ (respectively $(dd^c v)^{n-1}$) which pass through z. Since $u - v$ vanishes to second order on S, it follows that all second partial derivatives of $(u - v)$ except $(u - v)_{rr}$ vanish on S. From the non-characteristic condition and from $(dd^c u)^n = 0$, we may solve for u_{rr} in terms of all the other second partial derivatives at z. Thus it follows from (1.4) that $(u - v)_{rr} = 0$ on $W \cap S$.

Finally, if $\mathcal{F} \cap S$ and $\mathcal{F}' \cap S$ denote the foliations of S obtained by intersecting the leaves of $\mathcal{F}$ and $\mathcal{F}'$ with S, then we observe that these foliations have normal bundles $(dd^c u)^{n-1} \wedge dr$ and $(dd^c v)^{n-1} \wedge dr$ respectively. We have seen that for $z \in S \cap W$, the second derivatives of u and v at z are equal, and thus the foliations $\mathcal{F} \cap S$ and $\mathcal{F}' \cap S$ have the same tangent spaces. By the uniqueness theorem for ordinary differential equations, we see that $\mathcal{F} \cap S \cap W = \mathcal{F}' \cap S \cap W$. It follows that $\mathcal{F}$ and $\mathcal{F}'$ must agree in a neighborhood of z_0, since $z \in L \cap L'$ implies that $L \cap L'$ contains a curve in S; and $L \cap L'$ can contain a curve only if $L = L'$.

COROLLARY 1.2. *Let the function r defining S be strictly plurisubharmonic, and set*

$$\Omega = \{z \in C^n : r(z) < 0\} .$$

If $z_0 \in S = \partial\Omega$ *and there is a neighborhood* W *containing* z_0 *such that* $u, v \in C^3(\bar{\Omega})$ *satisfy* (1.4) *on* $W \cap \bar{\Omega}$ *and* $u(z) = v(z)$, $du(z) = dv(z)$ *for* $z \in W \cap S$, *then there is a smaller neighborhood* W' *containing* z_0 *such that* $u = v$ *on* $W' \cap \Omega$.

PROOF. Since S is strongly pseudoconvex at z_0, we may make a change of coordinates such that $z_0 = 0$, S is given near z_0 by a convex function $\operatorname{Re} z_1 = \varphi(\operatorname{Im} z_1, z_2, \ldots, z_n) \geqslant 0$, and $\varphi \geqslant \delta > 0$ on the set

$$(\operatorname{Im} z_1)^2 + |z_2|^2 + \ldots + |z_n|^2 = \varepsilon\,.$$

We claim that for $z \in \Omega$ such that $\operatorname{Re} z_1 < \delta$ and $(\operatorname{Im} z_1)^2 + |z_n|^2 < \varepsilon$, $u(z) = v(z)$. To see this, let L be the leaf of $\mathcal{F}(u)$ containing z. Since u satisfies (1.4) on $\bar{\Omega}$, it follows that L may be extended to $\bar{\Omega}$ with a continuous tangent plane. By the maximum principle, L must intersect S in a point $\zeta = (\zeta_1, \ldots, \zeta_n)$ with $\operatorname{Re} \zeta_1 < \delta$ and $(\operatorname{Im} \zeta_1)^2 + |\zeta_2|^2 + \ldots + |\zeta_n|^2 < \varepsilon$. Since S is strongly pseudoconvex, L can be tangent to S only at an isolated subset of $L \cap S$. Thus, moving to a nearby point $\zeta' \in L \cap S$ if necessary, we find a point where S is non-characteristic for u. By Proposition 1.1, it follows that $u = v$ on L.

COROLLARY 1.3 *Suppose* Ω_1, Ω_2 *are domains of the form* (*), *and suppose* Ω_1, Ω_2 *have the same inner or outer boundary. If* $\Omega_1 \subsetneqq \Omega_2$, *if the solution* u_j *of* (1.1) *is* $C^3(\bar{\Omega}_j)$ *for* $j = 1, 2$, *and if* $du_1 \wedge d^c u_1 \wedge (dd^c u_1)^{n-1} \neq 0$, *then* $N\{\Omega_1\} > N\{\Omega_2\}$.

PROOF. By the norm decreasing property of holomorphic mapping, it follows that $N\{\Omega_1\} \geqslant N\{\Omega_2\}$. To show that this inequality is strict, we observe (cf. equation (3.14) of [4])

$$N\{\Omega_j\} = \int_{\Gamma} \left(\frac{\partial u_j}{\partial r}\right)^n d^c r \wedge (dd^c r)^{n-1}$$

where $\Gamma = \{r = 0\}$, $dr \neq 0$ on Γ, the common boundary of Ω_1 and Ω_2. Since $\Omega_1 \subsetneqq \Omega_2$ it follows that $\partial u_1(z)/\partial r \geqslant \partial u_2(z)/\partial r$ for $z \in \Gamma$. If the norms are equal, then it follows that $\partial u_1/\partial r = \partial u_2/\partial r$ on Γ. By Proposition 1.1 it follows that $u_1 = u_2$, which is impossible since $\Omega_1 \neq \Omega_2$ and u_j satisfies (1.1).

We remark that this will be a special case of Theorem 2.1.

Uniqueness in the Cauchy problem also holds for solutions of

$$(dd^c u)^n = f\,dV \tag{1.6}$$

with $f > 0$, and dV is the volume form on C^n. (In our applications to holomorphic mappings, however, we will only be concerned with the case $f = 0$.)

PROPOSITION 1.4. *Let $u, u' \in C^\infty$ be plurisubharmonic and satisfy* (1.6) *for some $f > 0$. If $u(z) = u'(z)$, $du(z) = du'(z)$ for $z \in S$, then $u = u'$ in a neighborhood of S.*

PROOF. Since u, u' satisfy (1.6), then $w = u - u'$ satisfies

$$dd^c w \wedge [(dd^c u)^{n-1} + (dd^c u)^{n-2} \wedge dd^c u' + \dots + (dd^c u')^{n-1}] = 0 .$$

Let us write this as

$$dd^c w \wedge \omega = L(w) = 0 .$$

Since $(dd^c u)^n > 0$, it follows that ω is a strictly positive form, and so L is a strongly elliptic operator. It is known (see Nirenberg [20]) that uniqueness in the Cauchy problem holds for second order strongly elliptic operators with C^∞ coefficients, and so we conclude that $w = 0$ in a neighborhood of S.

PROPOSITION 1.5. *Let the function r defining S be real analytic, let φ_0, φ_1 be real analytic and let f be analytic in a neighborhood of S. If*

$$dr \wedge d^c r \wedge \big(dd^c(\varphi_0 + r\varphi_1)\big)^{n-1} \neq 0 \tag{1.7}$$

on S, then there exists a real analytic function u in a neighborhood of S such that

$$u(z) = \varphi_0(z) , \qquad \frac{\partial u}{\partial r}(z) = \varphi_1(z)$$

for $z \in S$ and $(dd^c u)^n = f$ in a neighborhood of S.

PROOF. If we write the solution $u = \sum_{j=0}^{\infty} \varphi_j r^j$ as a formal power series with φ_j analytic on S, then the condition (1.7) allows us to solve for φ_j in terms of $\varphi_0, \dots, \varphi_{j-1}$ in the equation

$$(dd^c u)^n = \left(\sum_{j=0}^{\infty} dd^c(r^j \varphi_j)\right)^n = f .$$

The proof that the formal power series converges is a special case of the Cauchy-Kovalevsky theorem.

From this, one may obtain special exhaustion functions for pseudoconvex domains with real analytic boundary.

COROLLARY 1.6. *Let Ω be a bounded, strongly pseudoconvex domain with real analytic boundary in C^n. Then there exists a function $\varrho \in C^\infty(\bar{\Omega})$ such that $\Omega = \{\varrho < 0\}$, $d\varrho \neq 0$ on $\partial\Omega$, and $(dd^c\varrho)^n = 0$ in a neighborhood of $\partial\Omega$.*

PROOF. By Proposition 1.5, there is a plurisubharmonic real analytic function u defined in a neighborhood of $\partial\Omega$ such that $(dd^cu)^n = 0$ and $\partial u/\partial r = 1$, $u = 0$ on $\partial\Omega$. For $\varepsilon > 0$ small, we let $\chi(t)$ be a convex function that is 0 for $t < -\varepsilon$ and such that $\chi'(t) = 1$ for $t > -\varepsilon/2$, and thus $\varrho = \chi(u) - \chi(0)$ is the desired function.

2. - A uniqueness theorem and some corollaries.

Here we show that the plurisubharmonic measure (i.e., solution of (1.1)) is the unique function that gives equality in (**).

THEOREM 2.1. *Let $\Omega \subset C^n$ be a domain of the form (*), and suppose that the solution u of (1.1) is smooth of class $C^4(\Omega) \cap C^3(\bar{\Omega})$ and satisfies (1.4). Let $v \in C^4(\Omega) \cap C^3(\Omega \cup \partial D_0)$ be another plurisubharmonic function with $0 < v < 1$, $(dd^cv)^n = 0$ and $(dd^cv)^{n-1} \neq 0$ off of a proper analytic subvariety of Ω. If $\Gamma = [\partial D_1]$ and*

$$\int_\Gamma d^cu \wedge (dd^cu)^{n-1} = \int_\Gamma d^cv \wedge (dd^cv)^{n-1} \tag{2.1}$$

then $u = v$.

Let us note some corollaries.

COROLLARY 2.2. *Let Ω_1, Ω_2 be annuli (satisfying (*)), and assume that the harmonic measure u_j of Ω_j is in $C^4(\bar{\Omega}_j)$ and satisfies (1.4). If $f\colon \Omega_1 \to \Omega_2$ is a holomorphic map with $N\{f_* \Gamma_1\} = N\{\Gamma_2\}$, then f is a proper, unramified covering of Ω_2 by Ω_1. Further, f has some smoothness at the boundary: if $u_2 \in C^{2k}(\bar{\Omega}_2)$, then $f \in C^{k-1,1}(\bar{\Omega}_1)$.*

PROOF. We apply Theorem 2.1 with $\Omega = \Omega_1$, $u = u_1$, and $v = f^* u_2$, and we conclude that $u_1 = u_2(f)$. Thus f is proper. To see that f is unramified, we use the argument of Kerzman, Kohn, and Nirenberg [16]: $\exp(u_1) = \exp(u_2(f))$ is a strictly plurisubharmonic function, and if we compute the complex hessian using the chain rule, we obtain $|f'| \neq 0$. Taking further derivatives with the chain rule and using another trick of [16], we obtain the boundary regularity of f. Let us note that if f is first known to be smooth at $\partial\Omega_1$, then Fornaess [10] shows that f is unramified.

REMARK. If Ω_1 and Ω_2 have C^m boundaries, then Naruki [19] shows that f is smooth of order $m-4$, provided that f is $C^4(\bar{\Omega}_1)$.

A large portion of the proof of Theorem 2.1 is spent proving Lemma 2.6. Without Lemma 2.6, it is still possible to conclude, in Corollary 2.2, that $u_1 \leqslant u_2(f)$, and thus f is proper.

Theorem 1 is a consequence of the following corollary, since for a Reinhardt domain satisfying (*) the solution of (1.1) is as smooth as $\partial D_0 \cup \partial D_1$.

COROLLARY 2.3. *Let Ω_1, Ω_2 satisfy* (*) *with $n \geqslant 2$, and let the solution u_j of* (1.1) *satisfy* (1.4) *and $u_j \in C^4(\bar{\Omega}_j)$, $j = 1, 2$. Let $N\{\Omega_1\} = mN\{\Omega_2\}$ with m an integer. If $f\colon \Omega_1 \to \Omega_2$ is a holomorphic mapping, then the degree of f* (i.e. $f_* \Gamma_1 = (\deg f)\Gamma_2$) *is an integer $0 \leqslant \deg f \leqslant m$. If Ω_2 is simply connected and if $\deg f = m$, then $m = 1$ and f is a biholomorphism. In general, if $\deg f = m = 1$, then f is a biholomorphism.*

PROOF. The degree of f is a non-negative integer since f is a holomorphic mapping and $n \geqslant 2$. Since the norm decreases, $\deg f \leqslant m$. If $\deg f = m$, then by Corollary 2.2, f is a covering which must be a biholomorphism since Ω_2 is simply connected. Thus $m = 1$.

We begin the proof of Theorem 2.1 with a sequence of lemmas. If

$$\Omega^{\pm} = \{z \in \Omega\colon \pm (u - v) > 0\},$$

it will ultimately be shown that $\Omega^+ = \Omega^- = \emptyset$.

LEMMA 2.4. *Let M be a leaf of $\mathcal{F}(u)$, the u-foliation. Then M reaches both the inner and outer boundaries of Ω,* i.e. *$\bar{M} \cap \partial D_j \neq 0$ for $j = 1, 0$.*

PROOF. M must reach the outer boundary since D_1 is pseudoconvex (see [2]). For the inner boundary, let $c = \inf\{u(z)\colon z \in M\} = \min\{u(z)\colon z \in \bar{M}\}$. If $c > 0$, let $z' \in \bar{M}$ be a point such that $u(z') = c$. Let M' be the $\mathcal{F}(u)$ leaf through z'. Now $M' \subset \bar{M}$, and so by the maximum principle $u = c$ on M', since u is harmonic on M'. This is a contradiction since M' reaches the outer boundary, and u comes arbitrarily close to 1 on M'.

LEMMA 2.5. *If $\Omega^- = \emptyset$, then $u = v$.*

PROOF. If $u \geqslant v$, then $v = 0$ on ∂D_0, and so $\partial u/\partial r \geqslant \partial v/\partial r \geqslant 0$ there.

However

$$\int_{\Gamma} d^c u \wedge (dd^c u)^{n-1} = \int_{\partial D_0} \left(\frac{\partial u}{\partial r}\right)^n d^c r \wedge (dd^c r)^{n-1} =$$

$$= \int_{\Gamma} d^c v \wedge (dd^c v)^{n-1} = \int_{\partial D_0} \left(\frac{\partial v}{\partial r}\right)^n d^c r \wedge (dd^c r)^{n-1}$$

where r is any smooth defining function for D_0. It follows that $u - v = = 0 = d(u - v)$ on ∂D_0, and so $u = v$ in a neighborhood of ∂D_0 by Proposition 1.1. Since the two foliations $\mathcal{F}(u)$ and $\mathcal{F}(v)$ must agree near ∂D_0, it follows by Lemma 2.4 that $u = v$ on Ω.

LEMMA 2.6. *If $\Omega^+ = \emptyset$, then $u = v$.*

PROOF. If $u \leqslant v$, then $v \in C(\bar{\Omega})$ and $v \equiv 1$ on ∂D_1. Let $u_\varepsilon = u/_{(1-\varepsilon)}$ for $\varepsilon > 0$ small, and set

$$\Omega_\varepsilon = \{z \in \Omega\colon 0 < u_\varepsilon < 1\} = \{0 < u < 1 - \varepsilon\}, \qquad \Gamma_\varepsilon = \{u_\varepsilon = 1\}.$$

We will show that

$$\int_{\Gamma} d^c v \wedge (dd^c v)^{n-1} \leqslant \int_{\partial\Omega} v\, d^c u \wedge (dd^c u)^{n-1} \tag{2.2}$$

holds. We recall that $\partial\Omega = \partial D_1 - \partial D_0$ and that $d^c u \wedge (dd^c u)^{n-1}$ is strictly positive everywhere on ∂D_0. By (2.1) it follows that $v = 0$ on ∂D_0. The solution of (1.1) is unique (see [4]), so $u = v$ on Ω.

Let us first establish an inequality

$$\begin{aligned} &\int_{\Omega_\varepsilon} d^c u_\varepsilon \wedge d^c v \wedge (dd^c v)^i \wedge (dd^c u_\varepsilon)^{n-i-1} \\ &\qquad \leqslant \int_{\Omega_\varepsilon} du_\varepsilon \wedge d^c v \wedge (dd^c v)^{i-1} \wedge (dd^c u_\varepsilon)^{n-i} + \int_{\Omega_\varepsilon} (u_\varepsilon - v)(dd^c v)^i \wedge (dd^c u_\varepsilon)^{n-i} \end{aligned} \tag{2.3}$$

for $i > 0$. This is obtained by repeated integration by parts:

$$\begin{aligned} &\int_{\Omega_\varepsilon} du_\varepsilon \wedge d^c v \wedge (dd^c v)^i \wedge (dd^c u_\varepsilon)^{n-i-1} \\ &\qquad = \int_{\Omega_\varepsilon} dv \wedge d^c u_\varepsilon \wedge (dd^c v)^i \wedge (dd^c v_\varepsilon)^{n-i-1} \\ &\qquad = \int_{\Omega_\varepsilon} d\big(v d^c u_\varepsilon (dd^c v)^i \wedge (dd^c u_\varepsilon)^{n-i-1}\big) - v (dd^c v)^i \wedge (dd^c u_\varepsilon)^{n-i} \end{aligned}$$

$$
\begin{aligned}
&= \int_{\partial\Omega_\varepsilon} v\, d^c u_\varepsilon \wedge (dd^c v)^i \wedge (dd^c u_\varepsilon)^{n-i-1} - \int_{\Omega_\varepsilon} v (dd^c v)^i \wedge (dd^c u_\varepsilon)^{n-i} \\
&\leqslant \int_{\Gamma_\varepsilon} d^c u_\varepsilon \wedge (dd^c v)^i \wedge (dd^c u_\varepsilon)^{n-i-1} - \int_{\Omega_\varepsilon} v (dd^c v)^i \wedge (dd^c u_\varepsilon)^{n-i} \\
&= \int_{\Gamma_\varepsilon} d^c v \wedge (dd^c v)^{i-1} \wedge (dd^c u_\varepsilon)^{n-i} - \int_{\Omega_\varepsilon} v (dd^c v)^i \wedge (dd^c u_\varepsilon)^{n-i} \\
&= \int_{\partial\Omega_\varepsilon} u_\varepsilon d^c v \wedge (dd^c v)^{i-1} \wedge (dd^c u_\varepsilon)^{n-i} - \int_{\Omega_\varepsilon} v (dd^c v)^i \wedge (dd^c u^i)^{n-} \\
&= \int_{\Omega_\varepsilon} du_\varepsilon \wedge d^c v \wedge (dd^c v)^{i-1} \wedge (dd^c u_\varepsilon)^{n-i} + \int_{\Omega_\varepsilon} (u_\varepsilon - v)(dd^c v)^i \wedge (dd^c u_\varepsilon)^{n-i} .
\end{aligned}
$$

The inequality arose because $v \leqslant 1$ on Γ_ε, $0 \leqslant v$ on ∂D_0, and $d^c u_\varepsilon \wedge (dd^c v)^i \wedge \wedge (dd^c u_\varepsilon)^{n-i-1}$ is a non-negative form on Γ_ε and ∂D_0.

Since u solves (1.1), we obtain

$$
\int_{\Gamma_\varepsilon} d^c v \wedge (dd^c v)^{n-1} = \int_{\partial\Omega_\varepsilon} u_\varepsilon d^c v \wedge (dd^c v)^{n-1} = \int_{\Omega_\varepsilon} du_\varepsilon \wedge d^c v \wedge (dd^c v)^{n-1} . \tag{2.4}
$$

By repeated application of (2.3) to (2.4) we obtain

$$
\begin{aligned}
\int_{\Gamma_\varepsilon} d^c v \wedge (dd^c v)^{n-1} \leqslant & \int_{\Omega_\varepsilon} du_\varepsilon \wedge dv \wedge (dd^c u_\varepsilon)^{n-1} \\
& + \sum_{i=1}^{n-1} \int_{\Omega_\varepsilon} (u_\varepsilon - v)(dd^c v)^i \wedge (dd^c u_\varepsilon)^{n-i} \\
& = \int_{\Omega_\varepsilon} dv \wedge du_\varepsilon \wedge (dd^c u_\varepsilon)^{n-1} + \sum_{i=1}^{n-1} \int_{\Omega_\varepsilon} (u_\varepsilon - v)(dd^c v)^i \wedge (dd^c u_\varepsilon)^{n-i} \\
& = \int_{\partial\Omega_\varepsilon} v\, d^c u_\varepsilon \wedge (dd^c u_\varepsilon)^{n-1} + \sum_{i=1}^{n-1} \int_{\Omega_\varepsilon} (u_\varepsilon - v)(dd^c v)^i \wedge (dd^c u_\varepsilon)^{n-i} .
\end{aligned}
$$

Thus, in order to prove (2.2) it remains only to show that the limit of the right hand term is $\leqslant 0$ as $\varepsilon \to 0$.

To prove that

$$
\overline{\lim_{\varepsilon \to 0}} \int_{\Omega_\varepsilon} (u_\varepsilon - v)(dd^c v)^i \wedge (dd^c u_\varepsilon)^{n-i} \leqslant 0 ,
$$

we proceed by induction on i. The case $i = 0$ is trivial. First we note that

$$\int_{\Omega_\varepsilon}(u_\varepsilon - v)(dd^c v)^i \wedge (dd^c u_\varepsilon)^{n-1} = \int_{\Omega_\varepsilon}[(u_\varepsilon - u) + (u - v)]\,(dd^c v)^i \wedge (dd^c u_\varepsilon)^{n-i}$$
$$\leqslant \int_{\Omega_\varepsilon}(u_\varepsilon - u)(dd^c v)^i \wedge (dd^c u_\varepsilon)^{n-i} = \frac{\varepsilon}{1-\varepsilon}\int_{\Omega_\varepsilon} u (dd^c v)^i \wedge (dd^c u_\varepsilon)^{n-i}.$$

Thus it suffices to show that

$$\lim_{\varepsilon\to 0} \varepsilon \int_{\Omega_\varepsilon} (dd^c v)^i \wedge (dd^c u_\varepsilon)^{n-i} = 0\,. \tag{2.5}$$

For this, we note that

$$\int_{\Omega_\varepsilon}(dd^c u_\varepsilon)^{n-i}\wedge (dd^c v)^i = \int_{\partial\Omega_\varepsilon} d^c v \wedge (dd^c v)^{i-1} \wedge (dd^c u_\varepsilon)^{n-i}$$
$$\leqslant \int_{\Omega_\varepsilon} dv \wedge d^c u_\varepsilon \wedge (dd^c v)^{i-1} \wedge (dd^c u_\varepsilon)^{n-i} + \int_{\Omega_\varepsilon} u_\varepsilon (dd^c v)^i \wedge (dd^c u_\varepsilon)^{n-i},$$

and thus

$$0 \leqslant \int_{\Omega_\varepsilon}(1 - u_\varepsilon)(dd^c u_\varepsilon)^{n-i}\wedge (dd^c v)^i$$
$$\leqslant \int_{\Omega_\varepsilon} du_\varepsilon \wedge d^c v \wedge (dd^c v)^{i-1} \wedge (dd^c u_\varepsilon)^{n-i}$$
$$\leqslant \int_{\Gamma_\varepsilon} d^c u_\varepsilon \wedge (dd^c u_\varepsilon)^{n-1} + \sum_{j=1}^{i-1} \int_{\Omega_\varepsilon}(u_\varepsilon - v)(dd^c v)^j \wedge (dd^c u_\varepsilon)^{n-j}\,.$$

Assuming, by induction, that (2.5) holds for $i - 1$, we take $\overline{\lim}_{\varepsilon\to 0}$ of both sides and get

$$\overline{\lim_{\varepsilon\to 0}} \int_{\Omega_\varepsilon}(1 - u_\varepsilon)(dd^c u)^{n-i}\wedge (dd^c v)^i \leqslant \int_{\Gamma} d^c u \wedge (dd^c u)^{n-1}\,.$$

By monotone convergence, we get

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}(1 - u - \varepsilon)(dd^c u)^{n-i}\wedge (dd^c v)^i = \int_{\Omega}(1 - u)(dd^c u)^{n-i}\wedge (dd^c v)^i \leqslant \int_{\Gamma} d^c u \wedge (dd^c u)^{n-1}$$

which implies (2.5), completing the proof.

Next we discuss the set

$$S = \{z \in \Omega \cup \partial D_0 : u(z) = v(z)\} .$$

LEMMA 2.7. *On S,*

$$d(u - v) \wedge d^c(u - v) \wedge (dd^c u)^{n-1} = 0 . \tag{2.6}$$

PROOF. Let $z_0 \in S$ be a point where the expression in (2.6) is strictly positive. We set $u_\varepsilon = (u - \varepsilon)/(1 - 2\varepsilon)$ and

$$\theta_\varepsilon = (dd^c u_\varepsilon)^{n-1} + (dd^c u_\varepsilon)^{n-2} \wedge dd^c v + \ldots + (dd^c v)^{n-1} .$$

An integration by parts and (2.1) gives

$$\left[\left(\frac{1}{1-2\varepsilon}\right)^n - 1\right] N\{\Gamma\} = \int_\Gamma d^c u_\varepsilon \wedge (dd^c u_\varepsilon)^{n-1} - \int_\Gamma dv \wedge (dd^c v)^{n-1} = \int_\Gamma d^c(u_\varepsilon - v) \wedge \theta_\varepsilon . \tag{2.7}$$

For $\frac{1}{2} > \varepsilon > 0$, $\gamma_\varepsilon = \{z \in \bar{\Omega} : u_\varepsilon(z) = v(z)\}$ is a compact subset of Ω. If we smooth u_ε and v with a non-negative smoothing kernel χ_ε^δ, then the smoothed function u_ε^δ and v^δ define a compact set $\gamma_\varepsilon^\delta$. By Sard's theorem, $\gamma_\varepsilon^\delta$ is smooth for almost all ε. Thus we may replace Γ by the smooth cycle $\gamma_\varepsilon^\delta$. By assumption, $d(u - v)(z_0) \neq 0$, so that $\{u = v\}$ is a smooth surface in a neighborhood U containing z_0. Furthermore, there are points $z_\varepsilon^\delta \subset \gamma_\varepsilon^\delta$ with $z_\varepsilon^\delta \to z_0$. Finally, since u and v are plurisubharmonic, $d^c(u_\varepsilon^\delta - v_\varepsilon^\delta) \wedge \theta_\varepsilon^\delta$ is a nonnegative form on $\gamma_\varepsilon^\delta$, and

$$\lim_{\delta,\varepsilon\to 0} \int_{\gamma_\varepsilon} d^c(u_\varepsilon^\delta - v^\delta) \wedge \theta_\varepsilon^\delta \geqslant \lim_{\delta,\varepsilon\to 0} \int_{U\cap\gamma_\varepsilon} d^c(u_\varepsilon^\delta - v^\delta) \wedge \theta_\varepsilon^\delta = \int_{U\cap\{u=v\}} d^c(u - v) \wedge \theta_0 \geqslant \int_{U\cap\{u=v\}} d^c(u - v) \wedge (dd^c u)^{n-1} > 0 .$$

This is a contradiction since the left hand side of (2.7) approaches 0 as $\varepsilon \to 0$, and so (2.6) holds at z_0.

LEMMA 2.8. *Let M be a leaf of $\mathcal{F}(u)$. If $M \cap \Omega^+ \neq \emptyset$, then $M \subset \bar{\Omega}^+$.*

PROOF OF THEOREM 2.1. This lemma will give a proof of the theorem, for by Lemma 2.4, M must reach some point $z_- \in \partial D_0$. Thus there is a neighborhood U of z_0 with $U \cap \partial D_0 \subseteq S \cap \bar{\Omega}^+$ and $U \cap \Omega \subset \Omega^+$. Since $u \geqslant v$, it follows that $d(v-u) = \alpha\, dr$ on $U \cap \partial D_0$. Since $dr \wedge d^c r \wedge (dd^c u)^{n-1} > 0$ on ∂D_0, it follows from Lemma 2.7 that $d(u-v) = 0$ on $U \cap \partial D_0$. Repeating now the argument of Lemma 2.5, we conclude that $u = v$ on M, which is a contradiction, proving the Theorem.

Proof of Lemma 2.8. We will suppose that $\Omega^- \cap M \neq \emptyset$ in order to derive a contradiction. By moving M slightly if necessary, we may assume that $(dd^c v)^{n-1}$ vanishes only at an isolated set of points of M. (Recall that $(dd^c v)^{n-1} = 0$ is contained in a variety.) Thus we may consider a point $z_0 \in M \cap S \cap \bar{\Omega}^+ \cap \Omega$ such that $\big(dd^c v(z_0)\big)^{n-1} \neq 0$. There are two cases to handle:

(*a*) $d(u-v) \neq 0$ at points of S arbitrarily near z_0;

(*b*) $d(u-v) = 0$ in a neighborhood of z_0 in S.

For case (*a*), we consider $z_j \in S$ close to z_0 such that $d(u-v)(z_j) \neq 0$, i.e. S is smooth at z_j. Let $M(z)$ be the leaf of $\mathcal{F}(u)$ containing z. The condition (2.6) says that the tangent space of $M(z)$ lies inside the tangent space of S. Since this holds for all $z \in S$ sufficiently near z_j that $d(u-v) \neq 0$, and since the foliation $\mathcal{F}(u)$ was obtained by integrating the $(2n-2)$ form $(dd^c u)^{n-1}$, it follows that in a neighborhood U_j of z_j, $M(z) \cap U_j \subset S$. Furthermore, $v = u$ is harmonic on $M(z) \cap U_j$, and it follows that $M(z) \cap U_j$ is also a leaf of $\mathcal{F}(v)$ since $(dd^c v)^{n-1} > 0$. Thus $M(z_j)$ is a leaf for both $\mathcal{F}(u)$ and $\mathcal{F}(v)$ and $u = v$ on $M(z_j)$. It follows that $u = v$ on $M(z_0)$, which means that $M(z_0) \subset \bar{\Omega}^+$, a contradiction for case (*a*).

In case (*b*), we want to show that $\mathcal{F}(u)$ and $\mathcal{F}(v)$ have the same tangent spaec at z_0, i.e. $dd^c u(z_0)$ and $dd^c v(z_0)$ have the same kernel. For $a \in S$, we let Tan (S, a) denote the tangent cone of S at a, i.e. the cone in $\boldsymbol{R}^{2n}$ generated by the limits of secants $(z_j - a)/|z_j - a|$ with $z_j \in S$, (see Federer [9], p. 233.) Let the tangent space $T_a S$ be the real linear span of Tan (S, a). We claim that on a dense set of points $a \in \bar{\Omega}^+ \cap \bar{\Omega}^-$, $\dim_{\boldsymbol{R}} T_z S \leqslant 2n-2$ for $z \in S$ in some neighborhood of a point $z_0 \in S$. In this case, one may show (see Federer [9], Lemma 3.3.5) that S is a countably rectifiable $(2n-2)$-dimensional set. This cannot happen in a neighborhood of $\bar{\Omega}^+ \cap \bar{\Omega}^-$ since a $(2n-2)$-dimensional set cannot disconnect Ω.

It suffices to show that $\ker dd^c u(z) = \ker dd^c v(z)$ for a point $z \in S$ arbitrarily close to z_0, so we may assume that $\dim T_{z_0} S \geqslant 2n-1$. If $\dim_{\boldsymbol{R}} T_{z_0} S = 2n$, then $dd^c u(z_0) = dd^c v(z_0)$, so we assume that $T_{z_0} S$ has dimension $2n-1$. Choose coordinates $z_j = x_j + iy_j$, $1 \leqslant j \leqslant n$, such that the set $\{\partial/\partial x_j,\ \partial/\partial y_j,\ 1 \leqslant j \leqslant n-1,\ \partial/\partial x_n\}$ spans $T_{z_0} S$. Since $u - v =$

$= d(u - v) = 0$ on S, it follows that all second derivatives of u and v are equal at z_0 except, possibly, the $\partial^2/\partial y_n^2$ derivative. If $\det (u_{i\bar{j}})_{1\leq i,j\leq n-1} \neq 0$, then we may solve for $\partial^2 u/\partial y_n^2 = \partial^2 v/\partial y_n^2$ in terms of the other second partial derivatives in the equation $\det (u_{i\bar{j}}) = \det (v_{i\bar{j}}) = 0$. If $\det (u_{i\bar{j}})_{1\leq i,j\leq n-1} = 0$, then $\det (v_{i\bar{j}})_{1\leq i,j\leq n-1} = 0$, and both $\mathcal{F}(u)$ and $\mathcal{F}(v)$ are orthogonal to $dz_n \wedge d\bar{z}_n$ at z_0. We may now proceed by induction on the space spanned by $\{\partial/\partial x_j, \partial/\partial y_j, 1\leq j\leq n-1\}$ to see that $dd^c u$ and $dd^c v$ have the same kernel at z_0.

Now we observe that if M were a leaf of the $\mathcal{F}(v)$ foliation, there would be nothing to prove, since by uniqueness in the Cauchy problem $u = v$ on M. The foliation $\mathcal{F}(v)$ is of class C^2 and, by the argument above, is tangent to M at all points of $\bar{\Omega}^+ \cap \bar{\Omega}^- \cap M$ near z_0. There are functions $\varphi_1, \dots, \varphi_{2n-2} \in C^2$ such that the leaves of $\mathcal{F}(v)$ are given near z_0 in the form $\{z : \varphi_1(z) = c_1, \dots, \varphi_{2n-2}(z) = c_{2n-2}\}$. It follows that $d(\varphi_j|_M) = 0$ on any point of $M \cap \bar{\Omega}^+ \cap \bar{\Omega}^-$. By A. P. Morse's theorem, the φ_j are constant on the connected components of $M \cap \bar{\Omega}^+ \cap \bar{\Omega}^-$. Since one of these components must have an accumulation point $\zeta_0 \in M$, it follows that the leaf $M'(\zeta_0)$ of $\mathcal{F}(v)$ must coincide with M. In other words, M is a leaf for both $\mathcal{F}(u)$ and $\mathcal{F}(v)$, completing the proof.

3. – Self maps of bounded domains.

If $H_{2n-1}(\Omega, \boldsymbol{R})$ has more than one generator, then the results of Section 2 cannot be applied directly (see Section 4). Much can be said, however, if the domain Ω is being mapped into itself. Theorem 2 will be shown to follow from a theorem of H. Cartan and the fact that the norm N is defined in the form (**). We will then discuss some similar norms obtained by changing the family $\mathcal{F}$ and the mapping theorems they provide.

A compact subset $E \subset \boldsymbol{C}^n$ will be said to be *negligible for* $\tilde{N}$ (*or* N) if there is a neighborhood ω of E such that the norm $\tilde{N}$ (or N) is identically zero on $H_{2n-1}(\omega\setminus E, \boldsymbol{R})$. Let us assume that $\tilde{\Omega}$ is a bounded domain in $\boldsymbol{C}^n$ with $H_{2n-1}(\tilde{\Omega}, \boldsymbol{Z}) = 0$, and that K, E are disjoint compact subsets of $\tilde{\Omega}$. We will assume that E is negligible and that $K = K_1 \cup \dots \cup K_l$ has finitely many connected components. By Alexander duality,

$$H_{2n-1}(\tilde{\Omega}\setminus K_j, \boldsymbol{Z}) = H^0(K_j, \boldsymbol{Z}) = \boldsymbol{Z}\,,$$

and without loss of generality, we assume that $\tilde{N}\{\partial\tilde{\Omega}, \tilde{\Omega}\setminus K_j\} > 0$, for otherwise we replace E by $E \cup K_j$.

Since $H_{2n-1}(\tilde{\Omega}, \boldsymbol{Z}) = 0$, the Mayer-Vietoris sequence gives

$$H_{2n-1}(\Omega, \boldsymbol{Z}) \cong H_{2n-1}(\tilde{\Omega}\setminus E) \oplus H_{2n-1}(\tilde{\Omega}\setminus K)$$

where $\Omega = \tilde{\Omega}\setminus(K \cup E)$. One way of viewing this direct sum is that if $\Gamma \in H_{2n-1}(\tilde{\Omega}\setminus E)$, then it may be represented by a cycle supported in a small neighborhood of E which is disjoint from K. Thus we may consider $\Gamma \in H_{2n-1}(\Omega)$. By the class of e.g., ∂K in $H_{2n-1}(\Omega)$, we mean the class of a smooth compact hypersurface in Ω, bounding a sufficiently small, compact region in $\tilde{\Omega}$.

Now we claim that the closure of 0 in the seminorm $\tilde{N}$ may be naturally identified with $H_{2n-1}(\tilde{\Omega}\setminus E)$. It is easily seen that if $\Gamma_1 \in H_{2n-1}(\Omega)$ and $\Gamma_2 \in H_{2n-1}(\tilde{\Omega}\setminus E)$, then

$$\begin{aligned}\tilde{N}\{\Gamma_1 + \Gamma_2\} &= \sup_{u\in\mathcal{F}'} T\, d^c u \wedge (dd^c u)^{n-1} = \\ &= \sup_{u\in\mathcal{F}'} T_1\, d^c u \wedge (dd^c u)^{n-1} + T_2\, d^c u \wedge (dd^c u)^{n-1} = \tilde{N}\{\Gamma_1\} + 0\,.\end{aligned}$$

We will write the induced (finite dimensional) normed space as

$$\tilde{H} = H_{2n-1}(\Omega, \boldsymbol{C})/\{\bar{0}\} \cong H_{2n-1}(\tilde{\Omega}\setminus K, \boldsymbol{C})\,.$$

Theorem 2 is a consequence of the following

THEOREM 3.1. *Let $\Omega = \tilde{\Omega}\setminus(K \cup E)$ where $\tilde{\Omega}$ is a bounded domain in $\boldsymbol{C}^n$ with $H_{2n-1}(\tilde{\Omega}, \boldsymbol{Z}) = 0$, and let K, E be disjoint compact subsets of $\tilde{\Omega}$. We assume that E is negligible and that $0 < \dim \tilde{H} < \infty$. If $f: \Omega \to \Omega$ is a holomorphic mapping then the following are equivalent:*

(i) *f is an automorphism of Ω;*

(ii) *f_* is an isomorphism of $H_{2n-1}(\Omega, \boldsymbol{R})$;*

(iii) *$f_*: \tilde{H} \to \tilde{H}$ is nonsingular;*

(iv) *if a subsequence of the iterates of f, $\{f^i\}$, converges uniformly on compact subsets to a map $F: \Omega \to \bar{\Omega}$, then F is an automorphism.*

(v) *$f_*: \tilde{H} \to \tilde{H}$ is an isometry.*

PROOF. The implication (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii). By the norm-decreasing property of f_*, it follows that

$$f_* H_{2n-1}(\Omega\setminus E, \boldsymbol{R}) \subset H_{2n-1}(\Omega\setminus E, \boldsymbol{R})\,.$$

It follows that $f_*: \tilde{H} \to \tilde{H}$. Since $\tilde{H}$ is finite dimensional, it is sufficient to show that f_* is onto. If $[\Gamma] \in \tilde{H}$, let $\Gamma \in H_{2n-1}(\Omega, \boldsymbol{R})$ be a representative. If $\Gamma' \in H_{2n-1}(\Omega, \boldsymbol{R})$ is any preimage under f_*, then $f_*[\Gamma'] = [\Gamma]$, and thus f_* is nonsingular on $\tilde{H}$.

(iii) $\Rightarrow$ (iv). Let $\{f^j\}$ be the sequence of iterates of f and let $F: \Omega \to \bar{\Omega}$ be a mapping to which a subsequence of the f^j converge uniformly on compact subsets of Ω. By a theorem of H. Cartan [5], (or see [18]) F is either an automorphism or the Jacobian determinant of F vanishes identically. In the latter case, we consider $[\Gamma] \in \tilde{H}$ and follow its image under f^j_*. Let Γ_j be the homology class of ∂K_j for $1 \leqslant j \leqslant l$. Thus there are integers c_{ij} such that

$$[f^j_* \Gamma] = \sum_{i=1}^{l} c_{ij}[\Gamma_i] .$$

We may assume that $c_{1j} \geqslant 1$ for infinitely many j and that $\tilde{N}\{\partial K_1\} > 0$. Thus there exists a current T_1 representing ∂K_1 compactly supported in Ω and a function $u_1 \in P(\tilde{\Omega} \diagdown K_1) \cap C(\tilde{\Omega} \diagdown K_1)$ such that $(dd^c u_1)^n = 0$ on $\tilde{\Omega} \diagdown K_1$ and $T_1 d^c u_1 \wedge (dd^c u_1)^{n-1} = \delta > 0$.

Thus integration by parts yields a bound for infinitely many j:

$$\sum_{1=i}^{l} [c_{ij}\Gamma_i] d^c u_1 \wedge (dd^c u_1)^{n-1}$$

$$\sum_{i=1}^{l} c_{ij} \int_{\Gamma_i} d^c u_1 \wedge (dd^c u_1)^{n-1} = c_{1j} T_1 d^c u_1 \wedge (dd^c u_1)^{n-1} + \sum_{i=2}^{l} c_{ij} \int_{\Gamma_i} (dd^c u_1)^n \geqslant \delta > 0 .$$

On the other hand, if T is a current representing Γ, then $F_* T = 0$ since the Jacobian determinant of F is zero and the dimension of T is $(2n-1)$. By uniform convergence,

$$\lim_j [f^j_* T] = [F_* T] = 0 ,$$

and so

$$\lim_j f^j_* T \, d^c u_1 \wedge (dd^c u_1)^{n-1} = 0 ,$$

which is a contradiction. Now (iv) $\Rightarrow$ (i), (i) $\Rightarrow$ (v), and (v) $\Rightarrow$ (iii) are obvious, and this completes the proof.

We will consider the class of functions for $j = 0$ or 2:

$$\mathcal{F}_j(\Omega) = \{v \in C^j(\tilde{\Omega}) \cap P(\tilde{\Omega}) : 0 < v < 1 \text{ and } (dd^c v)^n = 0 \text{ on } \Omega\} .$$

By Hartogs' Theorem, a holomorphic mapping $F: \Omega_1 \to \Omega_2$ extends to $\tilde{F}: \tilde{\Omega}_1 \to \tilde{\Omega}_2$. Thus the family $\mathcal{F}_j(\Omega_2)$ is mapped into $\mathcal{F}_j(\Omega_1)$ by F^*. It follows that the norm N_j defined by (∗∗) with $\mathcal{F}_j$ replacing $\mathcal{F}$ decreases under holomorphic maps.

A compact subset $E \subset \tilde{\Omega}$ without interior is easily seen to be negligible for the seminorm N_2. Theorem 3.1 with $\tilde{N}$ replaced by N_2 yields

COROLLARY 3.2. *Let $\Omega = \tilde{\Omega} \setminus K \Subset \boldsymbol{C}^n$, $n \geq 2$, be given where $H_{2n-1}(\tilde{\Omega}, \boldsymbol{Z}) = 0$ and* int K *is nonempty and has finitely many connected components. The conclusions* (i), ..., (v) *of Theorem* 3.2 *are equivalent if $\tilde{H}$ is replaced by H_2 (corresponding to N_2). In particular* (i) $\Leftrightarrow$ (ii).

Let us recall that a set $E \subset \boldsymbol{C}^n$ is *$\boldsymbol{C}^n$-polar* if for each $z \in E$ there is a neighborhood U of z and a function v plurisubharmonic on U such that $U \cap E \subset \{v = -\infty\}$. Josefson [15] has shown that if E is polar there is a function v plurisubharmonic on $\boldsymbol{C}^n$ with $E \subset \{v = -\infty\}$.

If K is a compact subset of $\Omega \subset \boldsymbol{C}^n$, we define

$$h(K, \Omega)(z) = \sup \{v(z): v \in P(\Omega), v \leq 0 \text{ on } K \text{ and } v \leq 1 \text{ on } \Omega\}.$$

It is well known that the upper regularization

$$h^*(K, \Omega)(z) = \limsup_{\zeta \to z} h(K, \Omega)(\zeta)$$

is plurisubharmonic. A function ψ on Ω is an *upper barrier* if it has the property that if $B \subset \Omega$ is a ball and $u \in P(\Omega)$, $u \leq \psi$ on ∂B, then $u \leq \psi$ on B. In the terminology of Hunt and Murray [14] an upper barrier is $(n-1)$-plurisuperharmonic. If E and Ω have regular boundaries, then the solution of the Dirichlet problem for the Laplacian is an upper barrier. A technique of Walsh [25] yields the following result.

PROPOSITION 3.3. *Let $\Omega \Subset \boldsymbol{C}^n$ be a pseudoconvex open set, and let there exist a function $p \in P(\Omega)$, $p < 0$, such that $\lim_{\zeta \to \partial\Omega} p(\zeta) = 0$. If there is an upper barrier ψ on $\Omega \setminus E$ such that $\lim_{\zeta \to E} \psi(\zeta) = 0$ and $\lim_{\zeta \to \partial\Omega} \psi(\zeta) = 1$, then $h(E, \Omega)$ is continuous, and $h(E, \Omega)$ is 0 on E, 1 on $\partial\Omega$.*

PROPOSITION 3.4. *Let $\Omega \subset \boldsymbol{C}^n$ be pseudoconvex, and let $E \subset \Omega$ be a compact polar set. If Γ is the homology class of $\partial\Omega$ in $\Omega \setminus E$, then $\tilde{N}\{\Gamma, \Omega \setminus E\} = N_0\{\Gamma, \Omega \setminus E\} = 0$.*

PROOF. Since Ω is pseudoconvex, it has a C^2, strictly plurisubharmonic exhaustion $p(z)$. We may assume that $E \subset \{p < 0\}$, and we may replace Ω by $\Omega = \{p < 0\}$. Let $w \in P(\Omega)$, $w < 0$, be such that $E \subset \{w = -\infty\}$.

For $k > 0$, we set $\omega(k) = \{z \in \Omega: w(z) < -k\}$. Since E is compact and $\omega(k)$ is open, there is a smoothly bounded open set U_k, $E \subset U_k \subset \omega(k)$. By Prop. 3.3, $h_k = h(\bar{U}_k, \Omega)$ is continuous. We define

$$\Omega(k) = \{z \in \Omega: 0 < h(\bar{U}_k, \Omega) < 1\} .$$

By Prop. 3.1 of [4],

$$\tilde{N}(\Gamma, \Omega(k)) = T\big(d^c h_k \wedge (dd^c h_k)^{n-1}\big)$$

where T is any compactly supported smooth current representing Γ. Since $w/k < h_k - 1 < 0$ it follows that the h_k converge uniformly to 1 on the support of T. Thus $0 = \lim_{k\to\infty} \tilde{N}(\Gamma, \Omega(k)) \geqslant \tilde{N}(\Gamma, \Omega/E) \geqslant N_0(\Gamma, \Omega \setminus E)$.

COROLLARY 3.5. *Let E be a compact polar set in Ω, a pseudoconvex subset of C^n. If $u \in C(\tilde{\Omega}) \cap P(\tilde{\Omega})$ and $(dd^c u)^n = 0$ on $\tilde{\Omega}\setminus E$, then $(dd^c u)^n = 0$ on $\tilde{\Omega}$.*

PROOF. Let T be a smooth current representing $\partial\Omega$ supported away from E, and let S be a current such that $bS = T$. It follows that

$$T\big(d^c u \wedge (dd^c u)^{n-1}\big) = S(dd^c u)^n \geqslant \int_E (dd^c u)^n \geqslant 0 .$$

COROLLARY 3.6. *Let E be a compact polar subset of a pseudoconvex domain Ω, then E is negligible for N_0.*

PROOF. Since $H_{2n-1}(\Omega, \boldsymbol{R}) = 0$, $H_{2n-1}(\Omega\setminus E, \boldsymbol{R})$ is generated by linear combinations of classes of the form $[\partial K]$. But if $v \in \mathcal{F}_0(\Omega\setminus E)$, then by Corollary 3.4 $(dd^c v)^n = 0$ on $\tilde{\Omega}$. Thus

$$\int_{\partial K} d^c v \wedge (dd^c v)^{n-1} = \int_K (dd^c v)^n = 0 \quad \text{and so } N_0\{\partial K\} = 0 .$$

PROPOSITION 3.7. *Let E be a compact subset of C^n, and let $\psi > 0$ be a continuous upper barrier defined on a neighborhood $\omega\setminus E$ such that $\lim_{\zeta\to z} \psi(\zeta) = 0$ for $z \in E$. Then*

$$\tilde{N}\{\partial\Omega, \Omega\setminus E\} \geqslant N_0\{\partial\Omega, \Omega\setminus E\} > 0$$

for any bounded open set Ω containing E.

PROOF. It suffices to take Ω to be a large ball containing E and to show that $N_0\{\partial\Omega, \Omega\setminus E\} > 0$. Since $\min(k\psi, 1)$ is an upper barrier on C^n, for k large, the continuity of $h(E, \Omega)$ follows from Proposition 3.6. It suffices to show

$$(+) \qquad T d^c h \wedge (dd^c h)^{n-1} > 0 \ ,$$

where T is a smooth current representing $\partial\Omega$ and $h = h(E, \Omega)$. Since Ω is pseudoconvex, it follows that $h = 1$ on $\partial\Omega$. Thus $\Omega(\delta) = \{z \in \Omega: 1 - \delta < h(z)\}$, is an open set and for $1 > \delta > 0$, $C^n \setminus \Omega(\delta)$ will contain a neighborhood of E. Since Ω is bounded, $N_0\{\Gamma, \Omega(\delta)\}$ is strictly positive. By Theorem 3.2 of [4],

$$\tilde{N}\{\Gamma, \Omega(\delta)\} = (1 - \delta)^{-n} T d^c h \wedge (dd^c h)^{n-1} ,$$

and thus $(+)$ holds.

EXAMPLES. Let us take E to be $\{(x_1, \dots, x_n) \in \mathbf{R}^n: \max_{1\le j\le n} |x_j| < 1\}$ a totally real cube in $\mathbf{C}^n$. To construct an upper barrier for E, we consider the one dimensional case of the interval $[-1, 1]$ in the disk of radius 2. There is a harmonic function $\varphi(z)$ on the set $\{z \in \mathbf{C}: |z| < 2, z \notin [-1, 1]\}$ such that $\varphi(z) = 1$ if $|z| = 2$ and $\varphi([-1, 1]) = 0$. Now we let $\Phi(z) = \sum_{j=1}^{n} \varphi(z_j)$, and we observe that $\Phi(z) > 0$ if $z \notin E$ and $\Phi(E) = 0$. Since Φ is n-harmonic it is an upper barrier.

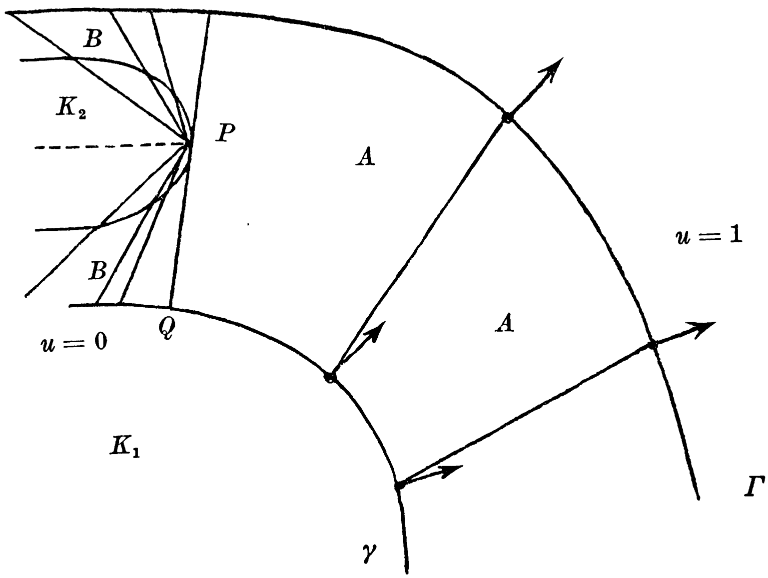
If M is a totally real smooth submanifold of $\mathbf{C}^n$ of dimension n, we may construct a similar upper barrier. For fixed $z_0 \in M$, there is a neighborhood ω of z_0 and a smooth diffeomorphism $T = (T_1, \dots, T_n)$ of $\omega \cap M$ with an open subset of $\mathbf{R}^n \subset \mathbf{C}^n$. Furthermore, since M is totally real T may be made « almost » holomorphic in the sense that $\bar{\partial}T$ will vanish to high order at $\omega \cap M$. If φ in the preceding example is taken to satisfy $\Delta\varphi = -1$, then sufficiently close to $\omega \cap M$, $\Phi(z) = \sum_{j=1}^{n} \varphi(T_j(z))$ will be superharmonic, and thus an upper barrier.

Let us conclude this section by computing the norm of a $(2n-1)$-dimensional homology class other than $\Gamma = [\partial D_0]$. If Ω is a Reinhardt domain of the form $\Omega = \tilde{\Omega} \setminus (K_1 \cup K_2)$ where K_1, K_2 are compact and disjoint, then $K_1 \cup K_2$ is not holomorphically convex in $\tilde{\Omega}$. If $\Gamma = \partial\tilde{\Omega}$, then

$$N\{\Gamma, \Omega\} = N\{\Gamma, \Omega \setminus \hat{K}\}$$

where $\hat{K}$ is the log convex hull of $K_1 \cup K_2$ (and it is assumed that $\hat{K} \subset \Omega$). Let us compute $N\{\gamma, \Omega\}$ where $\gamma = [\partial K_1]$. The extremal function u which gives the norm may be computed by looking at the logarithmic image of Ω (see [4] for details). We give here the construction in $\mathbf{C}^2$; the $\mathbf{C}^n$ case is similar.

In region A, the function is obtained by setting $u = 0$ on γ and $u = 1$ on Γ and requiring that u be linear along the straight line connecting the points on γ and Γ with equal normal vectors. On the region B, the function u is defined by $u = 0$ on γ and $u = 1$ on Γ, $u(P)$ is already determined,



and u is linear on all segments connecting P to γ and Γ. Note that u is not smooth of class C^1 at the segment PQ. Thus $N\{\gamma\} = \int_\gamma d^c u \wedge dd^c u$ is strictly greater than $N\{\gamma, \tilde{\Omega} \setminus K_1\}$. It remains an open question, however, whether $N\{\gamma\}$ can be strictly greater than $N\{\gamma, \tilde{\Omega} \setminus K_1\}$ if Ω satisfies (*).

4. – Structure of the (plurisub-) harmonic measure.

An obstacle to proving Theorem 2.1 for domains in C^n $(n \geq 2)$ with more than one hole is that solutions of (1.1) will not satisfy (1.4) on Ω (see the first Example below). If the solution of (1.1) is real analytic, the statements analogous to Theorem 2.1 and Corollary 2.2 remain true (see Corollary 4.8). Besides using the real analyticity of u, this depends on a more careful look at the geometric nature of the foliation.

EXAMPLE. Let $\Omega \subset C^2$ be defined as $\Omega = S \setminus \bigcup_{j=1}^{k} \bar{S}_j$ where S is an open ball and $\bar{S}_j$ are disjoint compact balls contained in S. Suppose that $u \in C^2$ satisfies (1.1) and (1.4) on $\bar{\Omega}$. Then the foliation $\mathcal{F}(u)$ exists everywhere on Ω and has tangent bundle $\mathfrak{J}$. There is a continuous global section $X = \lambda^1(\partial/\partial z_1) + \lambda^2(\partial/\partial z_2)$ of $\mathfrak{J}$ since $H^2(\Omega, \partial\Omega) = 0$. As was noted in Section 1, $\mathcal{F}(u)$ (and thus X) is transverse to $\partial\Omega$. Again, since $H^2(\Omega, \partial\Omega) = 0$, there is a continuous function $e^{i\theta}$ on $\bar{\Omega}$ such that the vector field $X_0 = \operatorname{Re}(e^{i\theta} X)$ is nonvanishing and points inward at points of $\partial\Omega$. It follows that the Euler characteristic of Ω must be zero, and thus Ω has only one hole. Now let us suppose that M is a closed leaf of $\mathcal{F}(u)$. Since X_0 is tangent to M, it follows that M, too, has vanishing Euler characteristic and must be an annulus.

In the following example, the outer boundary is not strictly pseudoconvex, and the foliation is nonsingular.

EXAMPLE. Let

$$\Omega_0 = \left\{z \in \boldsymbol{C}^2 : \frac{1}{16} < \left|z_1 - \frac{1}{2}\right|^2 + |z_2|^2 < 1\right\}$$

and let $f(z_1, z_2) = (z_1^2, z_2)$.

It follows that $f^{-1}(\Omega_0) = \Omega$ is a weakly pseudoconvex domain with two strongly pseudoconcave holes. The solution to (1.1) is

$$u(z_1, z_2) = \frac{\log\left(|z_1 - \frac{1}{2}|^2 + |z_2^2|\right)}{\log 4} + 1 ,$$

and the corresponding leaves are $\{z_2 = c(z_1^2 - \frac{1}{2})\}$.

Now let u satisfy (1.1) on a set Ω, and suppose that u satisfies (1.4) on a smaller open set Ω'. Let $\mathcal{N}$ denote the normal bundle to the foliation $\mathcal{F}$, which is defined over Ω'. Now we will consider the (1, 1) form $\omega = dd^c u$ as a metric on $\mathcal{N}$, and $\omega^{n-1} = (dd^c u)^{n-1}$ is its associated volume form. Let $\beta = (i/2)(dz_1 \wedge d\bar{z}_1 + \ldots + dz_n \wedge d\bar{z}_n)$ be the standard Kähler form on $\boldsymbol{C}^n$, and let * denote the Hodge *-operator. We recall that if $(g_{i\bar{j}})$ is a hermitian metric on a holomorphic vector bundle, then the associated Ricci form is $\operatorname{Ric} = dd^c \log\left(\det(g_{i\bar{j}})\right)$.

PROPOSITION 4.1. *The Ricci form of the metric* ω *is non-negative. In local coordinates, the function* $\log * \left(\beta \wedge (dd^c u)^{n-1}\right)$ *is subharmonic when restricted to a leaf of* $\mathcal{F}(u)$. *If the leaf* L *is locally a portion of the* z_1*-axis then* Ric *is given by* $dd^c \log * \left(\beta \wedge (dd^c u)^{n-1}\right)$ *restricted to* L.

PROOF. We choose coordinates so that a given leaf is the z_1-axis passing through 0, and thus on this leaf $u_{1\bar{j}} = u_{i\bar{1}} = 0$ for $1 < i,\ j \leqslant n$. We use $\partial/\partial z_2, \ldots, \partial/\partial z_n$ as a local basis for $\mathcal{N}$ along this leaf, and the matrix H of the metric has entries

$$H_{jk} = \left\langle \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k} \right\rangle = u_{j\bar{k}} , \qquad 2 \leqslant j, k \leqslant n .$$

It follows that

(4.1) $$*\beta\wedge(dd^c u)^{n-1} = (n-1)!\,4^{n-1}\det H\,.$$

Since dd^c on the z_1-axis is $\big(\partial^2/(\partial z_1\partial\bar z_1)\big)\,2i\,dz_1\wedge d\bar z_1$, it follows that the logarithm of (4.1) is subharmonic on the z_1-axis if and only if the Ricci form is non negative. In general coordinates $\zeta = \zeta(z)$, with Kähler form

$$\tilde\beta = (i/2)\sum d\zeta_j\wedge d\bar\zeta_j\,,$$

$$dd^c\log *\big(\tilde\beta\wedge(dd^c u)^{n-1}\big) = dd^c\log\det H + dd^c\log\left(\sum_{i=1}^{n}\left|\det_{\substack{2\leqslant j\leqslant n\\ l\neq i}}\left(\frac{\partial z_j}{\partial\zeta_l}\right)\right|^2\right).$$

We will compute the curvature 2-form of the hermitian connection of type (1, 0) on $\mathcal{N}$ associated to this metric. This is given by $\eta = 2iB\,dz_1\wedge d\bar z_1$ where

(4.2) $$B = -H^{-1}H_{1\bar 1} + H^{-1}H_1\cdot H^{-1}H_{\bar 1}\,,$$

the subscripts denoting entry-by-entry differentiation of H. We will show that B is a negative semi-definite hermitian endomorphism of $\mathcal{N}$. From this it follows that Ric is nonnegative since

(4.3) $$\big(\log(\det H)\big)_{1\bar 1} = -\operatorname{tr} B\,.$$

To show that B is nonpositive, we expand the identity

$$\det(u_{i\bar j})_{kl} = 0\,,\qquad 1\leqslant k,\ l\leqslant n\,.$$

On the z_1-axis, $u_{i\bar 1} = u_{1\bar j} = 0$, so that if $2\leqslant k,\ l\leqslant n$, then

$$0 \equiv \det(u_{i\bar j})_{k\bar l}\,,\quad 2\leqslant k,\ l\leqslant n$$

along the z-axis. Carrying out the differentiation, and noting that $u_{i\bar 1}\equiv 0$, $u_{1\bar j}\equiv 0$, $1\leqslant i,\ j\leqslant n$, along the z_1-axis, one gets

$$0 = \det(u_{i\bar j})_{k\bar l} = \sum_{r,s=1}^{n}\det\begin{bmatrix} u_{1\bar 1} & \cdots & u_{1\bar s k} & \cdots & u_{\bar 1 n}\\ \vdots & & \vdots & & \vdots\\ u_{r\bar 1\bar l} & \cdots & u_{r\bar s k\bar l} & \cdots & u_{r\bar n\bar l}\\ \vdots & & \vdots & & \vdots\\ u_{n\bar 1} & \cdots & u_{n\bar s k} & \cdots & u_{n\bar n}\end{bmatrix}$$

$$= \det \begin{bmatrix} u_{11k\bar l} & u_{1\bar 2k} \cdots u_{1nk} \\ u_{1\bar 2\bar l} & \\ \vdots & H \\ u_{n\bar 1\bar l} & \end{bmatrix} + \sum_{r,s\geq 2} \det \begin{bmatrix} 0 & 0 & \cdots & u_{\bar 1s\bar l} & \cdots & 0 \\ 0 & u_{2\bar 2} & \cdots & u_{2\bar s\bar l} & \cdots & u_{2\bar n} \\ \vdots & & & & & \\ u_{r\bar 1k} & u_{r\bar 2k} & \cdots & u_{r\bar sk\bar l} & \cdots & u_{r\bar nk} \\ \vdots & & & & & \\ 0 & u_{n\bar 2} & \cdots & u_{n\bar s\bar l} & \cdots & u_{n\bar 2} \end{bmatrix}$$

$$= u_{1\bar 1kl} \det(H) + \det \begin{bmatrix} 0 & u_{1\bar 2k} \cdots u_{n\bar k1} \\ u_{2\bar 1\bar l} & \\ \vdots & H \\ u_{n\bar 1\bar l} & \end{bmatrix} + \sum_{r,s=2} (-1)^{s+1+r} u_{\bar 1\bar s l} u_{r\bar 1\bar k} \det(H_{\hat r,\hat s}).$$

Here $H_{\hat r,\hat s}$ is the $(n-2)\times(n-2)$ matrix gotten by deleting the r-th row and s-th column of H. Thus, the cofactor matrix $C(H)$ of H has s-th–row, r-th–column entry $(-1)^{r+s}\det(H_{\hat r,\hat s})$. Continuing, we get

$$0 = u_{k\bar l1\bar 1}\det(H) - \sum_{r,s=2}^{n} u_{k\bar s1}(-1)^{r+s}\det(H_{\hat r,\hat s})u_{r\bar l\bar 1} - \sum_{r,s=2}^{n}(-1)^{s+r}u_{\bar 1\bar s l}u_{r\bar 1k}\det(H_{\hat r,\hat s}).$$

Thus,

$$-H_{1\bar 1}\det(H) + H_1\cdot C(H)H_{\bar 1} = \bar A^t\cdot\overline{C(H)}\cdot A,$$

where A is the matrix with (i,j) entry $u_{\bar 1\bar j i}$. Multiplying on the left by

$$\det(H)^{-1}H^{-1} \quad \text{gives} \quad B = -H^{-1}\bar A\bar H^{-1}A.$$

Without loss of generality, we may assume that $H(0) = I_{n-1}$, and thus $B\leqslant 0$, completing the proof.

REMARK. Notice that we have proved more, since the semi-definiteness of η gives Ric $= 0$ iff $\eta = 0$. Ric also measures the « anti-holomorphic twist » of $\mathcal{F}(u)$, for which we will give two interpretations. For the first, let us normalize so that $(u_{i\bar j}(0)) = I_{n-1}$. Then by (4.2) and (4.3),

$$\text{(4.4)} \qquad \operatorname{Ric}(0) = 2i \text{ trace } \bar A A\, dz_1\wedge d\bar z_1 = 2i\sum_{j,k=2}^{n} u_{1\bar j\bar k}u_{\bar 1kj}\,dz_1\wedge d\bar z_1.$$

We may parametrize the leaves of $\mathcal{F}(u)$ near 0 by letting $M(\alpha)$ be the leaf passing through $(0, \alpha_2, \dots, \alpha_n) = (0, \alpha)$. There exists a small $\delta > 0$ and a function

$$F(\zeta, \alpha) = \big(f_2(\zeta, \alpha), \dots, f_n(\zeta, \alpha)\big)$$

such that

$$\{|z_1| < \delta\} \cap M(\alpha) = \{(\zeta, F(\zeta, \alpha)) : |\zeta| < \delta\} .$$

We may write

$$f_j(\zeta, \alpha) = \alpha_j + \zeta\Big(\sum_{k=2}^{n} p_j^k \alpha_k + q_j^k \bar{\alpha}_k\Big) + O(\zeta\alpha^2) + O(\zeta^2) .$$

Thus the $\partial z_j/\partial z_1$ slope of $M(\alpha)$ at $(0, \alpha)$ is given by

$$\sum_{k=2}^{n} p_j^k \alpha_k + q_j^k \bar{\alpha}_k + O(\alpha^2) .$$

Since $M(\alpha)$ is the annihilator of the matrix $(u_{j\bar{k}})$, the $\partial z_j/\partial z_1$ slope of $M(\alpha)$ is also $- u_{1\bar{j}}/u_{j\bar{j}}$. With our normalizations,

$$\frac{\partial}{\partial \bar{z}_k}\left(\frac{u_{i\bar{j}}(0)}{u_{j\bar{j}}}\right) = u_{1\bar{k}\bar{j}} ,$$

so we have

$$\text{Ric}\,(0) = 2i \sum_{j,k=2}^{n} \frac{\partial f_j}{\partial \bar{\alpha}_k} \overline{\left(\frac{\partial f_k}{\partial \bar{\alpha}_j}\right)} dz_1 \wedge d\bar{z}_1 \tag{4.5}$$

which measures the non-holomorphic nature of $\mathcal{F}(u)$ at 0.

A foliation $\mathcal{F}$ of $\Omega \subset C^n$, whose leaves are complex submanifolds of dimension k is a holomorphic foliation if and only if $\mathfrak{J} = T^{1,0}(\mathcal{F}) \subset T^{1,0}(\Omega)$ is a holomorphic sub-bundle. For complex tangent vectors $\zeta \in \mathfrak{J}_z$ and $\xi \in T^{0,1}(\Omega)_z$, ξ transverse to $\mathcal{F}$, let $\tilde{\zeta}$ and $\tilde{\xi}$ denote extensions of ζ, ξ to local vector fields with $\tilde{\zeta}$ in $\mathfrak{J}$. The vector $[\tilde{\zeta}, \tilde{\xi}] \bmod \big(\mathfrak{J} \oplus T^{0,1}(\Omega)\big)$ in $\mathcal{N}_z$ depends only on ζ, ξ, call it $\mathfrak{L}(\zeta, \xi)$. $\mathfrak{L}$ is a linear map: $\mathfrak{J} \otimes \overline{\mathcal{N}} \to \mathcal{N}$, and $\mathcal{F}$ is holomorphic iff $\mathfrak{L} \equiv 0$. Indeed, $\mathfrak{J}$ is locally spanned by the vector fields

$$Z_i = \frac{\partial}{\partial z_i} + \sum_{j=k+1}^{n} b_{ij} \frac{\partial}{\partial z_i} , \qquad 1 \leqslant i \leqslant k ,$$

where the b_{ij} must be holomorphic on leaves of $\mathcal{F}$. Locally, we may take a leaf M of $\mathcal{F}$ to be a $(z_1, \dots, z_k)$-coordinate plane, so that $\overline{\mathcal{N}}|_M$ is spanned by $\partial/\partial\bar{z}_l$, $k+1\leqslant l\leqslant n$. On M, we may take brackets

$$\left[Z_i, \frac{\partial}{\partial\bar{z}_l}\right] = \sum_{j=k+1}^{n} \frac{\partial}{\partial\bar{z}_l}(b_{ij})\frac{\partial}{\partial z_j}.$$

Hence, $\mathfrak{L}\equiv 0$ if and only if the b_{ij} are holomorphic on Ω.

In the case at hand, we may compute along the z_1-axis to get

$$\mathfrak{L}\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial\bar{z}_l}\right) = \sum_{r,s=2}^{n}(-1)^{r+s}\,\overline{\det(H_{\hat{r},\hat{s}})}\,u_{1s\bar{l}}\frac{\partial}{\partial z_r}.$$

Hence, $\mathfrak{L}\equiv 0$ along the z_1-axis if and only if all $u_{\overline{1s}l}=0$ there. This gives:

PROPOSITION 4.2. *The foliation $\mathcal{F}(u)$ is holomorphic if and only if the Ricci form vanishes.*

Now we suppose that the solution u of (1.1) is real analytic on Ω. The hypersurface

$$S=\{r(z_0, z_1, \ldots, z_n)=\log z_0\bar{z}_0+u(z_1,\ldots,z_n)=0\} \tag{4.6}$$

in $\boldsymbol{C}\times\Omega$ is weakly pseudoconvex. If M is a complex manifold in S, and if $\pi\colon \boldsymbol{C}\times\Omega\to\Omega$ is projection, then $\pi(M)$ is a leaf of $\mathcal{F}(u)$ (see [2]). The technique of the proof of Theorem 4 of Diederich and Fornaess [7] applied to the surface S yields a method for extending leaves of $\mathcal{F}(u)$ over the set $\{(dd^cu)^{n-1}=0\}$.

PROPOSITION 4.3. *Let $a\in\Omega$ be given, and let u be real analytic and satisfy $(dd^cu)^n=0$, $(dd^cu)^{n-1}\not\equiv 0$. Then there exists $\varepsilon>0$ sufficiently small that if M is a component of $\mathcal{F}(u)$ on $\{|z-a|<\varepsilon\}$, there exists a (closed), irreducible variety $\tilde{M}$ in $\{|z-a|<\varepsilon\}$ with $M\subset\tilde{M}$.*

PROPOSITION 4.4. *Let the solution u of* (1.1) *be real analytic, and let ε, a be as above. Then $\tilde{M}\setminus M$ is a finite subset of $\{|z-a|<\varepsilon\}$.*

PROOF. Since $(dd^cu)^{n-1}\neq 0$ on M, it follows that $\tilde{M}$ is one-dimensional. Now $\varphi(z)=\log\left(*\beta\wedge(dd^cu)^{n-1}\right)$ is subharmonic on M, and $\lim_{z\to z_0}\varphi(z)=-\infty$ for $z_0\in\tilde{M}\setminus M$. Thus φ is subharmonic on $\tilde{M}$ and $\tilde{M}\setminus M=\{\varphi<-\infty\}=$ $=\tilde{M}\cap\{(dd^cu)^{n-1}=0\}$ is a real analytic polar set, which is finite.

PROPOSITION 4.5. *Let the solution u of* (1.1) *be real analytic and $C^2(\bar{\Omega})$. Then every leaf of $\mathcal{F}(u)$ reaches both the inner and outer boundaries of Ω.*

PROOF. Let us suppose that $\sup_M u=\max_{\bar{M}} u=c<1$. Let $z_0\in M$ be a point such that $u(z_0)=c$, and let $z_j\in M$ be points converging to z_0. By

Proposition 4.3, z_j is contained in a variety $\tilde{M}_j$ in an ε-ball about $z_0 = a$. By a theorem of J. E. Fornaess (see [27]), a sufficiently sparse subsequence of $\{\tilde{M}_j\}$ may be chosen so that the cluster set is a variety $\tilde{M}$, and $\tilde{M}$ is of the form $\pi(\tilde{M}_1)$, where $\tilde{M}_1$ is a variety in S of (4.6).

The set $E = \{z \in \Omega\colon (dd^c u)^{n-1} = 0\}$ is a compact real analytic subset of Ω since $(dd^c u)^{n-1} \neq 0$ on $\partial D_0 \cup \partial D_1$. So by Diederich-Fornaess [7], no component of $\tilde{M}$ is contained in E. Thus $\tilde{M}$ is contained in finitely many leaves of $\mathcal{F}(u)$ with a finite number of points added. Now we may use the fact that $\tilde{M} \subset \bar{M}$ and that $E \cap \tilde{M}$ (the set of singularities of the differential system defining $\tilde{M}$) is discrete to conclude that $u|_{\tilde{M}} \leqslant c$. By the maximum principle, $\tilde{M} \subseteq \{u = c\}$, but this is impossible by [7].

The proof that M reaches ∂D_0 is a similar argument, following the outline of Proposition 2.4, and we omit it.

REMARK. It follows that $E = \{z \in \Omega\colon (dd^c u)^{n-1} = 0\}$ has real codimension $\geqslant 2$ for any real analytic solution u of (1.1). For if $\dim E = 2n - 1$, then $H_{2n-1}(E, \mathbf{Z}_2) \neq 0$ and by Alexander duality $\tilde{H}_0(S^{2n} \setminus E, \mathbf{Z}_2) \neq 0$, i.e. E disconnects Ω, contradicting Prop. 4.5.

For the theorem below, a proof may be given which follows the proof of Theorem 2.1 in its essentials except that Lemma 2.4 is replaced by Proposition 4.5 and Lemma 2.8 may be modified to accomodate the changed hypotheses. We omit the details because too much duplication would be involved.

THEOREM 4.6. *Let u, v satisfy the hypotheses of Theorem* 2.1, *except that they need not satisfy* (1.4). *If u, v are real analytic in Ω, and if v satisfies* (1.4) *on $\Omega \setminus (Z \cup E)$ where Z is a proper analytic variety in Ω, and E does not contain any germ of a complex variety, then $u = v$.*

COROLLARY 4.7. *Let Ω_1, $\Omega_2 \subseteq \mathbf{C}^n$ be domains satisfying* (*), *and assume that the solution u_j of* (1.1) *is real analytic on Ω_j and C^3 on the inner boundary of Ω_j for $j = 1, 2$. Let Γ_j denote the outer boundary of Ω_j, and let $N\{\Gamma_1\} = N\{\Gamma_2\}$. If $f\colon \Omega_1 \to \Omega_2$ is a holomorphic mapping, and if $f_*\{\Gamma_1\} = \alpha\Gamma_2 + \gamma$ $\alpha \neq 0$ and γ is a positive linear combination of inner boundary components of Ω_2, then f is an unramified covering of Ω_2.*

PROOF. We set $u = u_1$ and $v = u_2(f)$ and observe that the set where $(dd^c v)^{n-1} = 0$ is $\{|f'| = 0\} \cup f^{-1}\{(dd^c u_2)^{n-1} = 0\} = Z \cup E$. By Hartogs' theorem, α is a nonnegative integer. Since $N\{\Gamma_1\} = N\{\Gamma_2\}$ and

$$\int_{\Gamma_1} d^c u \wedge (dd^c u)^{n-1} = \alpha \int_{\Gamma_2} + \int_{\gamma} d^c v \wedge (dd^c v)^{n-1},$$

it follows that $\alpha = 1$ and $\gamma = 0$. The rest of the conclusions follow as in Corollary 2.2.

REMARK. We have not explicitly constructed the solution to (1.1) for any example of a domain Ω satisfying (*) where D_0 has more than one component and $n \geqslant 2$.

The geometric notions of Propositions 4.1 and 4.2 were motivated by phenomena which appear more concretely for the real Monge-Ampère equation and seem to be related to work of Sacksteder [21] and Hartman [11].

If $u(x_1, \ldots, x_n)$ satisfies $\det(u_{ij}) = 0$ on a domain $D \subset \boldsymbol{R}^n$, then u satisfies $(dd^c u)^n = 0$ on the tube domain $D + i\boldsymbol{R}^n$. A leaf M of $\mathcal{F}(u)$ is a complex line, and the corresponding leaf for the solution on D is $M \cap D$.

Let us study the foliation by solving a certain « boundary value » problem. We will write a point in $\boldsymbol{R}^n$ as $(x_1, \ldots, x_{n-1}, y_n) = (x, y)$. Let $A = (a_{ij})$ and $B = (b_{ij})$ be nonsingular $(n-1)\times(n-1)$ matrices. We want to find $u(x, y)$ in a small neighborhood of the interval $I = \{(0, \ldots, 0, y)\colon -1 \leqslant y \leqslant 1\}$ such that rank $(u_{ij}) = n - 1$ in a neighborhood of I

$$\left\{\begin{aligned} u(x, -1) &= \Sigma a_{ij} x_i x_j + O(|x|^3) \\ u(x, 1) &= \Sigma b_{ij} x_i x_j + O(|x|^3)\,. \end{aligned}\right. \tag{4.7}$$

We may compute the foliation $\mathcal{F}$ near the interval I because ∇u is constant along the lines of $\mathcal{F}$. Thus $\mathcal{F}$ will contain the segment between $(x, -1)$ and $(\tilde{x}, 1)$ where

$$\tilde{x} = A^{-1} B x + O(|x|^2)\,. \tag{4.8}$$

Furthermore, since u is linear along this segment, it follows that

$$u(p(x), y) = \frac{1-y}{2} u(x, -1) + \frac{1+y}{2} u(x, 1)$$

where we set

$$p(x) = \left(\frac{1-y}{2}\right) x + \left(\frac{1+y}{2}\right) \tilde{x}\,. \tag{4.9}$$

From this we may calculate the hessian matrix of u

$$\left(\frac{\partial^2 u(0, y)}{\partial x_i \partial x_j}\right) = \left(\frac{1-y}{2} A^{-1} + \frac{1+y}{2} B^{-1}\right)^{-1}, \qquad 1 \leqslant i\,,\ j \leqslant n-1\,. \tag{4.10}$$

PROPOSITION 4.8. *Let* $u \in C^2(D)$ *be a function satisfying* $\operatorname{rank}(u_{ij}) = n-1$. *Let* L *be a segment of* $\mathcal{F}$, *and for* $p \in L$, *let* $\operatorname{Hess}(p)$ *denote the* $(n-1)\times(n-1)$ *hessian matrix of* u *taken in the orthogonal subspace to* L *at* p. *Then the entries of* $(\mathrm{Hess})^{-1}$ *are linear on* L. *In particular,* $\det(\mathrm{Hess})$ *does not tend to zero at either endpoint of* L.

PROOF. By an affine map, we may assume that the endpoints of L are $(0, \pm 2)$. By subtracting a linear function from u, we see that u solves (4.7) on I. Thus $(\mathrm{Hess})^{-1}$ is linear by (4.10). It follows that $\big(\det \operatorname{Hess}(0, y)\big)^{-1}$ is a polynomial of degree $(n-1)$, and thus it tends to a finite limit as $(0, y)$ approaches $(0, \pm 2)$.

Let us return to the solution of (4.7) and assume that $A > 0$, i.e. u is convex at $(0, -1)$. We perform a rotation and diagonalize A. Since $A > 0$, we may perform a change of scale and assume that $A = I$. Thus (4.10) takes the form

$$\operatorname{Hess} u(0, y) = \left(\frac{1-y}{2} I + \frac{1+y}{2} B'\right)^{-1}. \tag{4.11}$$

Performing another rotation in $\boldsymbol{R}^{n-1}$, we may take $B' = \operatorname{diag}(\lambda_1, \dots, \lambda_{n-1})$. Thus $\operatorname{Hess} u(0, y)$, in these new coordinates, is a diagonal matrix whose j-th entry is $\big((1-y)/2 + \lambda_j(1+y)/2\big)^{-1}$.

PROPOSITION 4.9. *Let* u *be a* C^2 *function that satisfies* $\det(u_{ij}) = 0$ *in a neighborhood of the entire y-axis. If the matrix* $\big(u_{ij}(0)\big)$ *has rank* $(n-1)$ *and is nonnegative, then* (u_{ij}) *is constant on the y-axis.*

PROOF. By the discussion above a coordinate system may be chosen so that the hessian matrix is diagonalized and the j-th entry is $\big((1-y)/2 + \lambda_j(1+y)/2\big)^{-1}$. Since this is always finite, it must be constant. Thus $\lambda_1 = \dots = \lambda_{n-1} = 1$.

REMARK. If the convexity assumption is dropped, then the Proposition is false. For example, by our previous discussion, there is a function $u(x, y, z)$ on $\boldsymbol{R}^3$ with $\det(u_{ij}) \equiv 0$, $u(0, y, z) = yz$, $u(1, y, z) = \frac{1}{2}(y^2 - z^2)$, and with foliation given by lines

$$y = y_0 - x(y_0 + z_0)$$

$$z = z_0 + x(y_0 - z_0).$$

Along the x-axis, $\operatorname{Hess}(x, 0, 0)^{-1} = \begin{bmatrix} x & 1-x \\ 1-x & -x \end{bmatrix}$.

For a fixed segment L of $\mathcal{F}$ it is natural to define a Poincaré map. For $p \in L$ let Σ_p be a neighborhood of p in the copy of $\boldsymbol{R}^{n-1}$ passing through p, which is orthogonal to L. By translation, Σ may be considered to be a neighborhood of 0 in $\boldsymbol{R}^{n-1}$. Following leaves of $\mathcal{F}$ near L, we have a map $f_{p_1 p_2}\colon \Sigma_{p_1} \to \Sigma_{p_2}$. Taking $f_{p_1 p_2}$ to map a neighborhood of 0 in $\boldsymbol{R}^{n-1}$ into $\boldsymbol{R}^{n-1}$, we see that the jacobian of $f_{p_1 p_2}$ at p_1 is $(\text{Hess}\, u(p_1))^{-1}\, \text{Hess}\, u(p_2)$. If u is convex at p_1, then this matrix can be diagonalized. If $\mathcal{N}^*$ is the dual bundle of $\mathcal{N}$, then the metric on $\mathcal{N}^*$ is given by $(\text{Hess}\, u)^{-1}$, and the jacobian of $f_{p_1 p_2}$ is an isometry.

REMARK. It seems that the complex version of problem (4.7) should give useful information about solutions of (1.1) and (1.4). Let a real function $a(\theta, z_2)$ be given with $a(\theta, 0) = a_2(\theta, 0) = 0$ and $a_{2\bar{2}}(\theta, 0) > 0$. We want to find a plurisubharmonic $u(z_1, z_2)$ such that

$$\text{(4.12)} \qquad \begin{cases} \text{rank}\,(u_{i\bar{j}}) = 1 & \text{for } |z_1| < 1 \text{ and } |z_2| \text{ small} \\ u(z_1, z_2) = a(\theta, z_2) & \text{for } z_1 = e^{i\theta} \text{ and } |z_2| \text{ small}\,. \end{cases}$$

Some further restrictions on $a(\theta, z_2)$ are necessary before (4.12) is solvable, consider

$$a(\theta, z_2) = b(\theta)\big(|z_2|^2 + \text{Re}\,(c z_2^2 e^{i\theta})\big)\,.$$

Let $A(z_1)$ be the unique nonvanishing analytic function on $|z_1| < 1$ such that $|A(e^{i\theta})|^2 = b(\theta)$. Changing coordinates by $z_1^* = z_1$, $z_2^* = A(z_1) z_2$ we may take $b(\theta) = 1$. To solve (4.12) we next look for the leaf of the foliation which passes through $(0, \zeta)$:

$$M(\zeta) = \big\{(z_1, F(\zeta, z_1))\colon |z_1| < 1\big\}\,.$$

Since u_2 must be holomorphic on $M(\zeta)$,

$$a_2\big(\theta, F(\zeta, e^{i\theta})\big) = \bar{F} + cFe^{i\theta}$$

must have no negative Fourier coefficients. Thus

$$F(\zeta, z_1) = \zeta - \bar{c}\bar{\zeta} z_1\,,$$

and on $M(\zeta)$,

$$u_2\big(z_1, F(\zeta, z_1)\big) = \bar{\zeta}(1 - |c|^2)$$

is constant. We conclude that

$$0 \leqslant u_{2\bar{2}}(0, 0) = 1 - |c|^2,$$

so c is not arbitrary.

5. – Cycles with minimal area.

The Carathéodory metric on a complex manifold Ω has the property of decreasing under holomorphic mappings. To a homology class $\Gamma \in H_l(\Omega, \boldsymbol{R})$, it is natural to assign the number $C\{\Gamma\}$, which is the infimum of the area (in the Carathéodory metric) of all cycles representing Γ. We will discuss this for the set $\Omega = \boldsymbol{B}^n \setminus K$, where K is a compact subset of $\boldsymbol{B}^n$, the unit ball in $\boldsymbol{C}^n$. In this case, the Carathéodory metric can be written as a Riemannian metric; at the point $(r, 0, ..., 0) \in \boldsymbol{B}^n$, we have

$$ds_c^2 = \frac{|dz_1|^2}{(1-r^2)^2} + \sum_{j=1}^{n} \frac{|dz|^2}{1-r^2}. \tag{5.1}$$

It is well known that this metric (which coincides with the Bergman and Kobayashi metrics) has negative curvature, and the unit ball is complete in this metric. A domain ω is convex in this metric if its second fundamental form is nonnegative. This can be checked in specific cases by mapping $p \in \partial\Omega$ to 0 via some $f \in \operatorname{Aut}(\boldsymbol{B}^n)$, since the second fundamental forms in the Euclidean and Carathéodory metrics will coincide at 0.

LEMMA 5.1. *Let ω be a relatively compact subset of $\boldsymbol{B}^n$ which is convex in the Carathéodory metric with $\partial\omega \in C^2$. Then $\partial\omega$ is the representative of the homology class $[\partial\omega] \in H_{2n-1}(\boldsymbol{B}^n \setminus \omega, \boldsymbol{Z})$ which has minimal surface area in the Carathéodory metric.*

PROOF. Let us define the map $R\colon \boldsymbol{B}^n \setminus \omega \to \partial\omega$ by $R(z) \in \partial\omega$ and $R(z)$ minimizes the Carathéodory distance between z and $\partial\omega$. Since $\boldsymbol{B}^n$ is contractible and negatively curved in the Carathéodory metric and ω is convex, R is well defined. It is well-known that

$$\operatorname{dist}\big(R(z_1), R(z_2)\big) \leqslant \operatorname{dist}(z_1, z_2)$$

since $\boldsymbol{B}^n$ has negative curvature.

Thus it follows that if γ is any $(2n-1)$-cycle representing $\partial\omega$, then $R(\gamma)$ has smaller area. Finally, it is clear that γ and $R(\gamma)$ are homologous, so that $R(\gamma)$ and γ represent the same class.

THEOREM 5.2. *Let ω_1, ω_2 be compact subsets of $\boldsymbol{B}^n$ that are convex in the Carathéodory metric. Let us set $\Omega_j = \boldsymbol{B}^n \setminus \bar{\omega}_j$, $j = 1, 2$, and let $f: \Omega_1 \to \Omega_2$ be a holomorphic mapping. If $C\{\partial\omega_1\} = C\{\partial\omega_2\}$, and if the homology class $f_*[\partial\omega_1]$ is non zero, then $f \in \operatorname{Aut}(\boldsymbol{B}^n)$.*

PROOF. By Hartogs' theorem, f extends to a holomorphic mapping $f: \boldsymbol{B}^n \to \boldsymbol{B}^n$. Since $f_*[\partial\omega_1] \neq 0$, $f(\partial\omega_1)$ is a nontrivial cycle in Ω_2. Since f decreases area, and since $C\{\partial\omega_1\} = C\{\partial\omega_2\}$, it follows by Lemma 5.1 that $f(\partial\omega_1) = \partial\omega_2$.

After composing with automorphisms of $\boldsymbol{B}^n$, we may assume that $0 \in \partial\omega_1 \cap \partial\omega_2$, that $f(0) = 0$, and that

$$T(\partial\omega_1)_0 = T(\partial\omega_2)_0 = \{\operatorname{Re} z_n = 0\}.$$

At the point 0, the Carathéodory and Euclidean metrics agree. If $J = f'(0)$ is the Jacobian matrix of f at 0, then by the distance decreasing property of f, $|J(v)| \leqslant |v|$ for all vectors v. On the other hand, f preserves the surface area of $\partial\omega$, so J is an isometry on $\{\operatorname{Re} z_n = 0\}$. Since J is complex linear, $|\det J| = 1$. Thus $J = \begin{pmatrix} \mathcal{U} & * \\ 0 & \lambda \end{pmatrix}$ where $|\lambda| = 1$ and $\mathcal{U}$ is unitary. Since $|J(0, \ldots, 0, c)| \leqslant |c|$, it follows that $* = 0$. By composing with a unitary map, f has the properties that $f(0) = 0$ and $f'(0) = I$. Thus it follows from a theorem of H. Cartan (see [18]) that $f(z) = z$, which completes the proof.

EXAMPLE. Let

$$E(a, b) = \left\{ z \in \boldsymbol{C}^2 : \left|\frac{z_1}{a}\right|^2 + \left|\frac{z_2}{b}\right|^2 < 1 \right\}$$

be an ellipsoid. We will compute the area of $\partial E(a, b)$ in the Carathéodory metric. At the point $(r, 0)$, it is easily seen from (5.1) that

$$|dy_1 \wedge dx_2 \wedge dy_2|_c = (1 - r^2)^{-2} \quad \text{and} \quad |dx_1 \wedge dy_1 \wedge dy_2|_c = (1 - r^2)^{-\frac{3}{2}}.$$

From this, the surface area at the point $(a \cos\theta e^{i\varphi_1}, b \sin\theta e^{i\varphi_2})$ may be shown to be

$$d\sigma_c = S_c(\theta)\, d\sigma = \frac{[(1 - r^2)a^2b^2 + (a^2 - b^2)^2(\sin 2\theta)/2)^2]^{\frac{1}{2}}}{r(1 - r^2)^{\frac{5}{2}}}\, d\sigma$$

where $d\sigma$ is the unit of Euclidean surface area on ∂E and

$$r = (a^2 \cos^2\theta + b^2 \sin^2\theta)^{\frac{1}{2}}.$$

If we perform the $d\varphi_1 d\varphi_2$ integration in $d\sigma$, we obtain the surface area

$$C(a, b) = (2\pi)^2 ab \int_0^{\pi/2} \sin\theta \cos\theta S_c(\theta)\, d\theta .$$

We observe that as was computed in [4], the norm of the homology class of ∂E is

$$N(a, b) = N\{\partial E(a, b)\} = (2\pi)^2 \int_0^{\pi/2} \left(\cos\theta \log\frac{1}{a} + \sin\theta \log\frac{1}{b}\right)^{-2} d\theta .$$

A numerical computation shows that there is no function f such that $N(a, b) = f\big(C(a, b)\big)$ for $0 < a,\ b < 1$.

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