

ANNALI DELLA  
SCUOLA NORMALE SUPERIORE DI PISA  
*Classe di Scienze*

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*Annali della Scuola Normale Superiore di Pisa, Classe di Scienze* 4<sup>e</sup> série, tome 6, n° 2 (1979), p. 367-377

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## Zeta Functions for Idelic Pseudo-Measures.

J. J. BENEDETTO (\*)

In [Benedetto, 1973] I introduced the notion of an «idelic pseudo-measure» in the context of investigating non-synthesizable phenomena. The Fourier analysis program that I envisaged for these pseudo-measures, and that I began in the above reference, stalled because I did not know enough about their structure. The theory of idelic pseudo-measures that I now develop is meant to correct this shortcoming. [Benedetto, 1976] complements the present paper and provides relations with some fundamental issues in analytic number theory.

Let  $A'(\mathbf{T})$  be the space of pseudo-measures on  $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$  with norm  $\| \cdot \|_{A'}$ . As is well-known, Hecke  $L$ -functions for the field  $\mathbf{Q}$  of rationals can be formed from characters of the locally compact abelian idele group  $\mathcal{J}_{\mathbf{Q}}$ ; and if these characters belong to the subgroup  $\mathbf{Q}^{\times\perp} \subseteq \mathcal{J}_{\mathbf{Q}}$  of idele class characters, then the associated  $L$ -functions comprise the usual Dirichlet zeta functions and have functional equations for purposes of analytic continuation. The Riemann  $\zeta$ -function is the  $L$ -function associated with the identity  $1 \in \mathbf{Q}^{\times\perp}$ .

We shall choose certain elements  $\gamma \in \mathcal{J}_{\mathbf{Q}}$  for which there is not only an associated  $L$ -function  $L(s, \gamma)$ ,  $s = \sigma + i\tau$ , but also a distribution  $T_s$ ,  $\sigma \leq 1$ , on  $\mathbf{T}$ . These distributions are called *idelic distributions* and if  $T_s \in A'(\mathbf{T})$  then  $T_s$  is an *idelic pseudo-measure*; IPM designates the space of such pseudo-measures. We shall prove the following analytic continuation and approximation result. There is a sequence  $\{\gamma_n\} \subseteq \mathcal{J}_{\mathbf{Q}} \setminus \mathbf{Q}^{\times\perp}$  and a sequence  $\{T_{n,s}\} \subseteq \text{IPM}$ ,  $\sigma > \frac{1}{2}$ , of corresponding idelic pseudo-measures having the following properties:

a) For each  $n$  and each  $s_0 \in \mathbf{C}$ ,  $\sigma_0 > \frac{1}{2}$ , there is a sequence  $\{A_m\} \subseteq A'(\mathbf{T})$  such that  $T_{n,s} = \sum A_m(s - s_0)^m$  converges in the  $A'(\mathbf{T})$  norm in some neighborhood of  $s_0$ ;

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Pervenuto alla Redazione il 7 Gennaio 1978.

b)  $\{\gamma_n\}$  converges to 1 in  $\mathcal{J}_Q$  and  $\|T_{n,s}\|_{A'} = \|T_{n+1,s}\|_{A'}$  for each  $n$ .

The result is also valid for  $L^p(\mathbf{T})$ ,  $p < \infty$ , cf., *Example 2.3*; although a major interest in these problems that I've posed is to construct pseudo-measures which are not measures. Part *a* is given in sect. 3 and somewhat more than part *b* is given in sect. 4.

Section 1 establishes criteria for determining elements of IPM and section 2 provides a selection of examples indicating the effectiveness of the criteria and the relation with number theoretic estimates. The setting of this paper is not intrinsically idelic and so it is necessary to explain our idelic language beyond the analytic reasons given in [Benedetto, 1973]. From a number theoretic point of view, it is natural to ask if knowledge about idele characters which are not idele class characters can ever provide any information about idele class characters. The results of sections 3 and 4 can be viewed in this context with the idea that IPM is an effective analytic space to study. In fact, we can construct idele characters  $\gamma$  which are not idele class characters such that the Riemann hypothesis is valid if and only if  $T_1 \in \text{IPM}$ .

*Acknowledgement.* In section 2 we have used D. Aharonov's development of the Lebedev-Milin inequalities.

### 1. - Criteria for idelic pseudo-measures.

For each  $\gamma \in \mathcal{J}_Q$  there is a function  $k: P \rightarrow \mathbf{R}$  and a conductor  $f \in N$  ( $P$  is the set of primes and  $N = \{1, 2, 3, \dots\}$ ) which determine the *Hecke-L series*

$$L(s, \gamma) = \sum_{\substack{n=1 \\ (n,f)=1}}^{\infty} \frac{1}{n^s} e^{ik(n)},$$

where  $k(1) = 0$  and  $k: N \rightarrow \mathbf{R}$  is defined by the rule  $k(pq) = k(p) + k(q)$  for each  $p, q \in P$ . If  $k(P) \subseteq \mathbf{Z}$  then the *idelic distribution*  $T_s$  (on  $\mathbf{T}$ ) corresponding to  $\gamma \in \mathcal{J}_Q$  is

$$T_s \sim \sum \hat{T}_s(n) e^{inz}$$

where

$$\hat{T}_s(n) = \sum_{\substack{k(m)=n \\ m \geq 1, (m,f)=1}} 1/m_s.$$

$T_s$  is *unbounded* if  $\hat{T}_1(0) = \infty$  and  $T_s \in \text{IPM}$  if  $\sup_n |\hat{T}_s(n)| < \infty$ . We shall write  $\gamma \sim k \sim T_s$  to designate the relation between idele character,  $k$ -func-

tion, and distribution. The above definitions and following result are developed in [Benedetto, 1976]. We shall assume that  $k$  is an increasing function  $P \rightarrow N \cup \{0\}$ ; the conditions on  $k$  can be weakened at the expense of a more technical presentation.

**PROPOSITION 1.1.** *Take  $\sigma > \frac{1}{2}$  and  $\gamma \sim k \sim T_s$ .  $T_s \in \text{IPM}$  if and only if the power series expansion of  $\exp\left(\sum_{p \in P} z^{k(p)} / p^\sigma\right)$  has bounded coefficients.*

**REMARK 1.** Given  $\gamma \sim k \sim T_s$  and take  $\sigma_0 \in (\frac{1}{2}, 1)$ . Set  $j_n = \inf \{p \in P: k(p) = n\}$  and  $m_n = \sup \{p \in P: k(p) = n\}$ . Assume that

$$\frac{m_n - (j_n - 1)}{\log j_n} \leq \frac{j_n^{\sigma_0}}{n}$$

for each  $n \geq 1$ . Then  $T_s \in \text{IPM}$  for every  $\sigma \in [\sigma_0, 1]$ .

**REMARK 2.** Given  $\gamma \sim k \sim T_s$  and take  $\sigma \in (\frac{1}{2}, 1)$ . Define the trigonometric series

$$f_\sigma \sim \sum_{p \in P} \frac{1}{p^\sigma} e^{ik(p)x}.$$

If  $\sum \|f_\sigma\|_{A'} / n! < \infty$  then  $T_s \in \text{IPM}$ .

**2. - Examples of idelic pseudo-measures.**

**EXAMPLE 2.1.** *a) Let  $k(p) = [\log p]$ ,  $p \in P$ . Then  $T_s \in \text{IPM}/M(T)$  for  $s = 1 + i\tau$ , e.g., [Benedetto, 1973, Example 4.1; 1976, Example 2.3; Holz-sager, 1975].*

*b) Using the number theoretic techniques necessary to prove part a we see that  $T_1 \notin \text{IPM}$  if  $k(p) = [\log^\alpha p]$  for  $\alpha < 1$ .*

*c) If  $k(p)$  increases faster than  $\log p$  then  $T_s \in \text{IPM}$ ,  $s = 1 + i\tau$ .*

**EXAMPLE 2.2.** Let  $k(p) = [\log^2 p]$ . To see that  $T_\sigma \notin \text{IPM}$ ,  $\sigma < 1$ , by means of Prop. 1.1 and the fact that  $\sum_0 z^n = \exp\left(\sum_1 z^n/n\right)$ , note that

$$\sum_{k(p)=n} 1/p^\sigma \geq (\pi(\exp \sqrt{n+1}) - \pi(\exp \sqrt{n})) / \exp(\sigma \sqrt{n+1})$$

and apply the prime number theorem ( $\pi(x) = \text{card} \{p \in P: p \leq x\}$ ). In particular, if  $k(p) = [\log p]$  then  $T_\sigma \notin \text{IPM}$  for  $\sigma < 1$ .

EXAMPLE 2.3. We shall show that (for most cases) if  $k(p)$  is strictly increasing and  $T_s \in \text{IPM}$  for each  $\sigma > \frac{1}{2}$  then  $T_s \in L^2(\mathbf{T})$ . Besides Prop. 1.1 we also need the inequality

$$|D_n|^2 \leq \exp\left(\sum_{k=1}^n k|A_k|^2 - \sum_{k=1}^n \frac{1}{k}\right), \quad D_0 = 1,$$

where  $\exp\left(\sum A_k z^k\right) = D_k z^k$  (Lebedev and Milin). The boundedness of  $\{|D_n|\}$  is assured if

$$\sum_{j=1}^n j A_j^2 < C \log n,$$

where  $A_j = 1/p^\sigma$  if  $j = k(p)$  and  $A_j = 0$  otherwise. Since  $\sigma > \frac{1}{2}$  we can apply Prop. 1.1 if there are at most  $C \log n$  elements in  $\{k(p) : k(p) \leq n\}$ . This latter condition essentially demands that  $k(p_j) \geq M e^j$  where  $p_j$  is the  $j$ -th prime. If  $k(p_j) \geq M e^j$  then each  $n = \prod p^r$  has at most one representation  $\sum r k(p)$ . Consequently,  $T_s \in L^2(\mathbf{T})$  for each  $\sigma > \frac{1}{2}$ .

EXAMPLE 2.4. a) Let  $p_j$  be the  $j$ -th prime and note that  $T_s$ ,  $\sigma \leq 1$ , is not in  $L^\infty(\mathbf{T})$  even if

$$(2.1) \quad k(p_{j+1}) > k(p_1) + \dots + k(p_j).$$

This is checked directly using the Fourier series criterion of  $\|\cdot\|_\infty$ -boundedness of Fejér partial sums, cf. part *d* which is weaker.

b) On the other hand if  $k$  satisfies (2.1) then an application of the Hausdorff-Young theorem shows that

$$\forall \sigma \in (\frac{1}{2}, 1] \quad \text{and} \quad \forall q < 1/(1 - \sigma), \quad T_\sigma \in L^q(\mathbf{T});$$

and, of course,  $T_s \in L^2(\mathbf{T})$  for each  $\sigma > \frac{1}{2}$ .

c) The significance of (2.1) is that the frequencies of

$$S_s(x) = \prod_{p \in \mathcal{P}} \left(1 + \frac{e^{ik(p)x}}{p^s}\right)$$

are distinct, noting that  $T_s = S_s \varphi_s$  where

$$\varphi_s(x) = \prod_{p \in \mathcal{P}} \left(\sum_{j=0}^{\infty} (e^{ik(p)x}/p^s)^{2j}\right) \in A(\mathbf{T}).$$

If  $k(p_j) = 2^{j-1}$  then  $N \cup \{0\}$  is the set of frequencies of  $S_s$ .

d) Set  $k(p_i) = 2^{i-1}$  and note that  $S_s, \sigma \in (\frac{1}{2}, 1]$ , does not have bounded variation on  $T$  since the Fourier coefficients of its distributional derivative are unbounded.

e) For perspective, recall that for each  $x \in (0, 2\pi)$ ,  $\sum (1/n^s) e^{inx}$  is an analytic function in the region  $\sigma > 0$ .

**3. – The analytic continuation problem.**

Given a function  $k: P \rightarrow N \cup \{0\}$  extended to  $N$  as in section 1. For each  $n$ , define

$$H(n) = \{m \geq 1: k(m) = n \text{ and } p|m \Rightarrow k(p) \neq 0\},$$

and  $m_n = \inf \{m: m \in H(n)\}$ . Note that if  $\{q_1, \dots, q_r\} \subseteq P$  is the zero set of  $k$  and if  $\lim_{\substack{p \\ p \neq q_i}} k(p) = \infty$  then  $\text{card } H(n) < \infty$  for each  $n$ . To prove this let  $k(p') > n$  and set

$$m_1 = \prod \{q^j: q \leq p', q \in P \setminus \{q_1, \dots, q_r\} \text{ and } j \in N \cup \{0\}\}.$$

where  $jk(q) > k(p')$ . Take  $m = \prod_{p \neq q_i} p^r \geq m_1$ , where the product is the prime decomposition of  $m$ . If some  $p > p'$  (in this product), then  $rk(p) \geq k(p') > n$  since  $k$  is increasing. If each  $p < p'$  (in this product) then some  $p = q$  has the property that  $r > j$ ; thus,  $k(m) \geq rk(p) \geq jk(p) = jk(q) > k(p') > n$ . Therefore there are at most  $m_1$  elements in  $H(n)$ .

**THEOREM 3.1.** *Given  $\sigma \in (\frac{1}{2}, 1]$  and  $k(p) \subseteq N \cup \{0\}$  where  $k(1) = k(2) = \dots = k(7) = 0$  and  $\text{card } \{p \in P: k(p) = 0\} < \infty$ . Assume that the power series expansion,  $\sum a_n t^n$ , of*

$$\exp \left( \sum_p \frac{\log p}{p^\sigma} t^{k(p)} \right)$$

has bounded coefficients, and that

$$(3.1) \quad \exists K_\sigma \text{ such that } \forall n \geq 0, \quad \frac{\text{card } H(n)}{m_n^\sigma} < K_\sigma$$

(in particular,  $\text{card } H(n) < \infty$ ). Extend  $k$  additively to  $N \cup \{0\}$  as in section 1. Then

$$(3.2) \quad \forall \sigma > 1/2, \quad \sup \left\{ \left| \sum_{k(m)=n} \frac{\log m}{m^\sigma} \right| : n \geq 0 \right\} < \infty.$$

PROOF. Clearly,  $\sum_0 z^j > e^z > 1 + z$ ; and so

$$(3.3) \quad \prod_p \left( \sum_{j=0}^{\infty} z_p^j \right) > \exp \left( \sum_p z_p \right) > \prod_p (1 + z_p) = \sum c_n t^n,$$

where  $z_p = ((\log p)/p^\sigma) t^{k(p)}$ .  $\{c_n\}$  is bounded since  $\{a_n\}$  is bounded. The left hand side of (3.3) can be written as

$$\prod_p (1 + z_p) \prod_p (1 + z_p^2 + z_p^4 + \dots) = \sum d_n t^n.$$

Some straight forward estimates and an occasional use of the prime number theorem yield the convergence of  $\sum b_n$ , where  $b_n$  is defined by  $\prod_p (1 + z_p^2 + \dots) = \sum b_n t^n$ . Combining these observations we see that  $\{d_n\}$  is bounded. The bound on (3.2) is then obtained in terms of  $\{d_n\}$ . The hypotheses in the statement that we have not mentioned in this outline of proof arise in the rather extensive technical verification of several of the above claims [Benedetto, 1974]. *q.e.d.*

Take  $\gamma \in \mathcal{J}_Q$  whose corresponding  $k$ -function takes values in  $N \cup \{0\}$  and satisfies (3.2). For each  $f \in A(\mathbf{T})$ , we define

$$\forall \sigma > \frac{1}{2}, \quad F_\varphi(s) = \langle T_s, \varphi \rangle = \sum_{n \in \mathbf{Z}} f(n) \hat{T}_s(n),$$

and we see that  $F_\varphi$  is analytic in the half-plane  $\sigma > \frac{1}{2}$  since its derivative

$$- \sum_{n \in \mathbf{Z}} f(n) \left( \sum_{k(m)=n} \frac{\log m}{m^s} \right)$$

is an absolutely convergent series. In this case we write for a fixed  $s_0, \sigma_0 > \frac{1}{2}$ ,

$$(3.4) \quad F_\varphi(s) = \sum_{n \geq 0} A_n(\varphi)(s - s_0)^n.$$

**THEOREM 3.2.** *Given  $\gamma \sim k \sim T_s$  and assume that  $k$  satisfies (3.2). Then for any fixed  $s_0 \in \mathbf{C}, \sigma_0 > \frac{1}{2}$ , there exists a sequence  $\{A_n: n = 0, 1, \dots\} \subseteq A'(\mathbf{T})$  depending on  $s_0$  such that*

$$(3.5) \quad T_s = \sum_{n=0}^{\infty} A_n(s - s_0)^n$$

converges in the  $\| \cdot \|_{A'}$ -norm in some neighborhood of  $s_0$ ; and

$$(3.6) \quad \sum_{n=0}^{\infty} \|A_n\|_{A'} < \infty$$

if  $\sigma_0 > \frac{3}{2}$ .

PROOF. Because of (3.4),

$$\forall \varphi \in A(\mathbf{T}), \quad A_n(\varphi) = \frac{1}{2\pi i} \int_C \frac{\langle T_s, \varphi \rangle}{(s - s_0)^{n+1}} ds,$$

where  $\sigma_0 > \frac{1}{2}$ ,  $C$  is the circle  $s_0 + re^{i\theta}$ , and  $\sigma_0 - r > \frac{1}{2}$ . In particular,  $A_n: A(\mathbf{T}) \rightarrow \mathbf{C}$  is linear and depends on  $s_0$  (and not on  $r$ ). Then,

$$|A_n(\varphi)| = \frac{1}{2\pi} \left| \int_0^{2\pi} \frac{F_\varphi(s_0 + re^{i\theta}) re^{i\theta} d\theta}{(re^{i\theta})^{n+1}} \right| \leq \left( \|\varphi\|_A \sup_{s \in C} \|T_s\|_{A'} \right) / r^n.$$

By the definition of  $T_s$ , we have

$$(3.7) \quad \|A_n\|_{A'} \leq \|T_{\sigma_0 - r}\|_{A'} / r^n.$$

Thus, if  $s \in C$  with  $\sigma > \frac{1}{2}$  has the property that  $|\sigma - \sigma_0|/r < 1$  then

$$\sum_{n=0}^{\infty} \|A_n\|_{A'} |s - s_0|^n < \infty.$$

This yields (3.5). (3.6) is clear from (3.7) and the requirements that  $r > 1$  and  $\sigma_0 - r > \frac{1}{2}$ . *q.e.d.*

EXAMPLE 3.1. Given  $\gamma \sim k \sim T_s$ , and assume that  $k$  satisfies (3.2). Then

$$\forall \sigma > \frac{1}{2}, \quad \lim_{|n| \rightarrow \infty} \hat{T}_s(n) = 0.$$

In fact,

$$\left| \sum_{k(m)=n} \frac{1}{m^\sigma} \right| \leq \frac{1}{\log m_n} \left| \sum_{k(m)=n} (\log m) / m^\sigma \right| \leq \frac{k}{\log m_n},$$

where  $m_n$  is the smallest  $m$  for which  $k(m) = n$ . Clearly  $\lim m_n = \infty$ .



#### 4. – A topological-arithmetic property of idele characters.

Given integers  $j, J \geq 1$ , and a finite set  $Y$  having at least  $([j^{J/j}] + 1)^j$  elements. The elements of  $Y$  will consist of  $j$ -tuples of integers satisfying the following property: if  $y = (y_1, \dots, y_j)$  and  $y' = (y'_1, \dots, y'_j)$  are in  $Y$  and  $y \neq y'$  then  $y - y'$  or  $y' - y$  is a  $j$ -tuple of *positive* integers.

EXAMPLE 4.1. Define  $Y = \{y_1, \dots\}$  in either of the following two ways

a)  $y_1 = (1, 1, \dots, 1), y_2 = (2, 2, \dots, 2), y_3 = (3, 3, \dots, 3), \dots$

b) Given  $x = (x_1, \dots, x_j)$  where  $x_h$  is a positive integer. Define  $y_h = hx$  for  $h = 1, 2, \dots$

The following elementary result in diophantine approximation depends only on the Dirichlet pigeon-hole principle.

LEMMA. *Given  $j, J$ , and  $Y$  as above. There are elements  $y, y' \in Y$  such that if  $k = y' - y > 0$ , where  $k = (k(p_1), k(p_2), \dots, k(p_j))$  and  $k(p_h) \geq 1$ , then for each  $j$ -tuple  $(n_{p_1}, \dots, n_{p_j}) \in \mathbf{Z} \times \dots \times \mathbf{Z}$  ( $j$ -times) with  $|n_{p_i}| \leq [J^{1/j}]$ , there is  $N = N(p_1, \dots, p_j)$  for which*

$$\left| \sum_{p \leq p_j} k(p) n_p - 2\pi N \right| < 1/j.$$

PROOF. Let  $M = [J^{1/j}]$  so that there are on the order of  $M^j = J$   $j$ -tuples  $(n_{p_1}, \dots, n_{p_j})$  of integers for which  $|n_p| \leq M$  if  $p \leq p_j$ . Consider the  $J \times j$  matrix  $(\alpha_{r,c})$  of numbers  $n/2\pi$  where  $n \in \mathbf{Z}$  and  $|n| \leq M$ .

For each  $y = (y_1, \dots, y_j) \in Y$  consider the  $J$ -tuple  $w_y = (w_1, \dots, w_j)$  defined by

$$w_r = w_r(y) = \sum_{c=1}^j \alpha_{r,c} y_c, \quad r = 1, \dots, J.$$

For any such  $w_y$ , let  $(x_1, \dots, x_j) \in \mathbf{Z} \times \dots \times \mathbf{Z}$  ( $J$ -times) have the property that  $\omega_y = (x_1 - w_1, \dots, x_j - w_j) \in [0, 1) \times \dots \times [0, 1_p)$  ( $J$ -times).

Divide  $[0, 1)$  into  $j$  intervals  $[h/j, (h+1)/j)$ ,  $h = 0, 1, \dots, j-1$ ; and in this way partition  $[0, 1) \times \dots \times [0, 1)$  ( $J$ -times) into  $j^J$  cubes.

We have

$$([j^{J/j}] + 1)^j > (j^{J/j})^j = j^J.$$

Since  $\text{card } Y \geq ([j^{J/j}] + 1)^j$  there is at least one of the  $j^J$  cubes, call it  $C$ , that has at least two of the  $\omega_y$  ( $y \in Y$ ) in it. Say  $\omega_y, \omega_{y'} \in C$ , where  $y \neq y'$ .

Because  $C$  is a cube of side  $1/j$  we have

$$\frac{1}{j} > |(x_r - w_r) - (x'_r - w'_r)| = \left| \sum_{c=1}^j \alpha_{r,c}(y'_c - y_c) - (x'_r - x_r) \right| \quad \text{for } r = 1, \dots, J.$$

By our hypothesis we can assume without loss of generality that  $y'_c - y_c > 0$  for each  $c = 1, \dots, j$ . Thus, finally, we can define  $k(p_c) = y'_c - y_c$ ,  $1 \leq c \leq j$ . *q.e.d.*

Choose  $\gamma \in \mathcal{J}_{\mathbf{Q}}$  so that if  $\gamma \sim k \sim T_s$  then  $T_s \in \text{IPM}$  for each  $\sigma > \frac{1}{2}$ . If  $j \geq 1$  define  $k_j(p) = jk(p)$  for  $p \in P$ , and write  $\gamma_j \sim k_j \sim T_{s,j}$ . By definition, we have

**PROPOSITION 4.1.** *Given  $T_{s,j} \in A'(\mathbf{T})$  where  $j$  is a positive integer.*

- a) *Take  $n \in \mathbf{Z}$ . If  $j \nmid n$ , then  $\hat{T}_{s,j}(n) = 0$ ; and if  $j \mid n$ , then  $\hat{T}_{s,j}(n) = \hat{T}_{s,1}(n/j)$ .*
- b)  $\|T_{s,j}\|_{A'} = \|T_{s,j+1}\|_{A'}$ .

**EXAMPLE 4.2.** a) Define  $k_j(p) = 0$  if  $p \leq p_j$  and  $k_j(p) = k(p) > 0$  if  $p > p_j$ , where  $\gamma \sim k \sim T_s \in \text{IPM}$  for all  $\sigma \in (\frac{1}{2}, 1]$ . Then

$$\hat{T}_{j,\sigma}(0) = \sum_{k(m)=0} 1/m^\sigma \geq \frac{1}{p_1^\sigma} + \dots + \frac{1}{p_j^\sigma}$$

and so  $\hat{T}_{j,\sigma}(0)$  increases indefinitely as  $j \rightarrow \infty$ . Thus,  $\lim_{j \rightarrow \infty} \|T_{j,\sigma}\|_{A'} = \infty$ . On the other hand, it is easy to compute that  $\hat{T}^{j,\sigma}(0) < \infty$  for such  $\sigma$ .

b) For the  $k_j$  defined in part a we let  $\gamma_j \in \mathcal{J}_{\mathbf{Q}}$  be the corresponding idele character; and note that  $\{\gamma_j\}$  converges to 1 in  $\mathcal{J}_{\mathbf{Q}}$ .

Given  $\gamma \sim k \sim T_s \in \text{IPM}$  for some  $\sigma \in (\frac{1}{2}, 1]$ , where  $k(p) > 0$  for each  $p \in P$ . Define  $k_j$  as we did before Prop. 4.1. Then

**THEOREM 4.1.** a) *There is a subsequence of  $\{\gamma_j\}$  which converges to 1 in  $\mathcal{J}_{\mathbf{Q}}$ .*

b) *Along with Prop. 4.1, we also have that  $\lim_{j \rightarrow \infty} T_{j,s} = 1$  in the  $\sigma(A', A)$  topology; in fact, for each  $j \hat{T}_{j,s}(0) = 1$  and for each  $n \neq 0 \lim_{j \rightarrow \infty} \hat{T}_{j,s}(n) = 0$ .*

**PROOF.** a.i. Let  $|\cdot|_p$  be the  $p$ -adic metric on the multiplicative group  $\mathbf{Q}_p^\times$  of the  $p$ -adic completion of  $\mathbf{Q}$ ; and let  $U_p$  be the  $p$ -adic units. We define

$$B_{p,n} = \left\{ r_p \in \mathbf{Q}_p^\times : \frac{1}{n} \leq |r_p|_p \leq n \right\},$$

$$K_n = \prod_{0 \leq p \leq p_n} B_{p,n} \times \prod_{p > p_n} U_p,$$

and

$$N_n = \{\gamma \in \mathcal{J}_Q : \forall x \in K_n, |(\gamma, x) - 1| < 1/n\} \quad (\text{where } \mathbf{Q}_0^\times = \mathbf{R}^\times).$$

From the definition of the topology in  $\mathcal{J}_Q$  we must verify that for each  $N_n$  there is  $J_n$  so that

$$(4.1) \quad \forall x = \{r_p\} \in K_n, \quad |(\gamma_j, x) - 1| < 1/n \quad \text{for each } j > J_n.$$

We know that

$$(\gamma_j, x) = \prod_p \exp i \frac{k_j(p)}{\log p} \log |r_p|_p,$$

where  $n_p \in \mathbf{Z}$ ,  $-\log n \leq -n_p \log p \leq \log n$ , and  $\log |r_p|_p = -n_p \log p$ ; in particular,  $|n_p| \leq \log n / \log p$ .

Taking  $x \in K_n$ , we have  $r_p \in U_p$  if  $p > p_n$  and so  $|r_p|_p = 1$  if  $p > p_n$ ; thus  $n_p = 0$  for each  $p > p_n$ .

Consequently, if  $x \in K_n$ , then

$$(\gamma_j, x) = \prod_{p \leq p_n} \exp -ik_j(p)n_p, \quad |n_p| \leq \log n / \log p.$$

Using the notation of the *Lemma*, we set  $M = [\log n]$  since

$$\max_{p \leq p_n} ([\log n / \log p]) \leq [\log n];$$

and we define  $J = M^n$ .

Note that if  $x \in K_n$ , then, by the above, the  $n$ -tuple  $(n_{p_1}, \dots, n_{p_n})$  corresponding to  $x$  has the property that  $|n_{p_i}| \leq M$  for each  $i$ .

Consequently, we apply the *Lemma* (and raise to the  $e^i$  power) to obtain (4.1).

*b* follows from *Prop 4.1. q.e.d.*

The analogous result holds for  $L^q(\mathbf{T})$ .

#### BIBLIOGRAPHY

- J. BENEDETTO, *Idele characters in spectral synthesis on  $\mathbf{R}/2\pi\mathbf{Z}$* , Ann. Inst. Fourier, **23** (1973) pp., 45-64.  
 J. BENEDETTO, *Zeta functions for idelic pseudo-measures*, U. of Maryland Math. Dept. Technical report 74-55 (1974), pp. 1-46.

- J. BENEDETTO, *Idele pseudo-measures and Dirichlet series*, Symposia mathematica (Istituto Nazionale di Alta Matematica, Roma). Conference in harmonic analysis, Academic Press, New York, 1976.
- R. HOLZSAGER, *A boundedness condition*, J. London Math. Soc., **11** (1975), pp. 179-180.
- J. TATE, *Fourier analysis in number fields and Hecke's zeta functions* (1950), *Algebraic number theory* (ed. J. Cassels and A. Fröhlich), Academic Press, New York, 1967.