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Variations on a theme of Carathéodory

*Annali della Scuola Normale Superiore di Pisa, Classe di Scienze* 4e série, tome 6, n° 1 (1979), p. 39-68

<http://www.numdam.org/item?id=ASNSP_1979_4_6_1_39_0>
Variations on a Theme of Carathéodory (*).

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odory and Kobayashi distances in terms of the Pták norm [14, 15]. These formulas yield a generalization of one of the main results in [23] from von Neumann algebras to C*-algebras with identity (Proposition 6.3).

1. – Preliminaries and plurisubharmonicity.

1. – Let \( A = \{ \zeta \in C : |\zeta| < 1 \} \) be the open unit disc in \( C \). The Poincaré-Bergman differential metric
\[
d s^2 = d \zeta d \bar{\zeta} (1 - |\zeta|^2)^2
\]
defines on \( A \) a distance
\[
w(\zeta_1, \zeta_2) = \frac{1}{2} \log \frac{1 + |(\zeta_1 - \zeta_2)/(1 - \zeta_1 \bar{\zeta}_2)|}{1 - |(\zeta_1 - \zeta_2)/(1 - \zeta_1 \bar{\zeta}_2)|}, \quad (\zeta_1, \zeta_2 \in A).
\]

Let \( E \) and \( E_1 \) be two complex, locally convex, Hausdorff vector spaces, and let \( A \) be a domain in \( E \). A holomorphic map \( F : A \to E_1 \) is, by definition [13, p. 25], a continuous map \( F \) of \( A \) into \( E_1 \) such that, for every choice of \( (x, y) \in A \times (E \setminus \{0\}) \) and every continuous linear form \( \lambda \) on \( E_1 \), the scalar-valued function \( \zeta \mapsto \lambda F(x + \zeta y) \) is holomorphic on the open set \( \{ \zeta \in C : x + \zeta y \in A \} \) of \( C \). If \( A_1 \) is a domain in \( E_1 \), we denote by \( \text{Hol}(A, A_1) \) the set of all holomorphic maps \( F : A \to E_1 \), such that \( F(A) \subseteq A_1 \).

The Kobayashi pseudo-distance \( d_A(x', x'') \) between two points \( x', x'' \) in \( A \) is defined as follows. Let \( \zeta'_1, \zeta'_2, \ldots, \zeta'_v, \zeta''_1, \zeta''_2, \ldots, \zeta''_v \) be \( v \) pairs of points in \( A \), and let \( f_1, \ldots, f_v \) be elements of \( \text{Hol}(A, A) \) such that \( f_j(\zeta'_j) = x' \), \( f_j(\zeta''_j) = x'' \), \( f_{j+1}(\zeta'_{j+1}) = f_j(\zeta'_j) \) for \( j = 1, \ldots, v-1 \).

The Kobayashi pseudo-distance \( d_A(x', x'') \) is, by definition,
\[
d_A(x', x'') = \inf \sum_{j=1}^v w(\zeta'_j, \zeta''_j),
\]
where the infimum is taken over all possible choices of \( v, \zeta'_1, \zeta'_2, \ldots, \zeta'_v, \zeta''_1, \zeta''_2, \ldots, \zeta''_v \).

A simple application of the triangle inequality and of the Schwarz-Pick lemma implies that, for every \( f \in \text{Hol}(A, A) \),
\[
w(f(x'), f(x'')) < d_A(x', x'').
\]

Thus, setting
\[
e_A(x', x'') = \sup \{ w(f(x'), f(x'')) : f \in \text{Hol}(A, A) \},
\]
we have [9]
\[
e_A(x', x'') < d_A(x', x''). \quad (1.1)
\]

The function \( (x', x'') \mapsto e_A(x', x'') \) is the Carathéodory pseudo-distance on \( A \).
Let $A_1$ be a domain in $\mathcal{E}_1$, and consider the Kobayashi and Carathéodory pseudo-distances $d_{A_1}$ and $e_{A_1}$. The above definitions imply that any $F \in \text{Hol}(A, A_1)$ is distance decreasing for both the Kobayashi and Carathéodory pseudo-distances, i.e.,

$$d_{A_1}(F(x'), F(x'')) < d_{A}(x', x''), \quad e_{A_1}(F(x'), F(x'')) < e_{A}(x', x'')$$

for all $x', x'' \in A$. In particular: 1) every bi-holomorphic diffeomorphism of $A$ onto $A_1$, is an isometry for both pseudo distances; 2) if $D$ is a domain in $\mathcal{E}$, such that $D \subset A$, then

$$d_{A}(x', x'') < d_{D}(x', x''), \quad e_{A}(x', x'') < e_{D}(x', x'') \quad (x', x'' \in D).$$

Furthermore, the Schwarz-Pick lemma yields [9]

(1.2) \[ e_d = \omega = d. \]

Let $p$ be a continuous semi-norm on $\mathcal{E}$, and let

$$B_p = \{ x \in \mathcal{E} : p(x) < 1 \}.$$

**Lemma 1.1.** For every $x \in B_p$,

$$e_{B_p}(0, x) = \omega(0, p(x)).$$

**Proof.** Let $x \in B_p$, with $p(x) > 0$. The (holomorphic) function $\zeta \mapsto (\zeta/p(x))x$ maps the unit disc into $B_p$, $0$ into $0$, and $p(x)$ into $x$. Thus

$$e_{B_p}(0, x) < d_{B_p}(0, x) < \omega(0, p(x)).$$

On the other hand, there exists a continuous linear form $\lambda$ on $\mathcal{E}$ such that $\lambda(x) = p(x)$ and $|\lambda(y)| < p(y)$ for all $y \in \mathcal{E}$. Thus $\lambda \in \text{Hol}(B_p, A)$, and therefore

$$\omega(0, p(x)) < e_{B_p}(0, x).$$

Let $x \neq 0$, but $p(x) = 0$. For any $t > 1$, the holomorphic function $f_t : \zeta \mapsto t\zeta x$ maps $A$ into $B_p$; moreover $f_t(0) = 0$, $f_t(1/t) = x$. Hence $e_{B_p}(0, x) < d_{B_p}(0, x) < \omega(0, 1/t)$.

Letting $t \to \infty$, we get $e_{B_p}(0, x) = d_{B_p}(0, x) = 0$. The proof of the lemma is complete. Q.E.D.
Let $r > 0$ and let $B_{p,r}$ and $A_r$ be the open discs $B_{p,r} = \{x \in \mathbb{C}: p(x) < r\}$, $A_r = \{\xi \in \mathbb{C}: |\xi| < r\}$. If $x \in \mathbb{C}$, and if $F: \mathbb{C} \to \mathbb{C}$ is the holomorphic map $\zeta \mapsto \zeta x$, then $F^{-1}(B_{p,r})$ is the disc $A_R$ of radius $R = r/p(x)$, where we set $R = \infty$ and $A_\infty = \mathbb{C}$ if $p(x) = 0$. In the latter case, both the Carathéodory and Kobayashi pseudo-distances on $A_\infty$ vanish identically. If $0 < R < \infty$, they can be obtained from (1.2) by a homotopy: they coincide, and

$$c_{\mathcal{A}_R}(0, \zeta) = d_{\mathcal{A}_R}(0, \zeta) = \omega \left( 0, \frac{\zeta}{R} \right) \quad (\zeta \in A_R).$$

Let $x \in B_{p,r} \setminus \{0\}$ and let $D$ be a domain in $\mathbb{C}$ such that $F(A_R) \subset D \subset B_{p,r}$.

By Lemma 1.1,

$$c_{D}(0, F^{-1}(x)) > c_{D}(0, x) > c_{B_{p,r}}(0, x) = \omega \left( 0, \frac{p(x)}{r} \right),$$

$$d_{\mathcal{A}_R}(0, F^{-1}(x)) > d_{D}(0, x) > d_{B_{p,r}}(0, x) = \omega \left( 0, \frac{p(x)}{r} \right).$$

That proves

**Corollary 1.2.** For all $x \in B_{p,r}$ and for any domain $D$ in $\mathbb{C}$ such that $F(A_R) \subset D \subset B_{p,r}$, we have

$$c_{D}(0, x) = d_{D}(0, x) = \omega \left( 0, \frac{p(x)}{r} \right).$$

Now let $p_1, \ldots, p_n$ be continuous seminorms on $\mathbb{C}$, and let $D_0$ be the domain

$$D_0 = B_{p_1, r_1} \cap \cdots \cap B_{p_n, r_n}$$

for some $r_1 > 0, \ldots, r_n > 0$. Let $x \in D_0$, and suppose that

$$\frac{p_1(x)}{r_1} > \frac{p_2(x)}{r_2} > \cdots > \frac{p_n(x)}{r_n}.$$

The function $\zeta \mapsto \zeta x$ maps $A_{r_1/r_1}(x)$ into $D_0$. Hence Corollary 1.2 yields

$$c_{D_0}(0, x) = d_{D_0}(0, x) = \omega \left( 0, \frac{p_1(x)}{r_1} \right),$$

i.e.

$$c_{D_0}(0, x) = d_{D_0}(0, x) = \max \left\{ \omega \left( 0, \frac{p_j(x)}{r_j} \right) : j = 1, \ldots, n \right\}.$$

$$= \max \left\{ c_{B_{p_j, r_j}}(0, x) : j = 1, \ldots, n \right\}.$$
For any $x_0 \in \mathcal{E}$, the domain

\begin{equation}
D_{x_0} = \{ x \in \mathcal{E} : p_1(x - x_0) < r_1, \ldots, p_n(x - x_0) < r_n \}
\end{equation}

is the image of $D_0$ by the translation defined by $x_0$. Thus

\begin{equation}
c_{D_{x_0}}(x_0, x) = d_{D_{x_0}}(x_0, x) = \max \left\{ \omega \left( 0, \frac{p_j(x - x_0)}{r_j} \right) : j = 1, \ldots, n \right\}.
\end{equation}

Since the open sets (1.3) generate a fundamental system of neighborhoods of $x_0$, then for any $x_0 \in A$ and any $\varepsilon > 0$, there is a neighborhood $U$ of $x_0$ in $A$ such that $d_A(x_0, x) < \varepsilon$ for all $x \in U$. Taking into account (1.4) we conclude with

**Proposition 1.3.** The functions $c_A : A \times A \to \mathbb{R}$, $d_A : A \times A \to \mathbb{R}$ are continuous.

2. In this section we shall show that, for any $x_0$ in the domain $A$, the Carathéodory pseudo-distance $c_A(x_0, x)$ is a logarithmically plurisubharmonic function of $x \in A$. We consider first the case $\mathcal{E} = \mathbb{C}$, $A = \Delta$.

**Lemma 2.1.** For any $\zeta_0 \in \Delta$, the function $\zeta \mapsto \log \omega(\zeta_0, \zeta)$ is subharmonic on $\Delta$.

**Proof.** Since the group of holomorphic automorphisms of $\Delta$ acts transitively on $\Delta$ and isometrically on the Poincaré-Bergman distance, it suffices to prove the lemma when $\zeta_0 = 0$.

The function $\zeta \mapsto \omega(0, \zeta)$ being continuous, we need only show that, for any $a \in \mathbb{C}$, the function $q_a : \zeta \mapsto e^{\omega(a, \zeta)}$ is subharmonic on $\Delta \setminus \{0\}$. Choosing a branch for log $\zeta$ on $\Delta \setminus \{0\}$, the function $\zeta \mapsto q_a(\zeta)$ is $C^\infty$ on $\Delta \setminus \{0\}$, and

\begin{align*}
\frac{\partial}{\partial \zeta} |\zeta|^2 &= \frac{1}{2} (\zeta/\bar{\zeta})^4, \\
\frac{\partial^2}{\partial \zeta^2} |\zeta|^2 &= \frac{1}{2} (\bar{\zeta}/\zeta)^4.
\end{align*}

Thus, for every $\zeta \in \Delta \setminus \{0\}$,

\begin{align*}
\frac{\partial q_a}{\partial \zeta} &= \frac{1}{2} \left( a q_a(\zeta) + \frac{1}{1 + |\zeta|^2} (\zeta/\bar{\zeta})^4 \right) e^{\omega(\zeta)}, \\
\frac{\partial^2 q_a}{\partial \zeta^2} &= \frac{1}{2} \left( \frac{|a|^2}{4} \log \frac{1 + |\zeta|^2}{1 - |\zeta|^2} + \frac{1}{1 - |\zeta|^2} \operatorname{Re} \left( a \left( \frac{\zeta}{\bar{\zeta}} \right)^4 \right) + \frac{1 + |\zeta|^2}{2 |\zeta|(1 - |\zeta|^2)^2} \right) \\
&> \frac{1}{2} \left( \frac{|a|^2}{4} \log \frac{1 + |\zeta|^2}{1 - |\zeta|^2} - \frac{|a|}{1 - |\zeta|^2} + \frac{1 + |\zeta|^2}{2 |\zeta|(1 - |\zeta|^2)^2} \right).
\end{align*}
We show now that for $0 < t < 1$ the trinomial in $\phi$

\[(2.1) \quad \phi^2 \log \frac{1 + t}{1 - t} - \frac{2\phi}{1 - t^2} + \frac{1 + t^2}{2t(1 - t^2)^2}\]

is positive. The discriminant is equal to $(2/t(1 - t^2)^2)\sigma(t)$ where

\[\sigma(t) = 2t - (1 + t^3) \log \frac{1 + t}{1 - t} \]

Since

\[\sigma'(t) = \frac{-2t}{1 - t^2} \left(2t + (1 - t^2) \log \frac{1 + t}{1 - t}\right) < 0 \quad \text{for} \ 0 < t < 1\]

the function $\sigma$ is strictly decreasing for $0 < t < 1$. Being $\sigma(0) = 0$, then $\sigma(t) < 0$ for $0 < t < 1$, and the trinomial (2.1) is positive definite. Thus $\partial^2 q_a/\partial \zeta \partial \bar{\zeta} > 0$ on $A \setminus \{0\}$. Since $q_a(\zeta) > 0$ on $A \setminus \{0\}$, then

\[q_a(0) = 0 < \frac{1}{2\pi} \int_0^{2\pi} q_a(re^{i\theta}) d\theta \quad \text{for any} \ 0 < r < 1.\]

Thus $q_a$ is subharmonic on $A$ for all $a \in C$. Q.E.D.

Going back to the general case, let $f \in \text{Hol}(A, d)$, and let $x_0 \in A$. Lemma 2.1 implies that the function $x \mapsto \log \omega(f(x_0), f(x))$ is a continuous plurisubharmonic function on $A$ [13, théorème 1.2.12, pp. 27-28].

Since the function $x \mapsto \log \omega(x_0, x)$ is a continuous function $A \to [-\infty, +\infty)$ (Proposition 1.2), which is by definition the upper envelope of a family of plurisubharmonic functions on $A$, then we have proved

**Theorem I.** For any $x_0$ in the domain $A$, the function $x \mapsto \log \omega(x_0, x)$ is a continuous plurisubharmonic function on $A$.

**3.** A bounded set $T \subset C$ is a polar set if there exists a subharmonic function $\varphi \neq -\infty$ such that $\varphi = -\infty$ on $T$. According to a theorem of H. Cartan [2], a bounded subset of $C$ is a polar set if, and only if, its exterior capacity is zero.

Theorem I yields

**Proposition 2.2.** Let $x_0$ be any point in the domain $A \subset C$, and let $f: A \to A$ be a holomorphic map with $x_0 \in f(A)$. The set $\{\zeta \in A: \omega(x_0, f(\zeta)) = 0\}$ is either the entire disc $A$, or a polar set.
In the latter case its exterior capacity is zero. This implies that, for any \( x_0 \in A \), the set \( \{ x \in A : c_A(x_0, x) = 0 \} \) has no interior points, unless it is \( A \) itself.

Although this paper is mainly devoted to the study of invariant metrics on domains in Banach spaces, it is worth noticing that the arguments leading to the proof of Theorem I hold, with no substantial change, in the case where \( A \) is a connected (finite dimensional, reduced) complex space. Thus \( \log c_A(x_0, \cdot) \) is a continuous plurisubharmonic function on a connected complex space \( A \), for any \( x_0 \in A \).

We list a few consequences of this fact.

In [1] A. Andreotti and R. Narasimhan gave a sufficient condition for a complex space \( A \) to be a Stein space, bearing on the existence on \( A \) of a suitable plurisubharmonic function. In view of this condition and of Theorem I, the following statement holds:

If the connected complex space \( A \) is \( K \)-complete and if, for some \( x_0 \in A \), the sets,

\[
A_k = \{ x \in A : c_A(x_0, x) < k \}
\]

are relatively compact in \( A \) for all \( k > 0 \), then \( A \) is a Stein space.

In particular, if a \( K \)-complete connected complex space \( A \) is finitely compact (i.e. every bounded closed subset is compact) or, more in particular, if \( c_A \) is Cauchy complete, then \( A \) is a Stein space.

In [7] H. Horstmann proved that any domain \( A \) in \( C^n \), for which \( A_k \) is relatively compact in \( A \) for every \( k > 0 \), is holomorphically convex. This fact—which was generalized by S. Kobayashi [9] to complex spaces—coupled with \( K \)-completeness, yields the above result by a classical theorem of K. Oka. Of course there are Stein spaces \( A \), like for instance \( C^n \), on which the Carathéodory distance degenerates completely, or for which the sets \( A_k \) are not relatively compact. However, if \( A \) is a Stein space and if the sets \( A_k \) are relatively compact for all \( k > 0 \), then by Theorem I and a theorem of R. Narasimhan [12], any \( A_k \) is a Stein space which is Runge in \( A \).

2. — « Mittelpunkttreu » automorphisms.

4. — Since holomorphic maps contract the Carathéodory and Kobayashi pseudo-distances, both these pseudo-distances have a built-in Schwarz lemma. In this and the following sections we shall examine explicit forms of this lemma for Banach spaces and Banach algebras, and discuss some applications.
Let $\mathcal{E}$ be a complex Banach space, with norm $\|\cdot\|$, and let $B$ be the open unit ball in $\mathcal{E}$.

Lemma 1.1 and Theorem I yield

**Lemma 4.1.** The function $x \mapsto \log \log \left( \frac{1 + \|x\|}{1 - \|x\|} \right)$ is plurisubharmonic on $B$.

Let $B_1$ be the open unit ball of a complex Banach space $\mathcal{E}_1$. If $F: B \to B_1$ is any holomorphic map such that $F(0) = 0$, then

$$c_{B_1}(0, F(x)) < c_B(0, x) \quad \text{for all } x \in B.$$  

Since the function $t \mapsto \log \left( \frac{1 + t}{1 - t} \right)$ is strictly increasing on $[0, 1)$, then Lemma 1.1 implies that

$$\|F(x)\| < \|x\| \quad \text{for all } x \in B.$$  

This weak form of the Schwarz lemma can also be obtained by applying the maximum principle to the subharmonic function $\zeta \mapsto \|\frac{1}{\zeta}F(\zeta x)\| \quad (\zeta \in \Delta, x \in B)$ (cf. [6]). A simple application of the maximum principle along the lines of the classical Schwarz lemma yields part i) of the following lemma. Before stating it, we recall the definition of a **complex extreme point**. Let $K$ be a convex subset of $\mathcal{E}$. A point $x \in K$ is a complex extreme point of $K$ if $y = 0$ is the only vector in $\mathcal{E}$ such that the function $\zeta \mapsto x + \zeta y$ maps $\Delta$ into $K$.

**Lemma 4.2.**

i) If equality holds in (4.1) at some point $x_0 \in B \setminus \{0\}$, then

$$\|F(\zeta x_0)\| = \|x_0\| \quad \text{for all } \zeta \in \mathcal{C} \text{ with } |\zeta| < \frac{1}{\|x_0\|}.$$  

ii) Assume that every point with norm one in $\mathcal{E}_1$ is a complex extreme point of the closure $\overline{B}_1$ of $B_1$. If equality holds in (4.1) at some point $x_0 \in B \setminus \{0\}$, then

$$F(\zeta x_0) = \zeta F(x_0) \quad \text{for all } \zeta \in \mathcal{C} \text{ with } |\zeta| < \frac{1}{\|x_0\|}.$$  

To prove part, ii) consider the subharmonic function

$$\zeta \mapsto \left\| \frac{1}{\zeta \|x_0\|} F(\zeta x_0) \right\| \quad \text{for } |\zeta| < \frac{1}{\|x_0\|},$$
reaching its maximum, 1, at $\zeta = 1 < \|x_0\|$. Since all points of norm one are complex extreme points of $B_1$, then the strong maximum principle [19] implies that the function $(1/\zeta \|x_0\|)F(\zeta x_0)$ is independent of $\zeta$, i.e. there is a vector $u$ of norm one in $E$ such that

$$F(\zeta x_0) = \zeta \|x_0\| \|u\| \quad \text{for all } |\zeta| < \frac{1}{\|x_0\|}.$$ 

Choosing $\zeta = 1$ we see that $\|x_0\| \|u\| = F(x_0)$, and that completes the proof of the lemma. Q.E.D

We shall now apply Lemma 4.2 to the study of a class of non-homogeneous bounded domains.

Let $(M, \mathcal{E}, \mu)$ be a measure space. Here $M$ is a set, $\mathcal{E}$ is a $\sigma$-algebra of subsets of $M$, and $\mu$ is a positive measure on $\mathcal{E}$. Let $E = L^1(M, \mu)$ and let $B$ be the open unit ball

$$B = \left\{ x \in E : \|x\| = \int_M |x| \, d\mu < 1 \right\}.$$

We will prove the following

**Theorem II.** If $\dim_c E > 1$, every holomorphic automorphism of $B$ is (the restriction to $B$ of) a linear isometry of $E$.

Let $H$ be a holomorphic automorphism of $B$. According to a theorem of H. Cartan [4], $H$ is a continuous linear map—and therefore a linear isometry of $E$—if (and only if) $H$ leaves the origin fixed. Hence all we have to prove is that $H(0) = 0$.

Let $y_0 = H(0)$, and suppose that $y_0 \neq 0$. We shall show that this assumption leads to a contradiction.

Consider the measure $d\psi = y_0(m) \, d\mu(m)$ $(m \in M)$, and let

$$d\psi = h|d\psi|$$

be its polar decomposition; $h$ is a measurable function such that $|h(m)| = 1$ for all $m \in M$. Then

$$|y_0| = \bar{h}y_0 \quad \text{a.e.}$$

The map $x \mapsto \bar{h}x$ is a linear isometry of $E$ onto $E$. Thus, composing $H$ with this isometry, we can assume that $y_0 = H(0)$ is a real positive element of $L^1(M, \mu)$. Since $\dim_c L^1(M, \mu) > 1$, the $\sigma$-algebra $\mathcal{E}$ contains at least two proper non-empty disjoint subsets on which $\mu$ takes finite, positive values. Hence there exists an element $K \in \mathcal{E}$, $K \neq M$, such that $\mu(M \setminus K) \in (0, +\infty]$
and that 

\[ \int_K y_0 \, d\mu > 0. \]

Let \( \varphi : M \to \mathbb{R} \) be the measurable function defined by: \( \varphi(m) = -1 \) if \( m \notin K \), \( \varphi(m) = 1 \) if \( m \in K \). The map \( x \mapsto \varphi x \) is a linear isometry of \( \mathcal{E} \) onto \( \mathcal{E} \), and the map \( B \ni x \mapsto \varphi H(x) \) is a (holomorphic) automorphism of \( B \) for which

\[ H(0) + \varphi H(0) = y_0 - y_0 = 0 \quad \text{a.e. on } M \setminus K, \]

\[ \int_K (H(0)(m) + \varphi(m)H(0)(m)) \, d\mu(m) = 2 \int_K y_0(m) \, d\mu(m) > 0. \]

Let \( x_0 = \frac{1}{2}(y_0 + \varphi y_0) \). Then \( x_0 = 0 \) a.e. on \( M \setminus K \), \( x_0 \in B \) and

\[ \int_K x_0 \, d\mu > 0. \]

Let \( \text{Aut}(B) \) be the group of all holomorphic automorphisms of B. W. Kaup and H. Upmeier have shown in [8] that there exists a closed complex sub-space \( \mathcal{F} \) of \( \mathcal{E} \) such that the orbit \( \text{Aut}(B)(0) \) is \( \mathcal{F} \cap B \). Hence there exists an automorphism \( F \in \text{Aut}(B) \) such that \( F(0) = x_0 \).

Let \( x \in B \). A subset \( \Gamma \subset B \), with \( x \in \Gamma \) will be called a complex geodesic curve at \( x \) in \( B \) if there exists a holomorphic map \( f : \Lambda \to B \) such that:

1) \( f(\Lambda) = \Gamma \), and thus \( x = f(\zeta_0) \) for some \( \zeta_0 \in \Lambda \);

2) \( c_\omega(x, f(\zeta)) = \omega(\zeta_0, \zeta) \) for all \( \zeta \in \Lambda \).

Note that, by applying first a suitable Moebius transformation of \( \Lambda \) we can always choose \( \zeta_0 = 0 \).

A result of E. Thorp and R. Whitley enables us to determine all complex geodesic curves at 0. In fact it was shown in [19] that every vector of norm one in \( \mathcal{E} \) is a complex extreme point of the closure \( \overline{B} \) of \( B \). Thus part ii) of lemma 4.2 shows that all complex geodesic curves at 0 are determined by linear maps \( C : \mathbb{R} \to \mathcal{E} \). More precisely, we have

**Lemma 4.3.** For every \( x \in B \), \( x \neq 0 \), the image of \( \Lambda \) by the linear map \( \zeta \mapsto (\zeta/\|x\|)x \) is the unique complex geodesic curve at 0, containing \( x \).

We will now construct a family of complex geodesic curves at \( x_0 \).

**Lemma 4.4.** Let \( a \) and \( b \) be two real vectors in \( \mathcal{E} \) such that

(4.2) \[ |a(m)| < b(m) \quad \text{a.e. on } M, \quad \int_M b(m) \, d\mu(m) = 1, \quad \int_M a(m) \, d\mu(m) = 0. \]
and let $f: \Delta \to \mathbb{E}$ be the holomorphic function on $\Delta$ defined by

\begin{equation}
(4.3) 
   f(\zeta) = \frac{1 + \zeta^2}{2} a + \zeta b \quad (\zeta \in \Delta).
\end{equation}

Then $f(\Delta) \subset B$, and $f(\Delta)$ is a complex geodesic curve at $f(\zeta)$ in $B$ for all $\zeta \in \Delta$.

**Proof.** For $\zeta = e^{i\theta}$ ($\theta \in \mathbb{R}$), $1 + \zeta^2 = 2 \cos \theta e^{i\theta}$, and therefore

\[ f(e^{i\theta}) = e^{i\theta}(\cos \theta \cdot a + b). \]

Since

\[ \cos \theta \cdot a + b > b - |a| > 0 \quad \text{a.e. on } M, \]

then

\[ \|f(e^{i\theta})\| = \int_M |\cos \theta \cdot a(m) + b(m)| d\mu(m) \]
\[ = \int_M (\cos \theta \cdot a(m) + b(m)) d\mu(m) = 1 \]

for all $\theta \in \mathbb{R}$. Since $f(0) = \frac{1}{2} a$, and $\|\frac{1}{2} a\| < \|b\| = \frac{1}{2}$, by the maximum principle $f(\zeta) \in B$ for every $\zeta \in \Delta$. Let $\gamma: B \to \Delta$ be the holomorphic map defined by $\gamma(x) = [x(m) d\mu(m)]$.

For any $\zeta \in \Delta$, $\gamma(f(\zeta)) = \zeta$. Thus, for all $\zeta_1, \zeta_2$ in $\Delta$,

\[ \omega(\zeta_1, \zeta_2) > \omega(f(\zeta_1), f(\zeta_2)) > \omega(\gamma \circ f(\zeta_1), \gamma \circ f(\zeta_2)) = \omega(\zeta_1, \zeta_2). \quad \text{Q.E.D.} \]

To obtain a complex geodesic curve of the above type at $x_0$ in $B$ we determine now $a$ and $b$ in such a way that $x_0 = f(\zeta_0)$ for some $\zeta_0 \in \Delta$. Since

\[ \|x_0\| = \int_M x_0(m) d\mu(m) = \gamma(x_0), \]

we must choose $\zeta_0 = \gamma(x_0) = \|x_0\|$, so that the vectors $a$ and $b$ are then related by

\[ a = \frac{2}{1 + \|x_0\|^2} (x_0 - \|x_0\| b). \]

Thus we choose any real $b \in \mathbb{E}$ such that the first two conditions (4.2) are fulfilled, and these are readily seen to be equivalent to

\begin{equation}
(4.4) 
\int_M b(m) d\mu(m) = 1, \quad \frac{2}{(1 + \|x_0\|^2)} x_0 < b \quad \text{a.e. on } M.
\end{equation}
The corresponding function expressed by (4.3), which will now be denoted by \( f_b \), is given by

\[
f_b(\xi) = \frac{1 + \xi^2}{1 + \|x_0\|^2} (x_0 - \|x_0\|b) + \xi b.
\]

Composing \( f_b \) with the Moebius transformation \( \xi \mapsto (\xi + \|x_0\|)/(1 + \|x_0\|\xi) \), we define the same complex geodesic curve by a new holomorphic function \( \Delta \to B \) satisfying conditions 1) and 2) and mapping 0 into \( x_0 \). This holomorphic function is expressed in terms of the real vector

\[
v = \frac{1 - \|x_0\|^2}{1 + \|x_0\|^2} \left( 2 \|x_0\| x_0 + (1 - \|x_0\|^2) b \right).
\]

In fact, let \( g_v : \Delta \to B \) be the function

\[
g_v(\xi) = f_v \left( \frac{\xi + \|x_0\|}{1 + \|x_0\|\xi} \right) \quad (\xi \in \Delta).
\]

Then \( g_v \) satisfies conditions 1) and 2), is such that \( g_v(0) = x_0 \), and has the power series expansion

\[
g_v(\xi) = x_0 + \xi \left\{ v + \sum_{n=0}^{+\infty} (-1)^n \|x_0\|^n \xi^{n+1} \left[ (n+1)(1-\|x_0\|^2) x_0 - (n+2)\|x_0\|^2 \right] \right\},
\]

(\( \xi \in \Delta \)).

Let \( V \) be the convex set

\[
V = \left\{ v \in \mathcal{E} : v \text{ real}, \quad \int_M v(m) d\mu(m) = 1 - \|x_0\|^4, \quad v(m) > 2(1 - \|x_0\|^2) x_0(m) \quad \text{a.e. on } M \right\}.
\]

Lemma 4.4 can be rephrased in terms of \( v \) as follows:

**Lemma 4.5.** For every \( v \in V \) the holomorphic map \( g_v : \Delta \to B \) defines a complex geodesic curve at \( g_v(\xi) \) in \( B \), for every \( \xi \in \Delta \). Moreover, \( g_v(0) = x_0 \).

In order to describe another family of complex geodesic curves at \( x_0 \) in \( B \), we shall consider the measure space \((\hat{M}, \mathcal{E}, \hat{\mu})\), where: \( \hat{M} = M \setminus K \), \( \mathcal{E} \) is the \( \sigma \)-algebra consisting of the intersections \( S \cap \hat{M} \) (\( S \in \mathcal{E} \)) and \( \hat{\mu} \) is the restriction of \( \mu \) to \( \mathcal{E} \). Let \( \hat{\mathcal{E}} = L^1(\hat{M}, \hat{\mu}) \) and let \( \hat{B} \) be the open unit ball in \( \hat{\mathcal{E}} \). Denoting by \( \lambda \) the continuous linear form on \( \mathcal{E} \),

\[
\lambda : x \mapsto \int_K x(m) d\mu(m),
\]
let $\alpha: B \to \mathbb{C}$ be the holomorphic map defined by
\[
\alpha(x(m)) = \frac{1}{1 - \lambda(x)} \cdot x(m) \quad (x \in B, \ m \in \mathcal{M}).
\]

Denoting by $\|\cdot\|$ the norm in $\mathbb{C}$, for any $x \in B$,
\[
\|\alpha x\| \leq \frac{1}{1 - \lambda(x)} \left( \int_{\mathcal{M}} |x(m)| \, d\mu(m) - \int_{K} |x(m)| \, d\mu(m) \right) < 1,
\]
i.e. $\alpha(B) \subset \bar{B}$.

Let $\beta: B \to \mathbb{C}$ be the map defined by:
\[
\begin{align*}
\beta(x(m)) &= (1 - \|x(m)\|) \cdot x(m) = (1 - \lambda(m)) \cdot x(m) \quad \text{if } m \in \mathcal{M} \setminus K, \\
\beta(m) &= x(m) \quad \text{if } m \in K.
\end{align*}
\]

First of all, for any $\tilde{x} \in \bar{B}$,
\[
\|\beta \tilde{x}\| = \int_{\mathcal{M} \setminus K} |\beta \tilde{x}(m)| \, d\mu(m) + \int_{K} |\beta \tilde{x}(m)| \, d\mu(m)
\]
\[
= (1 - \lambda(x)) \cdot \|\tilde{x}\| + \lambda(x) < 1
\]
i.e.
\[
\beta \subset B.
\]

Next we prove that $\beta$ is a holomorphic map. That amounts to showing [6, 13] that for every $\tilde{x} \in \tilde{B}$, $\tilde{y} \in \tilde{B} \setminus \{0\}$, $z \in L^2(M, \mathcal{E}, \mu)$, the scalar valued function on $\mathbb{C}$
\[
\varphi: \zeta \mapsto \int_{\mathcal{M}} \beta(\tilde{x} + \zeta \tilde{y})(m)z(m) \, d\mu(m)
\]
is holomorphic. We will prove this fact by applying Morera's theorem, i.e. by showing that, for any closed rectifiable curve $l$ in $\mathbb{C}$,
\[
\int_{l} \varphi(\zeta) \, d\zeta = 0.
\]

Indeed, by Fubini's theorem,
\[
\begin{align*}
\int_{l} \varphi(\zeta) \, d\zeta &= \int_{l} \left( (1 - \lambda(x)) \int_{\mathcal{M} \setminus K} (\tilde{x} + \zeta \tilde{y})(m)z(m) \, d\mu(m) + \int_{K} x_0(m)z(m) \, d\mu(m) \right) \, d\zeta \\
&= (1 - \lambda(x)) \int_{M \setminus K} \left( \int_{l} (\tilde{x} + \zeta \tilde{y})(m)z(m) \, d\zeta \right) \, d\mu(m) + \int_{K} x_0(m)z(m) \, d\mu(m) \int_{l} \, d\zeta \\
&= (1 - \lambda(x)) \cdot \int_{M \setminus K} \tilde{y}(m)z(m) \, d\mu(m) \cdot \int_{l} \zeta \, d\zeta = 0.
\end{align*}
\]
Thus $\beta$ is holomorphic. Finally for all $\tilde{x} \in \tilde{B}$, $\tilde{m} \in M$, we have
\[
((\alpha \circ \beta)\tilde{x})(\tilde{m}) = \frac{1}{1 - \lambda(\beta\tilde{x})} \lambda(\tilde{x})(\tilde{m}) = \frac{1 - \lambda(x)}{1 - \lambda(x)} \tilde{x}(\tilde{m}) = \tilde{x}(\tilde{m}),
\]
i.e.
\[
\alpha \circ \beta = \text{identity on } \tilde{B}.
\]

Consider now the Carathéodory pseudo-distances $c_B$ and $c_B^*$. For $\tilde{x}_1$, $\tilde{x}_2$ in $\tilde{B}$ we have
\[
c_B^*(\tilde{x}_1, \tilde{x}_2) > c_B(\beta\tilde{x}_1, \beta\tilde{x}_2) > c_B((\alpha \circ \beta)\tilde{x}_1, (\alpha \circ \beta)\tilde{x}_2) = c_B(\tilde{x}_1, \tilde{x}_2);
\]
hence,
\[
(4.8) \quad c_B^*(\tilde{x}_1, \tilde{x}_2) = c_B(\beta\tilde{x}_1, \beta\tilde{x}_2) \quad \text{for all } \tilde{x}_1, \tilde{x}_2 \in \tilde{B}.
\]

**Lemma 4.6.** Let $w \in \mathcal{E}$ be such that $w \not= 0$, but $w = 0$ a.e. on $K$. Then the holomorphic map of $\Lambda$ into $\tilde{B}$,
\[
\xi \mapsto x_0 + \frac{1 - \|x_0\|}{\|w\|} \xi w
\]
defines a complex geodesic curve at $x_0$ in $\tilde{B}$.

**Proof.** If $\tilde{w}$ is the restriction of $w$ to $\tilde{M}$, for every $\xi \in \Lambda$,
\[
x_0 + \frac{1 - \|x_0\|}{\|w\|} \xi w = \beta \left( \frac{\xi}{\|w\|} \tilde{w} \right).
\]
The lemma follows then from (4.8) and from Lemma 4.2. Q.E.D.

So far we have constructed two special families of complex geodesic curves at $x_0$ in $\tilde{B}$. On the other hand, the existence of the holomorphic automorphism $F$, mapping 0 into $x_0$, coupled with Lemma 4.3, yields a complete description of all the complex geodesic curves at $x_0$ in $B$. In fact, denoting by $dF(0)$ the differential of $F$ at 0, the following statement is a consequence of Lemma 4.3.

**Lemma 4.7.** Let $y \in \mathcal{E} \setminus \{0\}$, and let
\[
u_y = \frac{1}{\|dF(0)^{-1}y\|} dF(0)^{-1}y.
\]
For any $0 \in \mathbb{R}$, the holomorphic map $h_\theta: \Delta \to B$ expressed by

$$h_\theta(\zeta) = F(\zeta e^{i\theta} u_\zeta),$$

defines a complex geodesic curve at $x_\theta$ in $B$. Moreover, if $h: \Delta \to B$ is a holomorphic map such that

$$h(0) = x_0; \\
h(\Delta) \text{ is a complex geodesic curve at } x_0 \text{ in } B; \\
dh(0) = c\gamma \text{ for some } 0 \neq c \in \mathbb{C},$$

then $h = h_\theta$ for a suitable $\theta \in \mathbb{R}$.

We come now to the proof of Theorem II (*). Among the vectors $b$ satisfying (4.4) we choose two real vectors $b'$ and $b''$ such that

$$b'(m) = \frac{3x_\theta(m)}{(1 + \|x_\theta\|^2)z_\theta}, \text{ for } m \in \mathbb{K}, \ b'>0 \text{ on } \mathbb{M} \backslash \mathbb{K}, \ \int_M b'(m) d\mu(m) = 1,$$

$$b''(m) = \frac{4x_\theta(m)}{(1 + \|x_\theta\|^2)}, \text{ for } m \in \mathbb{K}, \ b''>0 \text{ on } \mathbb{M} \backslash \mathbb{K}, \ \int_M b''(m) d\mu(m) = 1.$$

The vectors

$$v' = \frac{1 - \|x_\theta\|^2}{1 + \|x_\theta\|^2}(2\|x_\theta\|x_\theta + (1 - \|x_\theta\|^2)b'),$$

$$v'' = \frac{1 - \|x_\theta\|^2}{1 + \|x_\theta\|^2}(2\|x_\theta\|x_\theta + (1 - \|x_\theta\|^2)b''),$$

belong to the convex set $V$ defined by (4.7). Since for $m \in \mathbb{K}$

$$v'(m) = \frac{1 - \|x_\theta\|^2}{1 + \|x_\theta\|^2}(3 - \|x_\theta\|^2)x_\theta(m),$$

$$v''(m) = 2 \frac{1 - \|x_\theta\|^2}{1 + \|x_\theta\|^2}(2 - \|x_\theta\|^2)x_\theta(m),$$

then, for $0 < t < 1$,

$$\lambda(tv' + (1 - t)v'')(m) = \int_K (tv' + (1 - t)v'')(m) d\mu(m) =$$

$$= \frac{1 - \|x_\theta\|^2}{1 + \|x_\theta\|^2}(4 - t - (2 - t)\|x_\theta\|^2 + 2\|x_\theta\|^2)\|x_\theta\|.$$

(*) Cf. the Note added in proof at the end of this paper.
Let \( \tau \) be the continuous linear form on \( E \)
\[
\tau(x) = \int_{E \setminus K} x(m) d\mu(m) = \int_{E} x(m) d\mu(m) - \lambda(x).
\]

Since \( tv' + (1-t)v'' \in V \) for \( 0 < t < 1 \), then
\[
\tau(tv' + (1-t)v'') = 1 - \|x_0\|^2 - \frac{1}{1 + \|x_0\|^2} (4 - t - (2-t)\|x_0\| + 2\|x_0\|^2)\|x_0\| = \frac{1 - \|x_0\|^2}{1 + \|x_0\|^2} (1 + (t-3)\|x_0\| + (3-t)\|x_0\|^2 - \|x_0\|^4).
\]

Let \( \mathcal{K} \) be the two dimensional complex subspace of \( E \) spanned by \( v' \) and \( v'' \). Since the restrictions of \( v' \) and \( v'' \) to \( K \) are linearly dependent, while \( v' \) and \( v'' \) are not, \( \mathcal{K} \) contains a vector \( w \neq 0 \) such that \( w = 0 \) a.e. on \( K \).

Consider the holomorphic map \( B \cap (dF(0)^{-1}\mathcal{K}) \to \mathbb{C}^2 \) defined by
\[
x \mapsto (\lambda \circ F(x), \tau \circ F(x)) \quad (x \in B \cap (dF(0)^{-1}\mathcal{K})).
\]

For \( u \in dF(0)^{-1}\mathcal{K}, \|u\| = 1 \), consider the power series expansions
\[
\lambda \circ F(\zeta u) = \|x_0\| + p_1(u)\zeta + p_2(u)\zeta^2 + \ldots,
\]
\[
\tau \circ F(\zeta u) = q_2(u)\zeta + q_3(u)\zeta^2 + \ldots,
\]
where \( p_r \) and \( q_r \) are homogeneous polynomials of degree \( r = 1, 2, \ldots \), on the two dimensional complex space \( dF(0)^{-1}\mathcal{K} \) and \( \zeta \in \Lambda \). Taking \( dF(0)u = v' \) or \( v'' \) and comparing with (4.6) we see that \( p_1 \neq 0 \). Let \( u_0 = 1/\|dF(0)^{-1}w\| \cdot dF(0)^{-1}w \). By Lemma 4.6 and 4.7,
\[
p_r(u_0) = 0 \quad \text{for} \ r > 1, \quad q_r(u_0) = 0 \quad \text{for} \ r > 2.
\]

Hence there exist homogeneous polynomials \( r_r \) and \( s_r \) of degree \( r = 1, 2, \ldots \), on \( dF(0)^{-1}\mathcal{K} \), such that
\[
p_r = p_r r_{r-1}, \quad q_r = p_r s_{r-1} \quad \text{for} \ r = 2, \ldots.
\]

Choose now any \( v \in \mathcal{K} \cap V \) and let \( u = dF(0)^{-1}v \). Then for \( \zeta \in \Lambda \),
\[
\lambda \circ F(\zeta u) = \|x_0\| + \zeta \lambda(v) + \sum_{n=0}^{\infty} (-1)^n \zeta^{n+2} \|x_0\|^n ((n+1)(1 - \|x_0\|^2)\|x_0\| - (n+2)\|x_0\|^2)\lambda(v),
\]
\[
\tau \circ F(\zeta u) = \zeta \tau(v) + \sum_{n=0}^{\infty} (-1)^{n+1} \|x_0\|^{n+1} \zeta^{n+2} (n + 2)\tau(v).
\]
Thus we must have \( p_1(u) = \lambda(v) \), \( q_1(u) = \tau(v) \), \( q_2(u) = -2\|x_0\|^2\tau(v) \), and therefore

\[
\lambda(v)\tau(v) = -2\|x_0\|^2\tau(v).
\]

Thus \( \tau(v)/\lambda(v) \) should depend linearly on \( t \), for \( v = tv' + (1 - t)v'' \) \((0 < t < 1)\), i.e. we should have

\[
\frac{1 + (t - 3)\|x_0\|^2 + (3 - t)\|x_0\|^2 - \|x_0\|^2}{4 - t + (t - 2)\|x_0\|^2 + 2\|x_0\|^2} = t \frac{1 - 2\|x_0\|^2 + 2\|x_0\|^2 - \|x_0\|^2}{3 - \|x_0\|^2 + 2\|x_0\|^2} + (1 - t) \frac{1 - 3\|x_0\|^2 + 3\|x_0\|^2 - \|x_0\|^2}{4 - 2\|x_0\|^2 + 2\|x_0\|^2}.
\]

But this is absurd, and this contradiction proves the theorem.

**Examples.** 1) Let \( G \) be a locally compact topological group (containing more than one element), let \( \mu \) be a left-invariant Haar measure on \( G \), and let \( B \) be the open unit ball of \( L^1(G, \mu) \). By Theorem II, every holomorphic automorphism \( F \) of \( B \) is a linear isometry. A theorem of J. G. Wendel [25] supplies a complete description of the isometric isomorphisms of \( L^1(G, \mu) \). According to this theorem, for every isometric isomorphism \( F' \) of \( L^1(G, \mu) \) onto itself, there exists a complex constant \( \beta \) with \( |\beta| = 1 \), a bi-continuous automorphism \( \gamma \) of \( G \) and a continuous character \( \chi \) of \( G \) such that,

\[
F(x)(\gamma g) = \beta \chi(g)x(g)
\]

for all \( g \in G \) and all \( x \in L^1(G, \mu) \).

2) Suppose that \( M \) consists of two points, \( m_1, m_2 \), and let \( \mu(m_1) = \mu(m_2) = 1 \). Then \( \mathcal{E} = L^1(M, \mu) \) can be identified with \( \mathbb{C}^2 \), and the unit ball of \( \mathcal{E} \) is

\[
B = \{ (\zeta_1, \zeta_2) \in \mathbb{C}^2 : |\zeta_1| + |\zeta_2| < 1 \}.
\]

In this case Theorem II was proved by N. Kritikos in [11], as one oft he first applications of the notion of Carathéodory's distance (*). His proof—which inspired ours—consisted in examining the Carathéodory metric

(* A different proof was given by Kritikos in [10], without appealing to the Carathéodory distance, but under the additional hypothesis that any automorphism of \( B \) could be extended to a holomorphic map of a neighborhood of \( B \) into \( \mathbb{C}^2 \). The proof consisted then in examining the behavior of this extension on the boundary of \( B \). Recent results by W. Kaup and H. Upmeier [8] show that every automorphism of \( B \) can be so extended, so that the additional hypothesis turns out to be automatically satisfied.
in neighborhoods of different points. However, the lack of a strong maximum principle prevented Kritikos from proving the uniqueness part of Lemma 4.3. Instead, the burden of the proof lay in a complicated analysis of the $2 \times 2$ matrix representing $dF(0)$. This result of Kritikos was re-obtained and generalized by P. Thullen in his classical article [20], in which he gives a complete classification of bounded Reinhardt domains in $\mathbb{C}^2$, containing the origin, in terms of their group of automorphisms. (For higher dimensional generalizations of some of Thullen's results and for the relevant bibliographical references cf. [18].)

3. - Spectral versions of the Schwarz lemma.

5. We shall now discuss some spectral versions of the classical Schwarz lemma. Let $\mathcal{A}$ and $\mathcal{A}'$ be complex Banach algebras; let $\varrho$ and $\varrho'$ be their spectral radii, and let

$$C = \{x \in \mathcal{A} : \varrho(x) < 1\}, \quad C' = \{x' \in \mathcal{A}' : \varrho'(x') < 1\}.$$ 

By the upper semi-continuity of the spectrum [17, p. 37], $C$ and $C'$ are open in $\mathcal{A}$ and $\mathcal{A}'$.

For every $x \in \mathcal{A}$ (or in $\mathcal{A}'$) we denote by $Sp x$ the spectrum of $x$, and by $P(x)$ the peripheral spectrum of $x$: $P(x) = \{\zeta \in Sp x : |\zeta| = \varrho(x)\}$.

**Proposition 5.1.** Let $f : C \to \mathcal{A}'$ be a holomorphic map such that $f(C) \subseteq C'$ (the closure of $C'$) and $f(0) = 0$. Then

$$\varrho'(f(x)) < \varrho(x) \quad \text{for all} \ x \in C.$$ 

If equality holds at some point $x \in C$, $x \neq 0$, then

$$\varrho'(f(\zeta x)) = \varrho(\zeta x) \quad \text{for all} \ \zeta \in C \ \text{with} \ |\zeta| < \frac{1}{\varrho(x)};$$

moreover the peripheral spectrum $P(f(\zeta x))$ of $f(\zeta x)$ is

$$P(f(\zeta x)) = |\zeta|P(f(x)) \quad \text{for all} \ \zeta \in C \ \text{with} \ |\zeta| < \frac{1}{\varrho(x)}.$$ 

**Proof.** Let $y \in \mathcal{A}$ with $0 < \varrho(y) < 1$. The function $\varrho_+: \zeta \mapsto (1/\varrho(y)) \cdot f(\zeta y)$ is a holomorphic map of the disc $A_{1/\varrho(y)}$ of radius $1/\varrho(y)$ in $C$ into $\mathcal{A}'$. Thus, by Theorem 1 of [21], the function $\varrho' \varrho_+: \zeta \mapsto \varrho'(\varrho_+(\zeta))$ is subharmonic.
on $A_{1/q(y)}$. Choosing $0 < r < 1/q(y)$, for $|\zeta| = r$, we have

$$q'(q_r(\zeta)) \leq \frac{1}{r q(y)}.$$

By the maximum principle this inequality holds for $|\zeta| < r$. Letting $r \neq 1/q(y)$, we obtain

(5.4) \hspace{1cm} q'(q_r(\zeta)) \leq 1 \quad \text{for all } |\zeta| < \frac{1}{q(y)}.

Let $x \in C$. If $q(x) > 0$, we choose a real $t > 1$ such that, for $y = tx$, $q(y) = t q(x) < 1$. Being $1/t < 1 < 1/q(y)$, for $\zeta = 1/t$ (5.4) yields (5.1). If $q(x) = 0$, then $\zeta x \in C$ for all $\zeta \in C$. The subharmonic function $\zeta \mapsto q'(f(\zeta x))$ is bounded by 1 on $C$, and therefore [22, Corollary 2.14] is constant. Being $q'(f(0)) = 0$, then $q'(f(\zeta x)) = 0$ for all $\zeta \in C$. This completes the proof of (5.1).

Suppose that equality holds in (5.1) at some $x \in C$ with $q(x) > 0$. Choosing as above a real $t > 1$ such that $y = tx \in C$, the function $q' \circ q_y$ attains its maximum, 1, at the point $1/t \in A_{1/q(y)}$. By the maximum principle, equality holds in (5.4) on $A_{1/q(y)}$. That proves (5.2).

According to [21, Proposition 2], if a holomorphic map $q$ of a domain $D \subset C$ into $\mathcal{A}'$ is such that $q' \circ q$ is constant on $D$, then the peripheral spectrum $P(q(\zeta))$ is independent of $\zeta \in D$. Hence, if equality holds in (5.1) at some point $x \in C$, $x \neq 0$, there exists a non-empty, compact subset $K$ of the unit circle, such that

$$P(q_r(\zeta)) = K \quad \text{for all } \zeta \in C, \text{ with } |\zeta| < \frac{1}{q(y)},$$

where $y = tx$ and $t > 1$, are chosen as above. Hence

$$P(f(\zeta y)) = q(\zeta y)K \quad \left(|\zeta| < \frac{1}{q(y)}\right).$$

For $\zeta = 1/t$, $P(f(x)) = q(x)K$, and (5.3) follows. Q.E.D.

For any $x \in C$, let

$$\tau(x) = \sup \{\omega(0, \zeta); \zeta \in \text{Sp } x\}.$$

We call $\tau(x)$ the hyperbolic spectral radius of $x$. Since the geodesic line, for the Poincaré-Bergman metric, from 0 to $\zeta \in A$ is the line-segment joining these two points, whose hyperbolic length is

$$\omega(0, \zeta) = \frac{1}{2} \log \frac{1 + |\zeta|}{1 - |\zeta|},$$
then
\[ \tau(x) = \frac{1}{2} \log \frac{1 + \varrho(x)}{1 - \varrho(x)} = \omega(0, \varrho(x)) \quad (x \in C). \]

The function \( t \mapsto \log \left( \frac{(1 + t)/(1 - t)}{1 - \varrho(x)} \right) \) being strictly increasing on \([0,1)\) then we obtain from Proposition 5.1, the following statement, where \( \tau' \) denotes the hyperbolic spectral radius on \( C' \).

**Proposition 5.2.** If \( f \) is as in Proposition 5.1, then
\[ \tau'(f(x)) < \tau(x) \quad \text{for all} \quad x \in C. \]

If equality holds at some \( x \in C, x \neq 0 \), then
\[ \tau'(f(\xi x)) = \tau(\xi x) \quad \text{for all} \quad \xi \in C \quad \text{for which} \quad |\xi| < \frac{1}{\varrho(x)}; \]
moreover (5.2) and (5.3) hold.

Since \( C \) and \( C' \) are not necessarily homogeneous, condition \( f(0) = 0 \) cannot be released, in general. However, if \( A' = C \), a similar argument to the classical proof of the Schwarz-Pick lemma implies the following

**Lemma 5.3.** Let \( f : C \to \Delta \) be a holomorphic map. Then
\[ \omega(f(x), f(0)) < \tau(x) \]
for all \( x \in C \). If equality holds at some \( x \in C, x \neq 0 \), then
\[ \omega(f(\xi x), f(0)) = \tau(\xi x) \quad \text{for all} \quad \xi \in C \quad \text{for which} \quad |\xi| < \frac{1}{\varrho(x)}. \]

Let \( d_c \) and \( \sigma_c \) be the Kobayashi and Carathéodory pseudodistances on \( C \). For \( x \in C \), with \( \varrho(x) > 0 \), consider the map \( f : \Delta \to C \) defined by \( f(\zeta) = (\zeta/\varrho(x)) x \). Since \( f(0) = 0, f(\varrho(x)) = x \), then, by (5.5),
\[ d_c(0, x) < \omega(0, \varrho(x)) = \tau(x). \]

If \( x \neq 0 \), but \( \varrho(x) = 0 \), then for every \( a \in \Delta \setminus \{0\} \) the function \( f: \xi \mapsto (\xi/a)x \) maps \( \Delta \) into \( C \); moreover \( f(0) = 0 \), and \( f(a) = x \). Hence
\[ d_c(0, x) < \omega(0, a) \]
and letting \( a \to 0 \), we obtain \( d_c(0, x) \). Thus (5.6) holds for every \( x \in C \),
and therefore
\[ c_d(0, x) < d_c(0, x) < \tau(x) \quad \text{for all } x \in C. \]

Thus, if \( \mathcal{A} \) contains non-trivial topologically nilpotent elements, both \( d_c \) and \( c_d \) are (pseudo-distances but) not distances on \( C \).

Since the function \( x \mapsto \rho(x) \) is not always continuous on \( C \) (cf. e.g. [17, pp. 282-283]), while \( d_c \) is continuous, then (5.7) is not always an equality. However, this is the case if \( \mathcal{A} \) is commutative.

**Lemma 5.3.** If \( \mathcal{A} \) is a commutative Banach algebra, then
\[ c_d(0, x) = d_c(0, x) = \tau(x) \quad \text{for every } x \in C. \]

**Proof.** Since \( \mathcal{A} \) is commutative, \( \rho \) is a continuous semi-norm on \( \mathcal{A} \). By the Hahn-Banach theorem, for any \( x \in C \) there is a continuous linear form \( \lambda \) on \( \mathcal{A} \) such that
\[ \lambda(x) = \rho(x), \quad |\lambda(y)| < \rho(y) \quad \text{for all } y \in \mathcal{A}. \]

Hence \( \lambda \) is a holomorphic map of \( C \) into \( \mathcal{A} \), and therefore \( \tau(x) = \omega(0, \rho(x)) = \omega(0, \lambda(x)) < c_d(0, x) \). Comparison with (5.7) yields the conclusion. Q.E.D.

Let \( D \) be a domain in \( C \), and let \( f \) be a holomorphic mapping of \( D \) into \( C \). By Theorem I and Lemma 5.3, if \( \mathcal{A} \) is commutative the function \( \log \tau \circ f \) is subharmonic on \( D \). We will now prove this fact for every Banach algebra \( \mathcal{A} \), thereby extending to the hyperbolic spectral radius Theorem 1' of [21].

**Proposition 5.4.** The function \( \zeta \mapsto \log \tau(f(\zeta)) \) is subharmonic on \( D \).

**Proof:** Since \( \rho \circ f \) is upper semi-continuous on \( D \), we need only show that, for every \( a \in C \), the function
\[ \varphi_a : \zeta \mapsto |e^{a \zeta}| \tau(f(\zeta)) \quad (\zeta \in D) \]
is subharmonic on \( D[16] \). Since \( \tau \circ f \) is upper semi-continuous, \( \varphi_a \) is upper semi-continuous too. Moreover, by (5.5) \( \varphi_a \) has a power series expansion, converging at every \( \zeta \in D \),
\[ \varphi_a(\zeta) = |e^{a \zeta}| \sum_{n=0}^{+\infty} \frac{\tau(f(\zeta))}{2n+1} = \sum_{n=0}^{+\infty} \frac{1}{2n+1} \left( e^{a \zeta/(2n+1)} f(\zeta) \right)^{2n+1}. \]

Since \( \zeta \mapsto e^{a \zeta/(2n+1)} f(\zeta) \) is a holomorphic map of \( D \) into \( \mathcal{A} \), then \( e^{a \zeta/(2n+1)} f(\zeta) \) is a subharmonic function of \( \zeta \in D \) for \( n = 1, 2, \ldots, [21] \), and
therefore also the function

\[ \zeta \mapsto \left( e^{\pi \xi (2n+1) f(\xi)} \right)^{2n+1} \]

is subharmonic on \( D \). Hence \( \varphi_a \) is the pointwise limit of an increasing sequence of subharmonic functions. Since \( \varphi_a \) is upper semi-continuous and \( \varphi_a(\zeta) < +\infty \) at every \( \zeta \in D \), then \( \varphi_a \) is subharmonic. Q.E.D.

6. Let \( \mathcal{A} \) be a complex Banach algebra with an identity \( e \), endowed with an involution \( * \). Let \( \mathcal{K}(\mathcal{A}) \) be the real linear subvariety consisting of all hermitian elements of \( \mathcal{A} \). We shall assume throughout the following that the involution is hermitian (i.e. that the spectrum of any hermitian element belongs to \( \mathbb{R} \)). No further hypothesis will be made on the involution. In particular we will not require \( * \) to be continuous, or equivalently, we will not require \( \mathcal{K}(\mathcal{A}) \) to be closed in \( \mathcal{A} \).

Let \( p: \mathcal{A} \to \mathbb{R}_+ \) be the function defined by

\[ p(x) = e(x^*x)^{\frac{1}{2}}. \]

We collect now a few known facts, that will be useful in the following.

I) \( p \) is a seminorm on \( \mathcal{A} \) which is submultiplicative, i.e. \( p(xy) \leq p(x)p(y) \) for all \( x, y \in \mathcal{A} \) [14; 15; 5];

II) \( e(x) < p(x) \) for all \( x \in \mathcal{A} \) [14; 15; 5];

III) \( p \) is continuous, i.e. there is a constant \( k > 0 \) such that \( p(x) < k\|x\| \) for all \( x \in \mathcal{A} \) [15, (8.2), p. 32].

Let \( Q_0 \) be the set of positive elements of \( \mathcal{K}(\mathcal{A}) \), that is

\[ Q_0 = \{ x \in \mathcal{K}(\mathcal{A}) : \text{Sp} x \in \mathbb{R}_+ \}. \]

IV) If \( x_1, x_2 \in Q_0 \), then \( x_1 + x_2 \in Q_0 \) [15, (5.6), p. 24].

By IV), \( Q_0 \) is a convex cone in \( \mathcal{K}(\mathcal{A}) \). Let \( \Omega \) be the interior part of \( Q_0 \) for the topology in \( \mathcal{K}(\mathcal{A}) \). If \( x \in Q_0 \) and if \( 0 \in \text{Sp} x \), then \( x - (1/n)e \notin Q_0 \) for \( n = 1, 2, \ldots \). Since \( x - (1/n)e \) tends to \( x \) as \( n \to +\infty \), then \( x \notin Q_0 \). Conversely, if \( \text{Sp} x \subset \mathbb{R}_+ = \{ t \in \mathbb{R} : t > 0 \} \), then, by the upper semi-continuity of the function \( x \mapsto \text{Sp} x \) [17, p. 35], there is a neighborhood of \( x \) in \( \mathcal{K}(\mathcal{A}) \) all of whose points have their spectra in \( \mathbb{R}_+ \). In conclusion

\[ \Omega = \{ x \in \mathcal{A} : \text{Sp} x \subset \mathbb{R}_+ \}. \]
V) If $x \in \Omega_0 (\Omega)$, there is an element $v \in \Omega_0 (\Omega)$ such that $v$ commutes with $x$, and $v^* = x$ ([3], [15, (1.5), p. 7]). If $x \in \Omega$, then $v$ is invertible and therefore $v \in \Omega$. We shall call such a $v$ a square root of $x$, and we shall denote it by $x^*$. Every $z \in \mathcal{A}$ can be written in a unique way as

$$z = x + iy,$$

where $x = \frac{1}{2}(x + x^*)$, $y = (1/2i)(x - x^*)$ both belong to $\mathcal{H}(\mathcal{A})$. Let

$$D(\Omega) = \left\{ z \in \mathcal{A} : \frac{1}{2i} (x - x^*) \in \Omega \right\} = \{ z = x + iy : x \in \mathcal{H}(\mathcal{A}), y \in \Omega \}.$$

Since $\Omega$ is convex, $D(\Omega)$ is convex too, hence connected. We shall prove that $D(\Omega)$ is an open homogeneous domain, biholomorphically equivalent to the open unit ball $B_1$:

$$B_1 = \{ w \in \mathcal{A} : p(w) < 1 \}.$$

For any $w \in B_1$, $p(w) < p(w) < 1$, hence $1 \notin \text{Sp } w$. Let $U_1 = \{ w \in \mathcal{A} : 1 \notin \text{Sp } w \}$. Then $B_1 \subset U_1$. By the upper-semicontinuity of the function $w \mapsto \text{Sp } w [17, p. 35]$, $U_1$ is open in $\mathcal{A}$. Let $\zeta : U_1 \to \mathcal{A}$ be the holomorphic map defined by

$$(6.1) \quad \zeta(w) = i(e + w)(e - w)^{-1} \quad (w \in U_1).$$

Since $e + w$ and $e - w$ commute,

$$(6.2) \quad \zeta(w) = i(e - w)^{-1}(e + w).$$

Let $w \in B_1$. Then

$$\zeta_0(w) - \zeta_0(w)^* = i[(e + w)(e - w)^{-1} + (e - w^*)(e + w^*)^{-1} - (e + w^*)(e - w^*)]$$
$$= i(e - w^*)(e + w + (e + w^*)(e - w))(e - w)^{-1}$$
$$= 2i(e - w^*)^{-1}(e - w^* w)(e - w).$$

Being $\rho(w^* w) < 1$, then $\text{Sp } (e - w^* w) \subset (0, 1]$. Let $v \in \Omega$ be a square root of $e - w^* w$. Then

$$\zeta_0(w) - \zeta_0(w)^* = 2i(v(e - w)^{-1})(v(e - w)^{-1}).$$
Taking into account the fact that $v$ is invertible, we see that
\[ \frac{1}{2i} (\bar{\mathcal{C}}_0(w) - \mathcal{C}_0(w)^*) \in \Omega , \]
i.e.
\[ \mathcal{C}_0(B_\rho) \subset D(\Omega) . \]

For $w \in U_1$, (6.1) and (6.2) yield
\[ \mathcal{C}_0(w + ie) = (\mathcal{C}_0(w) + ie)w = \mathcal{C}_0(w) - ie . \]

**Lemma 6.1.** If $z \in D(\Omega)$, then $z$ is invertible.

**Proof.** Let $z = x + iy$, with $x \in \mathcal{K}(\mathcal{A})$, $y \in \Omega$; let $y^1 \in \Omega$ be a square root of $y$, and let $y^{-1} = (y^1)^{-1}$. Then $y^{-1} \in \Omega$, and $z$ can be represented as
\[ z = x + iy = iy^1(e - iy^{-1}xy^{-1})y^1 . \]

Since $y^{-1}xy^{-1}$ is hermitian, then
\[ \text{Sp} (e - iy^{-1}xy^{-1}) \subset \{ 1 - it : t \in \mathbb{R} \} , \]
showing that $e - iy^{-1}xy^{-1}$ is invertible. Q.E.D.

Let $z = x + iy$ with $x \in \mathcal{K}(\mathcal{A})$, $y \in \Omega_0$. Then $z + ie \in D(\Omega)$. By Lemma 6.1, $z + ie$ is invertible, i.e., $-i \notin \text{Sp } z$.

Let $U_{-i} = \{ z \in \mathcal{A} : -i \notin \text{Sp } z \}$. Then $D(\Omega) \subset U_{-i}$. By the upper semi-continuity of the function $z \mapsto \text{Sp } z$, $U_{-i}$ is open in $\mathcal{A}$. Let $\mathcal{G}_i : U_{-i} \to \mathcal{A}$ be the holomorphic map defined by
\[ \mathcal{G}_i(z) = (z - ie)(z + ie)^{-1} \quad (z \in U_{-i}) . \]

Since $z - ie$ and $z + ie$ commute, then $\mathcal{G}_i(z)$ can also be written
\[ \mathcal{G}_i(z) = (z + ie)^{-1}(z - ie) \quad (z \in U_{-i}) . \]

We prove now that
\[ \mathcal{G}_i(D(\Omega)) \subset B_\rho . \]
In fact, let \( z = x + iy \in D(\Omega) \), with \( x \in \mathcal{K}(\mathcal{A}), \ y \in \Omega \). Then

\[
e - \mathcal{C}_1(z)^* \mathcal{C}_1(z) = e - (z^* - ie)^{-1}(z^* + ie)(z - ie)(z + ie)^{-1}
\]

\[
= (z^* - ie)^{-1}[(z^* - ie)(z + ie) - (z^* + ie)(z - ie)](z + ie)^{-1}
\]

\[
= 2i(z^* - ie)^{-1}(z^* - z)(z + ie)^{-1} = 4(z^* - ie)^{-1}y(z + ie)^{-1}
\]

\[
= 4(z^* - ie)^{-1}y^ty^t(z + ie)^{-1}
\]

\[
= 4(y^t(z + ie)^{-1})^*(y^t(z + ie)^{-1})
\]

Thus \( e - \mathcal{C}_1(z)^* \mathcal{C}_1(z) \in \Omega \), and therefore \( \text{Sp}(\mathcal{C}_1(z)^* \mathcal{C}_1(z)) \subset [0, 1) \). In conclusion \( p(\mathcal{C}_1(z)) = \rho(\mathcal{C}_1(z)^* \mathcal{C}_1(z))^t < 1 \), i.e. \( \mathcal{C}_1(z) \in B_\rho \). That proves (6.6).

Comparing (6.1) and (6.5) (or (6.2) and (6.4)) we see that

\[
\mathcal{C}_1 \circ \mathcal{C}_1 \text{ is identity on } B_\rho,
\]

\[
\mathcal{C}_0 \circ \mathcal{C}_1 \text{ is identity on } D(\Omega).
\]

It is readily checked on (6.4) and (6.5) that \( \mathcal{C}_1 \) is injective. By consequence, if \( z \in U_{-i} \) is such that \( \mathcal{C}_1(z) \in B_\rho \), then \( z = \mathcal{C}_0(\mathcal{C}_1(z)) \in D(\Omega) \). That proves that \( D(\Omega) = \mathcal{C}_1^{-1}(B_\rho) \). Since \( \mathcal{C}_1 \) is continuous, and \( B_\rho \) is open, then \( D(\Omega) \) is open.

Denoting by \( \mathcal{C} \) the restriction of \( \mathcal{C}_1 \) to \( D(\Omega) \), the restriction of \( \mathcal{C}_0 \) to \( B_\rho \) is \( \mathcal{C}^{-1} \). Thus the map \( \mathcal{C} : D(\Omega) \to B_\rho \) is a bi-holomorphic diffeomorphism of \( D(\Omega) \) onto \( B_\rho \); \( \mathcal{C} \) will be called the Cayley transform.

We shall prove now that \( D(\Omega) \) is affine-homogeneous. Let \( z = x + iy \in D(\Omega), \ (x \in \mathcal{K}(\mathcal{A}), \ y \in \Omega) \) and let \( F_s : \mathcal{A} \to \mathcal{A} \) be the affine automorphism of the Banach space \( \mathcal{A} \) defined by

\[
(6.7)
F_s(w) = y^{-1}(w - x)y^{-1},
\]

where \( y^i \in \Omega \) is a square root of \( y \), and \( y^{-1} = (y^i)^{-1} \). For \( w = u + iv \) \( (u, v \in \mathcal{K}(\mathcal{A})) \), then

\[
F_s(u + iv) = y^{-1}(u - x)y^{-1} + iy^{-1}v^ty^{-1},
\]

where both \( y^{-1}(u - x)y^{-1} \) and \( y^{-1}v^ty^{-1} \) are hermitian elements. If \( v \in \Omega \), denoting by \( v^i \in \Omega \) a square root of \( v \), we have

\[
y^{-1}v^ty^{-1} = y^{-1}v^iv^ty^{-1} = (v^i y^{-1})^*(v^i y^{-1}).
\]

Since both \( v^i \) and \( y^{-1} \) are invertible, then \( y^{-1}v^ty^{-1} \in \Omega \), i.e. \( F_s(u + iv) \in D(\Omega) \).
Vice versa, let \( y^{1}v^{1}y^{-1} = v' \in \Omega \). If \( v' \in \Omega \) is a square root of \( v \) then
\[
v = y^{1}v^{1}v'^{1}y^{1} = (v'^{1}y^{1})^*(v'^{1}y^{1}),
\]
and therefore \( v \in \Omega \), i.e. \( w = u + iv \in D(\Omega) \). In conclusion, \( F_{z}(w) \in D(\Omega) \)
if, and only if, \( w \in D(\Omega) \). That proves that \( F_{z} \) defines an affine automorphism of \( D(\Omega) \).
Since, for any \( z \in D(\Omega) \), \( F_{z}(z) = iz \), then \( D(\Omega) \) is affine homogeneous.

Summarizing the above results, we state

**Proposition 6.2.** Let \( \mathcal{A} \) be a Banach algebra with unit, endowed with a hermitian involution. The Cayley transform maps the convex domain \( D(\Omega) \) bi-holomorphically onto the domain \( B_{z} \). The domain \( D(\Omega) \) is affine-homogeneous. Thus \( D(\Omega) \) and \( B_{z} \) are homogeneous.

**Lemma 1.1** implies that the Kobayashi and Carathéodory pseudodistances coincide on \( B_{z} \), and therefore also on \( D(\Omega) \):

\[
e_{B_{z}} = d_{B_{z}}, \quad e_{D(\Omega)} = d_{D(\Omega)}.
\]

For \( z_{1}, z_{2} \in D(\Omega) \)
\[
e_{D(\Omega)}(z_{1}, z_{2}) = e_{D(\Omega)}(F_{z_{1}}(z_{1}), F_{z_{1}}(z_{2})) = e_{D(\Omega)}(z_{1}, z_{2}).
\]

Since \( \mathcal{C}(iz) = 0 \), Lemma 1.1 yields
\[
e_{D(\Omega)}(z_{1}, z_{2}) = \omega \left( 0, p \left( \mathcal{C}(F_{z_{1}}(z_{2})) \right) \right).
\]

Let \( z_{2} = x_{2} + iy_{2}, \ z_{2} \in \mathcal{K}(\mathcal{A}), \ y_{2} \in \Omega \). Let \( y_{2}^{1} \in \Omega \) be a square root of \( y_{2} \)
and let \( y_{2}^{-1} = (y_{2}^{1})^{-1} \). Then, by (6.4) and (6.7),
\[
\mathcal{C}(F_{z_{1}}(z_{2})) = (y_{1}^{-1}(z_{2} - x_{1}) y_{1}^{-1} - iz)(y_{1}^{-1}(z_{2} - x_{1}) y_{1}^{-1} - iz)^{-1}
\]
\[
= y_{1}^{-1}(z_{2} - x_{1} - iy_{1}) y_{1}^{-1}(y_{1}^{-1}(z_{2} - x_{1} + iy_{1}) y_{1}^{-1})^{-1}
\]
\[
= y_{1}^{-1}(z_{2} - x_{1})(z_{2} - x_{1}^{*})^{-1} y_{1}^{-1},
\]
and therefore
\[
e_{D(\Omega)}(z_{1}, z_{2}) = d_{D(\Omega)}(z_{1}, z_{2}) = \omega \left( 0, p \left( y_{1}^{-1}(z_{2} - x_{1})(z_{2} - x_{1}^{*})^{-1} y_{1}^{-1} \right) \right),
\]

\[
(z_{1}, z_{2} \in D(\Omega)).
\]

In general \( p \) is only a semi-norm. If it is a norm and if \( \mathcal{A} \) is complete with respect to \( p \), then—\( B_{z} \) being homogeneous—the Carathéodory distance on \( B_{z} \),
is complete [24, théorème 2, p. 279], and therefore also the Carathéodory
distance on $D(\Omega)$ is complete.

For example, if $\mathcal{A}$ is a $C^*$-algebra with identity, then for $z \in \mathcal{A}$,

$$p(z) = \phi(z^* z)^{1/2} = \|z^* z\|^{1/2} = \|z\|.$$  

Hence $B_p$ is the open unit ball $B$ for the norm $\|\cdot\|$. $D(\Omega)$ is biholomorphically equivalent to $B$, and all previous requirements are fulfilled. Thus we have

**Proposition 6.3.** If $\mathcal{A}$ is a $C^*$-algebra with identity, then $B$ and $D(\Omega)$ are complete metric spaces for their Carathéodory (and Kobayashi) distances.

This proposition extends Theorem IV of [23] from von Neumann algebras to $C^*$-algebras with identity.

**Examples.** 1) Let $\mathcal{A}$ be a commutative Banach algebra with identity, endowed with a hermitian involution. In this case $\phi$ is a submultiplicative norm on $\mathcal{A}$. Hence, by II), we have

$$\phi(z) < p(z) = \phi(z^* z)^{1/2} < \phi(z^{*})^{1/2} \phi(z)^{1/2} = \phi(z) \quad (z \in \mathcal{A}),$$

whence $\phi(z) = p(z)$. Thus $B_p = C = \{z \in \mathcal{A} : \phi(z) < 1\}$, and by Proposition 6.2, $C$ is homogeneous. Since $\phi(z) < \|z\|$, then $B \subset C$.

2) Let $G$ be a discrete abelian group containing more than one element. Let $\mu$ be the counting measure on $G$, and let $\mathcal{A}$ be the convolution algebra on $L^1(G, \mu)$. Then $C$ is homogeneous, while $B$ is not, by Theorem II. Is there any homogeneous domain $D$ such that $B \subset D \subset C$?

3) If $G$ consists of two elements, $e$ and $g$, and $\mu(e) = \mu(g) = 1$, then $L^1(G, \mu) \simeq C^*$,

$$B = \{ (\zeta^1, \zeta^2) \in C^2 : |\zeta^1| + |\zeta^2| < 1 \}.$$  

The convolution in $L^1(G, \mu)$ is defined as follows. For $z' = (\zeta'^1, \zeta'^2)$, $z'' = (\zeta''^1, \zeta''^2)$ in $C^2$

$$(z' \star z'')(e) = \zeta'^1 \zeta''^1 + \zeta'^2 \zeta''^2 \quad (z' \star z'')(g) = \zeta'^1 \zeta''^2 + \zeta'^2 \zeta''^1.$$  

The dual group of $G$ is $G$ itself. For any $z = (\zeta^1, \zeta^2)$, the Gelfand transform $\hat{z}$ is defined by

$$\hat{z}(e) = \zeta^1 + \zeta^2, \quad \hat{z}(g) = \zeta^2 - \zeta^1.$$  

Thus
\[ \varrho(z) = \max \left( |z^1 + z^2|, |z^1 - z^2| \right), \]
and
\[ C = \{ z = (z^1, z^2) : |z^1 + z^2| < 1, \quad |z^1 - z^2| < 1 \}. \]

Hence \( C \) is a polydisc, and there is no bounded homogeneous domain \( D \subset C^2 \) such that \( B \subset D \subsetneq C \).

4) Going back to formula (6.8) in the general case, let \( z_1 = ie, z_2 = iy \), with \( y \in \Omega \). Since \( y - e \) and \( y + e \) commute, then
\[ p((z_2 - ie)(z_2 + ie)^{-1}) = p((y - e)(y + e)^{-1}) = \varrho((y - e)(y + e)^{-1}). \]

By the spectral mapping theorem,
\[ \varrho((y - e)(y + e)^{-1}) = \max \left\{ \left| \frac{t-1}{t+1} \right| : t \in \text{Sp} y \right\} = \max \left\{ \max \left\{ \frac{t-1}{t+1} : t \in \text{Sp} y \right\}, -\min \left\{ \frac{t-1}{t+1} : t \in \text{Sp} y \right\} \right\}. \]

Since
\[ \min \{ t : t \in \text{Sp} y \} = \frac{1}{\varrho(y^{-1})}, \quad \max \{ t : t \in \text{Sp} y \} = \varrho(y), \]
then
\[ p((y - e)(y + e)^{-1}) = \max \left\{ \varrho(y) - 1, \frac{\varrho(y^{-1}) - 1}{\varrho(y) + 1}, \frac{\varrho(y^{-1}) - 1}{\varrho(y^{-1}) + 1} \right\}. \]

A simple discussion shows then that
\[ e_{p(\Omega)}(ie, iy) = d_{p(\Omega)}(ie, iy) = \frac{1}{2} \max (\log \varrho(y), \log \varrho(y^{-1})). \]

This formula was obtained in [23, Theorem II and (8.4)] under the additional condition that the involution \( * \) be locally continuous.

Note added in proof, October 1978.

The proof of theorem II is considerably simplified by the following result established by T. J. Suffridge (Starlike and convex maps in Banach spaces, Pacific J. Math., 46 (1973), pp. 575-589; cf. theorem 8, pp. 584-586).

With the same notations as in theorem II, let \( f \in \text{Hol}(B, L'(M, \mu)) \) be such
that $f(B)$ is an open convex subset of $L^1(M, \mu)$ and that $f$ is a bi-holomorphic map of $B$ onto $f(B)$. If $\dim\mathbb{C} L^1(M, \mu) > 1$, then the map $x \mapsto f(x) - f(0)$ is the restriction to $B$ of a continuous linear map of $L^1(M, \mu)$ onto itself.

In view of this result and of lemma 4.3, the image by $F$ of any complex geodesic curve at 0 in $B$ belongs to a complex affine line through $x_0 = F(0)$. Hence any complex geodesic curve at $x_0$ must belong to a complex affine line. If $x_0 \neq 0$, that contradicts lemmas 4.4 and 4.5.

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