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**A remark on runge approximation of meromorphic functions**

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# A Remark on Runge Approximation of Meromorphic Functions (*).

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## 0. – Introduction.

Let $\Omega_1$ be an open subset of the complex manifold $\Omega_2$. In [2] Hirschowitz calls the pair $(\Omega_1, \Omega_2)$ meromorphic-convex if every function holomorphic in $\Omega_1$ may be uniformly approximated by functions meromorphic in $\Omega_2$. He calls the pair $(\Omega_1, \Omega_2)$ $\mu$-convex if even every function meromorphic in $\Omega_1$ may be uniformly approximated by functions meromorphic in $\Omega_2$. In [2, Theorem 5.1] it is claimed that a meromorphic-convex pair of Stein manifolds is $\mu$-convex if and only if the natural homomorphism $H_2(\Omega_1, \boldsymbol{R}) \to \to H_2(\Omega_2, \boldsymbol{R})$ is injective. In this paper I shall prove by means of a counterexample that this condition is not necessary.

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## 1. – An approximation theorem.

PROPOSITION 1. *Let $\Omega_1$ be an open and Stein subset of $\Omega_2$ such that the pair $(\Omega_1, \Omega_2)$ is meromorphic convex. Assume for each hypersurface $h \subset \Omega_1$ and for each $\alpha \in H_2(\Omega_1, \boldsymbol{Z}_n)$, $n \in \boldsymbol{N}_0$, that the intersection number $S(h, \alpha)$ vanishes. Then $(\Omega_1, \Omega_2)$ is $\mu$-convex.*

PROOF. Let $m$ be an in $\Omega_1$ meromorphic function which is holomorphic in a neighbourhood of the compact set $K$. For a given $\varepsilon > 0$ we shall have to construct a meromorphic function $\tilde{m}$ which is also holomorphic in a

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neighbourhood of $K$, such that $\|m - \tilde{m}\|_K < \varepsilon$. Since $\Omega_1$ is Stein we can exhaust it by special analytic polyhedra. Therefore we can choose such polyhedra $P_1$, $P_2$ with

$$K \subset P_1 \subset \overline{P}_1 \subset P_2 \subset \Omega_1 .$$

Let $h$ be the set of poles of $m$. According to our hypothesis we have $S(h, \alpha) = 0$ for all $\alpha \in H_2(\Omega_1, \boldsymbol{Z}_n)$ where $n$ is an arbitrary non-negative integer. It then follows from [7] that the Poincarè-problem has a solution for $m$ on $P_2$, i.e. there are functions $f, g \in \mathcal{O}(P_2)$ which are relatively prime, s. th. $m|_{P_2} = f/g$. In particular we have $M_g := \inf_{x \in K} |g(x)| > 0$. According to [9] $f$ and $g$ can be uniformly approximated by functions holomorphic in $\Omega_1$. We can choose $f_1, g_1 \in \mathcal{O}(\Omega_1)$ with $\|f - f_1\|_K < \varepsilon$ and $\|g - g_1\|_K < \varepsilon$.

Since $(\Omega_1, \Omega_2)$ is meromorphic-convex, there are functions $m_f$ and $m_g$ meromorphic in $\Omega_2$, which are holomorphic in a neighbourhood of $K$, s. th.

$$\|f_1 - m_f\|_K < \varepsilon \quad \text{and} \quad \|g_1 - m_g\|_K < \varepsilon .$$

Put $\tilde{m} := m_f/m_g$. For sufficiently small $\varepsilon$, $\tilde{m}$ is holomorphic in a neighbourhood of $K$ and for $\varepsilon < \operatorname{Min}\{M_g/4, \|f\|_K, \|g\|_K\}$ we have

$$\begin{aligned}
\|m - \tilde{m}\|_K &= \left\| m - \frac{m_f}{m_g} \right\|_K \leqslant \left\| \frac{f}{g} - \frac{f_1}{g_1} \right\|_K + \left\| \frac{f_1}{g_1} - \frac{m_f}{m_g} \right\|_K \leqslant \\
&\leqslant \frac{1}{M_g(M_g - \varepsilon)} \|fg_1 - gf_1\|_K + \frac{1}{(M_g - \varepsilon)(M_g - 2\varepsilon)} \|f_1 m_g - g_1 m_f\|_K \leqslant \\
&\leqslant \frac{4}{M_g^2} \left( \|fg_1 - fg\|_K + \|fg - f_1 g\|_K + \|f_1 m_g - f_1 g_1\|_K + \|f_1 g_1 - g_1 m_f\|_K \right) \leqslant \\
&\leqslant \varepsilon \frac{12}{M_g^2} \left( \|f\|_K + \|g\|_K \right) . \qquad ***
\end{aligned}$$

COROLLARY. *Again let $\Omega_1$ be open and Stein in $\Omega_2$, such that $(\Omega_1, \Omega_2)$ is meromorphic-convex. If $H_2(\Omega_1, \boldsymbol{Z})$ is divisible and $H_1(\Omega_1, \boldsymbol{Z})$ is torsion free, then $(\Omega_1, \Omega_2)$ is $\mu$-convex.*

PROOF. Because of Proposition 1 it suffices to prove that for each hypersurface $h \subset \Omega_1$ and each $\alpha \in H_2(\Omega_1, \boldsymbol{Z}_n)$ we have $S(h, \alpha) = 0$. For $n = 0$ this is an immediate consequence of the divisibility of $H_2(\Omega_1, \boldsymbol{Z})$. For $n \neq 0$ the universal coefficient theorem and our hypothesis yield

$$H_2(\Omega_1, \boldsymbol{Z}_n) \cong H_2(\Omega_1, \boldsymbol{Z}) \otimes \boldsymbol{Z}_n \oplus \operatorname{Tor}\left(H_1(\Omega_1, \boldsymbol{Z}), \boldsymbol{Z}_n\right) = 0 . \qquad ***$$

## 2. – Construction of a counterexample.

Let $D := \{(z_1, z_2) \in \boldsymbol{C}^2;\ |z_1| < 1,\ |z_2| < 1\}$ be the standard dicylinder in $\boldsymbol{C}^2$.

PROPOSITION 2. *There exists a domain of holomorphy $G \subset D$ with the following properties*:

(i) $(G, D)$ *is meromorphic-convex.*

(ii) $H_1(G, \boldsymbol{Z}) \cong \boldsymbol{Q}$.

(iii) $H_2(G, \boldsymbol{Z}) = 0$.

PROOF. In carrying out this construction we follow ideas of Pontrjagin, Stein and Ramspott (see [4], [8] and [5]). We shall construct a sequence of biholomorphic mappings $f_n\colon \boldsymbol{C}^2 \to \boldsymbol{C}^2$ with $f_n(0) = 0$, of smooth analytic sets $B_n \subset D_n := f_n(D)$ and of neighbourhoods $V_n$ of $B_n$ such that with $A_n := f_n^{-1}(B_n)$ and $U_n := f_n^{-1}(V_n)$ the following conditions are fulfilled:

(1) $B_n = \{(z_1, z_2) \in D_n;\ z_2^n - c_n z_1 = 0\}$ for some $c_n \in \boldsymbol{R}_+$. There is a smooth neighbourhood of $\{z_2^n - c_n z_1 = 0\} \cap \partial D_n$ in $\partial D_n$ and the two manifolds intersect transversally.

(2) $0$ is deformation retract of $B_n$.

(3) $D_n - V_n$ and $D_n - B_n$ have the same homotopy type.

(4) $\overline{U}_n \subset U_{n-1}$ where $\overline{U}_n$ is the closure of $U_n$ in $D$.

(5) $d(D - U_n, A_n) > 0$ where $d$ is the Euclidean distance.

(6) $\tilde{d}(A_n, A_{n+1}) < 1/2^n$ where $\tilde{d}$ denotes the Hausdorff metric (see [1], [2]).

(7) $\tilde{d}(\overline{U}_n, A_n) < 1/2^n$.

The conditions (6) and (7) imply that the sequences $(A_n)_{n\in\boldsymbol{N}}$ and $(U_n)_{n\in\boldsymbol{N}}$ converge to a common limit $A$. (2), (3) and (4) will enable us to compute the homology of $G := D - A$, the other conditions are necessary for the induction.

To start the induction we choose

$$f_1 := id\,,$$

$$A_1 := B_1 := \{(z_1, z_2) \in D;\ z_2 - \tfrac{1}{2} z_1 = 0\}\,.$$

$$U_1 := V_1 := \{(z_1, z_2) \in D;\ |z_2 - \tfrac{1}{2} z_1| < \tfrac{1}{4}\}\,.$$

Now we assume that $f_n$, $B_n$ and $V_n$ are given.

Put $g_n\colon \boldsymbol{C}^2 \to \boldsymbol{C}^2;\ (z_1, z_2) \mapsto (z_2, z_2^n - c_n z_1)$.

We take $f_{n+1} := g_n \circ f_n$ and get $D_{n+1} = f_{n+1}(D) = g_n(D_n)$.
Furthermore

$$f_{n+1}(A_n) = g_n(B_n) = \{(z_1, z_2) \in D_{n+1};\ z_2 = 0\} .$$

The plane $\{z_2 = 0\}$ intersects $\partial D_{n+1}$ transversally. We define

$$B_{n+1} := \{(z_1, z_2) \in D_{n+1};\ z_2^{n+1} - c_{n+1} z_1 = 0\} .$$

For sufficiently small $c_{n+1} \in \boldsymbol{R}_+$ the conditions (1) and (6) are clearly fulfilled. A retraction of $B_n$ to 0 gives a retraction of $g_n(B_n)$ to 0 and out of this we can construct a retraction for $B_{n+1}$, hence (2) is valid. We also can choose $c_{n+1}$ sufficiently small such that $B_{n+1} \subset g_n(V_n)$ and $d(B_{n+1}, D_{n+1} - g_n(V_n)) > 0$. We now have to find a suitable neighbourhood $V_{n+1}$ of $B_{n+1}$. To do this we look at

$$g_{n+1}\colon \boldsymbol{C}^2 \to \boldsymbol{C}^2 ; \qquad (z_1, z_2) \mapsto (z_2, z_2^{n+1} - c_{n+1} z_1) .$$

Again we have
$g_{n+1}(B_{n+1}) = \{(z_1, z_2) \in D_{n+2};\ z_2 = 0\}$ where the plane $\{z_2 = 0\}$ intersects $\partial D_{n+2}$ transversally. Put

$$W_{n+1} := \{(z_1, z_2) \in D_{n+2};\ |z_2| < \varrho_{n+1}\} \quad \text{for some } \varrho_{n+1} \in \boldsymbol{R}_+ .$$

For sufficiently small $\varrho_{n+1}$ we have according to the above $\overline{W_{n+1}} \subset g_{n+1}(g_n(V_n))$ and $d(D_{n+2} - g_{n+1}(B_{n+1}), D_{n+2} - W_{n+1}) > 0$.

Moreover we can acquire

$$\tilde{d}(\overline{f_{n+1}^{-1}(W_{n+1})}, A_{n+1}) < \frac{1}{2^{n+1}} .$$

The sets

$$D_{n+2} - g_{n+1}(B_{n+1}) = \{(z_1, z_2) \in D_{n+2};\ z_2 = 0\}$$

and

$$D_{n+2} - W_{n+1} = \{(z_1, z_2) \in D_{n+2};\ |z_2| < \varrho_{n+1}\}$$

have the same homotopy type. If we put $V_{n+1} := g_{n+1}^{-1}(W_{n+1})$ then the conditions (3), (4), (6) and (7) are fulfilled, i.e. $V_{n+1}$ is a suitable neighbourhood of $B_{n+1}$.

Let $A$ be the limit of the sequence $(A_n)_{n \in N}$. $A$ is non-empty. We claim that $G := D - A$ has the desired properties. We shall first prove that $G$ is connected. Take two points $(z_1^{(1)}, z_2^{(1)}), (z_1^{(2)}, z_2^{(2)}) \in G$. For some big $n_0$ we have $(z_1^{(1)}, z_2^{(1)}) \notin U_{n_0} \not\ni (z_1^{(2)}, z_2^{(2)})$. Because of $A \subset \overline{U_{n_0+1}} \subset U_{n_0}$ it is sufficient to prove that $D - U_{n_0}$ is pathwise connected. But this is a consequence of the fact that $D - A_{n_0}$ is connected and that both sets have the same homotopy-type. $G$ is a domain of holomorphy. To see this, consider

$$G_m := D - A_m \qquad \text{and} \qquad \hat{G}_n := \overset{\circ}{\widehat{\bigcap_{m \geq n} G_m}} = D - \overline{\bigcup_{m \geq n} A_m}\,.$$

As above, one sees that $\hat{G}_n$ is connected. Being the open kernel of an intersection of domains of holomorphy $\hat{G}_n$ is a domain of holomorphy itself. Moreover $\hat{G}_n \subset \hat{G}_{n+1}$ and $G = \bigcup_{n \in N} \hat{G}_n$. Hence $G$ is a domain of holomorphy. (See [3, p. 38]). $A$ is a limit of hypersurfaces, hence it is a limace in the terminology of Hirschowitz. It follows from [2; Theorem 3.5] that $(G, D)$ is meromorphic conex. The next step will be to prove $H_2(G, \boldsymbol{Z}) = 0$. Let $\beta$ be a 2-cycle in $G$. For sufficiently big $n_0$, $\beta$ is contained in $D - U_{n_0}$. Thus it suffices to prove $H_2(D - U_{n_0}, \boldsymbol{Z}) \cong H_2(D - A_{n_0}, \boldsymbol{Z}) = 0$. The exact homology sequence of the pair $(D, D - A_{n_0})$ yields

$$\ldots \to H_3(D, D - A_{n_0}, \boldsymbol{Z}) \to H_2(D - A_{n_0}, \boldsymbol{Z}) \to \underset{\substack{\| \\ 0}}{H_2(D, \boldsymbol{Z})} \to \ldots\,.$$

On the other hand Alexander-Pontrjagin duality implies $H_3(D, D - A_{n_0}, \boldsymbol{Z}) \cong$ $\cong H^1_*(A_{n_0}, \boldsymbol{Z})$, where the star denotes cohomology with compact support. Since $A_{n_0}$ has no singularities Poincarè duality gives $H^1_*(A_{n_0}, \boldsymbol{Z}) \cong$ $\cong H_1(A_{n_0}, \boldsymbol{Z}) = 0$, since $A_{n_0}$ is contractible. Hence $H_3(D, D - A_{n_0}, \boldsymbol{Z}) = 0$, and this clearly implies $H_2(D - A_{n_0}, \boldsymbol{Z}) = 0$.

It remains to prove $H_1(G, \boldsymbol{Z}) \cong \boldsymbol{Q}$. According to [5, Satz 2] we have $H_1(D - U_n, \boldsymbol{Z}) \cong H_1(D - A_n, \boldsymbol{Z}) \cong \boldsymbol{Z}$. We want to construct a generating cycle for these homology groups. Therefore consider

$$f_{n+1}(A_n) = g_n(B_n) = \{(z_1, z_2) \in D_{n+1};\, z_2 = 0\}$$

and

$$W_n = f_{n+1}(U_n) = \{(z_1, z_2) \in D_{n+1};\, |z_2| < \varrho_n\}\,.$$

As a generating cycle for $H_1(D_{n+1} - f_{n+1}(A_n), \boldsymbol{Z}) \cong H_1(D_{n+1} - W_n, \boldsymbol{Z})$ we can choose

$$\alpha_n := \{(0, \varrho_n \exp[2\pi i t]);\, 0 \leq t \leq 1\}\,.$$

Put $t_n := f_{n+1}^{-1}(\alpha_n)$, denote by $\bar{t}_n$ the homology class in $H_1(D - U_n, \boldsymbol{Z})$ and by $\bar{\bar{t}}_n$ the homology class in $H_1(G, \boldsymbol{Z})$. The classes $\bar{\bar{t}}_n$ generate $H_1(G, \boldsymbol{Z})$.

To see this take a 1-cycle $\alpha$ with homology class $\bar{\bar{\alpha}}$. Then for some $n_0$, $\alpha$ is contained in $D - U_{n_0}$ and there it is homologous to some $m\cdot t_{n_0}$. In particular $\bar{\bar{\alpha}} = m\cdot\overline{\overline{t_{n_0}}}$. Now we have to find the relations between the $\overline{\overline{t_n}}$. Therefore consider

$$g_{n+2}(\alpha_n) = \{(\varrho_n \exp[2\pi it], \varrho_n^{(n+1)} \exp[2\pi i(n+1)t]); 0\leqslant t\leqslant 1\}.$$

In $D_{n+2} - W_{n+1}$ the cycle $g_{n+2}(\alpha_n)$ is homologous to $(n+1)\cdot\alpha_{n+1}$. This implies $\overline{\overline{t_n}} = (n+1)\cdot\overline{\overline{t_{n+1}}}$. Moreover $m\cdot\overline{\overline{t_n}} \neq 0$ for all $m\neq 0$. Because, if we assumed $m\cdot\overline{\overline{t_n}} = 0$ this would imply that $m\cdot t_n$ was homologous to 0 in some set $D - U_{n_0}$, $n_0 \geqslant n$. But this would mean $m\cdot(n+1)\dots n_0\cdot\overline{t_{n_0}} = 0$, a contradiction to $H_1(D - U_{n_0}, \boldsymbol{Z}) \cong \boldsymbol{Z}$. This also means that apart from the relations $\overline{\overline{t_n}} = (n+1)\cdot\overline{\overline{t_{n+1}}}$ there are no other relations between the $\overline{\overline{t_n}}$. The map $\overline{\overline{t_n}} \mapsto 1/n!$ gives an isomorphism $H_1(G, \boldsymbol{Z}) \cong \boldsymbol{Q}$. ***

We can now deliver our counterexample. Take $\dot{E} := \{z\in\boldsymbol{C}; 0<|z|<1\}$ to be the punctured unit-disc in $\boldsymbol{C}$. The pair $(\dot{E}, \boldsymbol{C})$ is meromorphic-convex. $(G\times\dot{E}, D\times\boldsymbol{C})$ is meromorphic-convex since it is the product of meromorphic-convex pairs. The Künneth formula yields

$$H_2(G\times\dot{E}, \boldsymbol{Z}) \cong \boldsymbol{Q}.$$

$$H_1(G\times\dot{E}, \boldsymbol{Z}) \cong \boldsymbol{Z}\oplus\boldsymbol{Q}.$$

By virtue of our corollary $(G\times\dot{E}, D\times\boldsymbol{C})$ is $\mu$-convex. On the other hand it follows from the universal coefficient theorem that

$$H_2(G\times\dot{E}, \boldsymbol{R}) \cong \boldsymbol{R}.$$

Since

$$H_2(D\times\boldsymbol{C}. \boldsymbol{R}) = 0$$

the canonical homomorphism $H_2(G\times\dot{E}, \boldsymbol{R}) \to H_2(D\times\boldsymbol{C}, \boldsymbol{R})$ cannot be injective.

## 3. – Remarks

As A. Hirschowitz has pointed out in a discussion, it is the first sentence that contains the mistake in the proof of [2; Theorem 5.1]. There it is assumed that the mapping $H_2(\Omega_1, \boldsymbol{R}) \to \mathrm{Hom}\,(H^2(\Omega_1, \boldsymbol{Z}), \boldsymbol{R})$ is injective. This is not true in general. If however the homology of $\Omega_1$ is of finite type there is an exact sequence

$$0\to \mathrm{Ext}\,(H^3(\Omega_1, \boldsymbol{Z}), \boldsymbol{R}) \to H_2(\Omega_1, \boldsymbol{R}) \to \mathrm{Hom}\,(H^2(\Omega_1, \boldsymbol{Z}), \boldsymbol{R}) \to 0.$$

(Cf. [6, p. 248]). Since $\boldsymbol{R}$ is divisible $\operatorname{Ext}(H^3(\Omega_1, \boldsymbol{Z}), \boldsymbol{R}) = 0$. Under this condition as well as under other conditions which imply the injectivity of $H_2(\Omega_1, \boldsymbol{R}) \rightarrow \operatorname{Hom}(H^2(\Omega_1, \boldsymbol{Z}), \boldsymbol{R})$ the arguments given in [2] remain true.

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