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Holomorphic Perturbation of a System of Micro-Differential Equations (*).

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dedicated to Hans Lewy

The purpose of this paper is to study some interesting phenomena which we encounter when we holomorphically change the coefficients of a system of (micro)-differential equations. Roughly speaking, our results claim that solutions of the system in question cannot change holomorphically with respect to parameters $t = (t_1, \ldots, t_N)$ even when the coefficients of the system depend holomorphically on t. Such phenomena were observed by Zerner [4] for linear differential operators with constant coefficients and by Lewy [2] for a class of linear differential operators of the first order. The method employed in this article is completely different from theirs and it reveals the general mechanism hidden behind their very ingenious arguments. The results in this paper will also be of their own interest in connection with a recent result (Kawai [1]) on the stability of the cohomology groups under the deformation of *elliptic* systems. (See Remark 4 at the end of this paper.)

The notations used in this article are the same as those used in Sato-Kawai-Kashiwara [3] (1). Micro-differential operators were called pseudo-differential operators in that article.

First we show a theorem to the effect that holomorphic character in some variables of a solution of the system in question automatically entails the analyticity in all variables. Its connection to a recent result of Lewy [2] will be explained later as its corollary. In the sequel $t = (t_1, \ldots, t_N)$ denotes a point in an open set $\Omega \subset \boldsymbol{C}^N \simeq \boldsymbol{R}^{2N}$, $\bar{t} = (\bar{t}_1, \ldots, \bar{t}_N)$, $r = (r_1, \ldots, r_N)$

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(1) This article shall be referred to as S-K-K [3] in the sequel for short.
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and $s=(s_1, \dots, s_N)$ denote its complex conjugate, real part and imaginary part, respectively. Their dual variables are denoted by $\tau=(\tau_1, \dots, \tau_N)$, $\bar{\tau}=(\bar{\tau}_1, \dots, \bar{\tau}_N)$, $\varrho=(\varrho_1, \dots, \varrho_N)$ and $\sigma=(\sigma_1, \dots, \sigma_N)$, respectively. We also denote by $\bar{\partial}_t$ the operator

$$\left(\frac{\partial}{\partial \bar{t}_1}, \dots, \frac{\partial}{\partial \bar{t}_N}\right) \equiv \left(\frac{1}{2}\left(\frac{\partial}{\partial r_1}+\sqrt{-1}\frac{\partial}{\partial s_1}\right), \dots, \frac{1}{2}\left(\frac{\partial}{\partial r_N}+\sqrt{-1}\frac{\partial}{\partial s_N}\right)\right).$$

THEOREM 1. *Let $\mathcal{M}$ be an admissible system of micro-differential equations defined in a neighborhood of $(r, s, x; \sqrt{-1}(\varrho, \sigma, \eta)\infty)=(0,0,x_0; \sqrt{-1}\cdot(0,0,\eta_0)\infty)\equiv p_0^* \in \sqrt{-1}S^*M$, where M is a neighborhood of $(0, x_0)\in \boldsymbol{R}^{2N}\times\boldsymbol{R}^n$. Assume that system $\mathcal{M}$ commutes with the operator $\bar{\partial}_t$, i.e. the coefficients of the defining equations of $\mathcal{M}$ are independent of $\bar{t}$. Assume that there exists an analytic function $\varphi(t, \bar{t}, x; \tau, \bar{\tau}, \eta)$ which is homogeneous with respect to $(\tau, \bar{\tau}, \eta)$ and defined in a complex neighborhood of p_0^* and which satisfies following conditions*:

(1) $\varphi=0$ *on the (complex) characteristic variety V of $\mathcal{M}$.*

(2) $$\frac{\partial\varphi}{\partial t}\equiv\left(\frac{\partial\varphi}{\partial t_1}, \dots, \frac{\partial\varphi}{\partial t_N}\right)\neq 0 \text{ at } p_0^*.$$

Then any microfunction solution f of $\mathcal{M}$ that is defined near p_0^* and that depends holomorphically on t is necessarily zero in a neighborhood of p_0^*.

PROOF. The holomorphy of f in t implies that f should satisfy

(3) $$\bar{\partial}_t f=0\,.$$

On the other hand, the assumption on the commutativity of $\mathcal{M}$ and $\bar{\partial}_t$ entails that we can find a well-defined system $\tilde{\mathcal{M}}$ by adding equation (3) to $\mathcal{M}$. Clearly the (complex) characteristic variety W of $\tilde{\mathcal{M}}$ is contained in $\{\varphi=0, \bar{\tau}=0\}$ by condition (1). Here $\bar{\tau}$ denotes a point in $(\boldsymbol{R}^{2N}_{\varrho,\sigma})^{\boldsymbol{C}}\cong \boldsymbol{C}^{2N}_{\varrho,\sigma}$. By virtue of assumption (2) we may assume without loss of generality that

$$\mathrm{Re}\left(\frac{\partial\varphi}{\partial t_l}\right)(p_0^*)<0$$

holds for some $l(1\leqq l\leqq N)$.

For the sake of definiteness, we assume

$$\mathrm{Re}\left(\frac{\partial\varphi}{\partial t_1}\right)(p_0^*)<0\,.$$

Then, choosing a positive constant C sufficiently large, we find

$$\{\varphi + C\bar{\tau}_1, \bar{\varphi} + C\tau_1\}(p_0^*) < 0 . \tag{4}$$

Since W is contained in $\{\varphi + C\bar{\tau} = 0\}$, we can apply Theorem 2.3.8 of S-K-K [3] to conclude

$$Ext^0(\tilde{\mathcal{M}}, \mathcal{C})_{p_0^*} = 0 \tag{5}$$

holds. This is equivalent to saying that any microfunction solution of $\tilde{\mathcal{M}}$, i.e. any microfunction solution of $\mathcal{M}$ that depends holomorphically on t, is inevitably zero as a microfunction near p_0^*. This is the required result. Q.E.D.

COROLLARY 1 (Lewy [2]). *Consider a linear differential operator*

$$L = \frac{\partial}{\partial t_1} + a_1(t_1, x_1, x_2)\frac{\partial}{\partial x_1} + a_2(t_1, x_1, x_2)\frac{\partial}{\partial x_2},$$

where a_1 and a_2 are analytic in (t_1, x_1, x_2) and holomorphic in t_1. Assume

$$\frac{\partial a_1}{\partial t_1} a_2 \neq a_1 \frac{\partial a_2}{\partial t_1} . \tag{6}$$

Let $f(t_1, x_1, x_2)$ be a (hyper)function defined in a neighborhood of $(t_1, x_1, x_2) = (0, 0, 0)$. Assume that f satisfies the equation $Lf = 0$ and that f depends holomorphically on t_1. Then f is analytic near the origin.

PROOF. It suffices to choose $\tau_1 + a_1\eta_1 + a_2\eta_2$ as φ in Theorem 1. Actually, $\partial\varphi/\partial t_1 \neq 0$ on the characteristic variety V associated with L under assumption (6). Recall that the nullity as a microfunction at every point is equivalent to the analyticity of a hyperfunction. Q.E.D.

REMARK 1. Lewy [2] also gives another sufficient condition (7) that guarantees the same analyticity property of f.

$$a_1\bar{a}_2 - \bar{a}_1 a_2 \neq 0 . \tag{7}$$

One can easily verify that condition (7) is equivalent to the claim the real locus of the characteristic variety of the following system $\mathfrak{L}$ of differential equations is void.

$$\mathfrak{L}: \begin{cases} Lf = 0 , \\ \dfrac{\partial f}{\partial \bar{t}_1} = 0 . \end{cases} \tag{8}$$

Therefore, in this case the analyticity of f follows from a less subtle theorem on the regularity of solutions, i.e. regularity of solutions of an elliptic system. (See S-K-K [3], Chapter III, Theorem 2.1.1, for example.) Actually, the original argument of Prof. Lewy was also easier in this case.

The next corollary gives an answer to the question of Prof. Lewy how to generalize Corollary 1 to the case with several parameters $t=(t_1, \dots, t_N)$.

COROLLARY 2. *Consider a linear differential operator*

$$\tilde{L}=\sum_{l=1}^{N} b_l(t, x_1, \dots, x_n)\frac{\partial}{\partial t_l}+\sum_{j=1}^{n} a_j(t, x_1, \dots, x_n)\frac{\partial}{\partial x_j},$$

where $(b_1, \dots, b_N)$ *and* $(a_1, \dots, a_n)$ *are analytic in* (t, x) *and holomorphic in* t. *Assume*

$$\text{rank}\begin{pmatrix} a_1, & \dots, & a_n \\ \dfrac{\partial a_1}{\partial t_1}, & \dots, & \dfrac{\partial a_n}{\partial t_1} \\ \cdots & \cdots & \cdots \\ \dfrac{\partial a_1}{\partial t_N}, & \dots, & \dfrac{\partial a_n}{\partial t_N} \end{pmatrix}=n \tag{9}$$

holds at the origin. Let $f(t, x)$ *be a hyperfunction solution of the equation* $\tilde{L}f=0$ *that is defined in a neighborhood of* $(t, x)=(0, 0)$ *and that depends holomorphically on* t. *Then* f *is analytic near the origin.*

The proof of Corollary 2 is the same as that of Corollary 1 and is omitted.

Theorem 1 suggests that a system of micro-differential equations with holomorphic parameters might not admit « plenty of solutions with holomorphic parameters ». This guess is most neatly embodied by the following theorem.

THEOREM 2. *Consider a micro-differential operator* $L(t_1, x, D_x)$ *of order* m *which is defined in a neighborhood of* $(t_1, x, \sqrt{-1}\eta\infty)=(0, x_0, \sqrt{-1}\eta_0\infty)\equiv \equiv p_0^* \in \Omega\times\sqrt{-1}S^*M$, *where* Ω *is an open set in* $\mathbf{C}$, *and which depends holomorphically on* t_1. *Assume*

$$\begin{cases} L_m(p_0^*)=0, \\ \dfrac{\partial}{\partial t_1}L_m(p_0^*)\neq 0, \end{cases} \tag{10}$$

holds. Here L_m *denotes the principal symbol of* L. *Then the equation* $Lf=g$ *does not, in general, admit a microfunction solution* f *near* p_0^* *that depends* *holomorphically on* t_1, *even if* g *depends holomorphically on* t_1.

PROOF. First consider following system $\mathfrak{L}$.

$$\mathfrak{L}:\begin{cases} Lf = 0 , \\ \dfrac{\partial f}{\partial \bar{t}_1} = 0 . \end{cases} \tag{11}$$

The holomorphic dependence of L on t_1 guarantees that $\mathfrak{L}$ is a well-defined system of micro-differential equations. Since the eigenvalues of the generalized Levi-form associated with $\mathfrak{L}$ (S-K-K [3], Chapter III, Definition 2.3.1) are, by the definition, the solutions of the following equation:

$$\begin{vmatrix} \lambda - \dfrac{1}{2\sqrt{-1}}\{L_m, \bar{L}_m\} & \dfrac{1}{2\sqrt{-1}}\dfrac{\partial L_m}{\partial t_1} \\ -\dfrac{1}{2\sqrt{-1}}\overline{\left(\dfrac{\partial L_m}{\partial t_1}\right)} & \lambda \end{vmatrix} = 0 . \tag{12}$$

Since $(\partial/\partial t_1)L_m(p_0^*) \neq 0$ by the assumption, equation (12) has one strictly positive solution and one strictly negative solution. Then Theorem 2.3.6 in S-K-K [3], Chapter III, claims that

$$Ext^1(\mathfrak{L}, \mathcal{C}) \neq 0 \tag{13}$$

holds at $(r_1, s_1, x; \sqrt{-1}(\varrho_1, \sigma_1, \eta)\infty) = (0, 0, x_0; \sqrt{-1}(0, 0, \eta_0)\infty)$.

This is equivalent to saying that

$$\begin{cases} Lf = g , \\ \dfrac{\partial g}{\partial \bar{t}_1} = 0 , \end{cases} \tag{14}$$

is not (even micro-locally) solvable in general. This is the required result. Q.E.D.

REMARK 2. The generalization of Theorem 2 to general overdetermined systems shall be discussed elsewhere, as it involves some technical complexity.

REMARK 3. Theorem 2 generalizes a result of Zerner [4] for linear differential equations with *constant* coefficients. Note, however, that in the case with constant coefficients, more precise result related to the higher order derivatives of L_m with respect to t_1 can be obtained as Zerner [4] did for distibution solutions. Since the argument used to obtain such a more precise result for equations with constant coefficients is completely different from the one employed here, it shall be discussed elsewhere.

REMARK 4. Although the results in this article might have already convinced the reader that the ellipticity of the equation under deformation should be crucial in obtaining the stability of the cohomology groups, we indicate the following system $\mathcal{N}$ of (micro-)differential equations as an example which makes this point clearer:

$$\mathcal{N}: \begin{cases} \left(\frac{\partial}{\partial x_1} + \sqrt{-1}\, t_1 (\sin x_1) \frac{\partial}{\partial x_2}\right) f = 0 , \\ \left(\frac{\partial}{\partial t_1} - \sqrt{-1}\, (\cos x_1) \frac{\partial}{\partial x_2}\right) f = 0 . \end{cases}$$

It is clear that the tangential system $\mathcal{N}_c$ induced from $\mathcal{N}$ onto $\{t_1 = c\}$ is $\mathcal{M}(c)$ given by

$$\mathcal{M}(c): \left(\frac{\partial}{\partial x_1} + \sqrt{-1}\, c (\sin x_1) \frac{\partial}{\partial x_2}\right) g = 0 .$$

On the other hand, it is known (S-K-K [3], Chapter III, Theorem 2.3.6) that the structure of cohomology groups $Ext^j\,(\mathcal{M}(c), \mathcal{C})$ does depend on the sign of c. This implies that even micro-local structure of cohomology groups cannot be preserved under the deformation discussed in Kawai [1], if the ellipticity assumption is omitted. Note also that $\mathcal{N}$ may be considered as an equation on $\boldsymbol{R}_t \times (\boldsymbol{R}/2\pi \boldsymbol{Z})^2_x$ due to the periodicity of its coefficients.

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