WALTER LITTMAN

Boundary control theory for hyperbolic and parabolic partial differential equations with constant coefficients


<http://www.numdam.org/item?id=ASNSP_1978_4_5_3_567_0>
1. - Introduction.

Suppose we have a well-posed initial-boundary value problem for a hyperbolic or parabolic equation $Lu = 0$ in a cylindrical domain $\Omega = D \times [0, \infty)$, where $D$ is a bounded domain in $\mathbb{R}^n$. One basic problem of boundary control theory is that of null-controllability: given initial data in $D$ at $t = 0$, can this data be supplemented with appropriate non-homogeneous time dependent boundary data, prescribed on the lateral boundary of $\Omega$, such that the solution of the resulting initial boundary value problem will vanish for $t > T$? Furthermore, how small can $T$ be chosen? For the wave equation, the heat equation and more general second order equations this problem has been treated by Russell[9], Fattorini[2], Seidman[10] and others. For additional references see these works.

The present work was to a large extent inspired by two sources, which we are happy to acknowledge: A number of very instructive conversations with L. Markus which acquainted us with the basic problems and work done in the area; secondly, a lecture given by Frank Jones in November 1973, the subject of which we shall describe shortly.

Once instance where the problem of boundary control theory has a rather straight-forward solution is the wave equation in an odd number of space

(*) Partially supported by NSF Grant MCS 76-05863 the University of Minnesota.
(**) University of Minnesota, School of Mathematics, Minneapolis, Minnesota. Pervenuto alla Redazione il 28 Luglio 1977.
dimensions greater than one. (See Russell [9]). One simply extends the initial data outside \( D \) with support in a slightly larger set, and proceeds to solve the pure initial value problem with this data. Since the Riemann function for this problem has support on the surface of a (forward) cone (Huygen's principle), the solution restricted to \( \Omega \) will vanish for \( t > T \), where \( T \) can be determined from the geometry of \( D \). The appropriate boundary data for the control problem is then simply read off from the solution determined as described above. Let us note that this procedure works whenever the Riemann function has an appropriate lacuna.

Frank Jones' result [7], alluded to earlier, consists in constructing a fundamental solution to the heat equation which can play the same role in the boundary control problem for the heat equation that the ordinary Riemann function plays for the (odd space dimensional) wave equation—a role for which the ordinary fundamental solution of the heat equation is not suited. Jones constructs a fundamental solution with support in \( 0 < t < \epsilon \), with \( \epsilon > 0 \) arbitrary.

The methods just described have several advantages. One is that the boundary control problem is solved once and for all without reference to any specific initial boundary value problem. Another is that one need only have uniqueness for the mixed problem. The existence of a solution to the control problem is a consequence of the procedure. (Uniqueness of the solution of the control problem is not asserted here and is, in general, not valid under our hypotheses.) A third advantage of the procedure is that it is not restricted to cylindrical domains.

We briefly describe the content in the remainder of the paper. Section 2 reviews some elementary facts about hyperbolic operators for \( n = 1 \). In section 3 we treat the boundary control problem for strictly hyperbolic equations in one space variable (which curiously enough, is not quite as immediate as the case \( n \) odd \( > 1 \)). We dwell at greater length on the one dimensional case than might seem necessary mainly because the basis of the method is encountered here already. In section four we use a plane wave decomposition to treat the \( n \) dimensional hyperbolic case. Section five deals with a parabolic equation in one dimension, while section six deals with the \( n \)-dimensional case, again via a plane wave decomposition. The main theorem for the parabolic case is stated at the end of that section.

This paper emphasizes methods and qualitative results. Thus, for the sake of simplifying proofs, we assume strict hyperbolicity, although results continue to hold for general hyperbolic operators with constant coefficients with slight modifications in the proofs. We are also content with stating all results in the context of \( C^\infty \) functions, at least in the hyperbolic case.
We intend to discuss estimates in Sobolev norms (for the hyperbolic case), more general equations, variable coefficients and other aspects of the problem at another time.

The subject of this paper was presented at the Conference on Partial Differential Equations held at Oberwolfach, March 1975.

2. - Some elementary facts concerning hyperbolic equations in one space dimension.

We first treat the case of a strictly hyperbolic operator $L$ corresponding to a homogeneous polynomial (i.e., with no lower order terms)

$$P_m(\xi, \tau) = a \prod_{i=1}^{m} (\xi_i + \tau),$$

where we assume that $a \neq 0$, the $\xi_i$'s are real and, for the sake of simplicity, distinct. The differential operator $L = \alpha(\partial/\partial x) + \partial/\partial t$ represents a directional derivative in the direction $(\alpha, 1)$ and has two fundamental solutions, $g_-$ and $g_+$, i.e., solutions to the equation $Lg = \delta(\alpha) \delta(t)$, both being measures with support on the line $\alpha t = x$; $g_+$ vanishing for negative $t$, $g_-$ for positive $t$.

Both represent signals propagating with velocity $\alpha$ and beginning or ending at $x = 0, t = 0$.

Let us assume that none of the $\alpha_i$ vanish, i.e., there is no zero velocity of propagation. Then we gain complete symmetry between the $x$ and $t$ variables. Moreover each operator $L_i = \alpha_i(\partial/\partial x) + \partial/\partial t$ will have two fundamental solutions each of which can be uniquely determined by specifying support either in the upper or lower (closed) half plane—or alternately, in the right or left half plane. Convoluting the fundamental solutions for $L_i$ with support in the (upper, lower, right, left) half plane we obtain, apart from a constant factor, a fundamental solution $(G_+, G_-, G_R, G_L)$ to the operator $L$ with support in that half plane. The actual support of $G_+$ is the sector in the upper half plane bounded by the two lines (passing through the origin) corresponding to the algebraically lowest and highest velocity of propagation; $\min \alpha_i$ and $\max \alpha_i$. Letting $\lambda_i = 1/\alpha_i$ we see that $G_R(G_L)$ will have support in the sector in the right (left) half plane bounded by the rays corresponding to the algebraically lowest and highest reciprocal velocity; i.e., $\min \lambda_i$ and $\max \lambda_i$. It will be of importance to us to notice that if we let $\tilde{\lambda} = \max |\lambda_i|$ then the support of $G_R(G_L)$ is contained in the intersection of the set $|t/|x| < \tilde{\lambda}$ with the right (left) half plane.
Next we consider operators with lower order terms

\[ L = P(D_x, D_t) = P_m(D_x, D_t) + Q(D_x, D_t), \quad \left( D_x = \frac{\partial}{\partial x}, D_t = \frac{\partial}{\partial t} \right), \]

where \( Q(\xi, \tau) \) is a polynomial with complex coefficients of degree \( < m \).

\( L \) has four fundamental solutions, \( y \), which we again denote by \( G_L, G_R, G_L, \) having the same supports as those for \( P_m(D) \). Furthermore these will vary continuously with the coefficients of the operator, as long as the speeds of propagation remain separated.

3. - The hyperbolic case, \( n = 1 \).

Let \( L = P(D_x, D_t) \) be a strictly hyperbolic operator \( (n = 1) \), and let \( \lambda \) be the reciprocal of the slowest speed of propagation, i.e., \( \lambda = 1/\min |x_i| \).

**Theorem 1.** Let \( C^\infty \) Cauchy data be prescribed at \( t = 0 \) in the closed interval \( I \) on the \( x \) axis. Then there exist \( C^\infty \) solution \( v \) to \( Lv = 0 \) in \( I \times [0, \infty) \), assuming this Cauchy data, vanishing for \( t > T_1 \), where \( T_1 \) is any number exceeding \( T_0 = \text{diameter } I \times \lambda \).

We recall that by \( C^\infty(I) \) (for a closed interval \( I \)) we mean functions extendable to \( C^\infty \) in a larger interval.

**Proof.** Extend the initial data to a \( C^\infty \) function with support in a slightly larger interval \( I_\epsilon \) contained in an \( \epsilon \) neighborhood of \( I \), and solve the equation \( Lu = 0 \) with the extended Cauchy data for all \( -\infty < x < \infty, \ t > 0 \). Call the solution \( u \). Let \( \varphi(t) \) be a \( C^\infty \) function equal to one for \( t < \frac{1}{2} T_1 - \epsilon \) and zero for \( t > \frac{1}{2} T_1 + \epsilon \), and set \( L(u\varphi) = f \).

If \( m \) denotes the mid-point of \( I \), let \( \Psi \) be a \( C^\infty \) function equal to one for \( x < m - \epsilon \), and zero for \( x > m + \epsilon \), and set

\[ f_L = f \cdot \Psi, \quad f_R = f \cdot (1 - \Psi) \]

and

\[ U_L = f_L \ast G_L, \quad U_R = f_R \ast G_R, \]

\( \ast \) denoting involution, and \( G_L \) and \( G_R \) the fundamental solutions described in the last section. For \( \epsilon \) sufficiently small (which we now so choose) \( U_L \) and \( U_R \) are \( C^\infty \) and each vanish in neighborhoods of the sets

\[ \{ t = 0, x \in I \} \quad \text{and} \quad \{ t > T_1, x \in I \}. \]
(This is of course a consequence of the estimates of the supports of $G_L$ and $G_R$ of the previous section.) Hence the sum $U = U_L + U_R$ does also, and moreover satisfies $Lu = f$. Next, set $v = u_0 - U$. We have $Lv = 0$, for $x \in \bar{I}$, $v$ agrees with $u$ near $t = 0$, has the originally assigned Cauchy data in $\bar{I}$, and vanishes for $t > T_1$. Hence it is a desired solution.

4. -- The $n$ dimensional hyperbolic case.

We shall accomplish the transition from one to several space dimensions by a standard device—a plane wave decomposition. To that effect, let us recall some basic facts concerning such decompositions, as described for example by Ludwig [8]. For $f \in \mathcal{S}(\mathbb{R}^n)$ ($\mathcal{S}$ is the Schwartz space of rapidly decreasing $C^\infty$ functions) the Radon transform is defined by

$$(Rf)(s, \omega) = \int f(x) \, ds \quad x \cdot \omega = s$$

where $\omega$ is a unit vector and $S$ represents $n - 1$ dimensional surface area, while $-\infty < s < \infty$. For $g = g(s, \omega)$ define

$$(Kg)(s, \omega) = \begin{cases} \frac{1}{2(2\pi)^{n-1}} \left( \frac{1}{i} \frac{\partial}{\partial s} \right)^{n-1} g(s, \omega) & (n \text{ odd}), \\ \frac{1}{2(2\pi)^{n-1}} i H \left( \frac{1}{i} \frac{\partial}{\partial s} \right)^{n-1} g(s, \omega) & (n \text{ even}), \end{cases}$$

$H$ denoting the (one dimensional) Hilbert transform. Then a plane wave decomposition

$$f(x) = \int f_\omega(x \cdot \omega) \, d\omega$$

is accomplished by setting

$$f_\omega(s) = (KRf)(s, \omega).$$

If $f$ belongs to $\mathcal{S}(\mathbb{R}^n)$ then $f_\omega(s)$ is infinitely differentiable in $s$ and $\omega$. In our applications the function will depend on another parameter, $t$, and part of the last statement has a more quantitative version:

$$\max_{|s| < k} |D_s^r D_t^\nu f_\omega(s, t)| < \text{Const} \max_{|s| < k + r} (1 + |x|)^\nu |D_s^r D_t^\nu f(x, t)|.$$
Here \( v \) and \( y \) are integers depending on the dimension \( n \); \( k \) is an arbitrary non-negative integer; the maximum on the left is taken over all \( s \in \mathbb{R}^1 \) while the one on the right is taken over all \( x \in \mathbb{R}^n \). Both maxima may also be taken over the same closed \( t \) interval. « Const. » depends only on \( n \).

We are now ready to treat the case of a strictly hyperbolic operator with constant coefficients in \( n \)-space dimensions. Let \( L = P(D_{x_1}, \ldots, D_{x_n}, D_t) \) be such an operator, and \( D \) a bounded domain in \( \mathbb{R}^n \). For each (spacial) direction \( \omega \) (i.e., unit vector in \( x \)-space) let \( \tilde{\lambda}_\omega \) denote the reciprocal of the slowest speed of propagation in the \( \omega \) direction. Let \( d_\omega \) be the diameter of the \( \omega \)-projection of \( D \), i.e., \( D \) projected on a line parallel to the \( \omega \) vector. Let \( T_0 = \sup_{\omega} \tilde{\lambda}_\omega \cdot d_\omega \).

**Theorem 2.** Given assigned \( C^\infty \) Cauchy data at \( t = 0 \) on \( \overline{D} \) (which can be extended as \( C^\infty \) functions in a neighborhood of \( \overline{D} \)); then for each \( T_1 > T_0 \) there exists a solution \( v \) to \( Lv = 0 \) in \( \overline{D} \times \{ t > 0 \} \) assuming the prescribed Cauchy data in \( D \) and vanishing in \( \overline{D} \times \{ t > T_1 \} \).

(Note: \( \overline{D} = \text{closure of } D \).)

**Proof.** Let us extend the Cauchy data so as to be \( C^\infty \) and have support in an \( \epsilon \) neighborhood of \( D \), and let us solve the resulting Cauchy problem in all of \( \mathbb{R}^n \times (t > 0) \), calling the solution \( u \). Let \( T_1 \) be as in the statement of the theorem and let, as before,

\[
L(u \varphi(t)) = f,
\]

where \( \varphi(t) \) is a \( C^\infty \) function such that

\[
\varphi(t) = \begin{cases} 
1 & \text{for } t < \frac{1}{2} T_1 - \epsilon, \\
0 & \text{for } t > \frac{1}{2} T_1 + \epsilon.
\end{cases}
\]

For each unit vector \( \omega \) let

\[
I_\omega = \{ s \in \mathbb{R}^1 : s = x \cdot \omega, x \in \overline{D} \}
\]

i.e., the \( \omega \) projection of \( \overline{D} \), and let \( m_\omega \) denote the mid point of \( I_\omega \). Let \( \Psi(s) \) be \( C^\infty(\mathbb{R}^1) \) such that

\[
\Psi(s) = \begin{cases} 
1 & \text{for } s < -1, \\
0 & \text{for } s > 1.
\end{cases}
\]
and set \( \Psi_s(s) = \Psi_{m_s}(s) = \Psi((s - m_s)/c) \). Now let

\[
f(x, t) = \int f_\omega(s, t) d\omega \quad (s = x \cdot \omega)
\]

be the plane wave decomposition of \( f \), and let

\[
f_{sL}(s, t) = f_\omega(s, t) \Psi_s(s),
\]

\[
f_{sR}(s, t) = f_\omega(s, t)(1 - \Psi_s(s)).
\]

We note that

\[ P(D_x, D_t) U(x, \omega, t) = P_\omega(D_x, D_t) U(s, t), \]

where \( P_\omega(D_x, D_t) \) is a strictly hyperbolic operator in \( s, t \) space depending in a \( C^\infty \) way of \( \omega \), having the strict hyperbolicity maintained uniformly with respect to \( \omega \). As a matter of fact, the polynomial \( P_\omega(\sigma, \tau) \) is nothing else but the polynomial \( P(\xi, \tau) \) restricted to the 2-plane in \( \xi - \tau \) space which contains the \( t \)-axis and the \( \omega \) vector, with \( \sigma = \omega \cdot \xi \).

From the case \( n = 1 \) we know that the operator \( P_\omega(D_x, D_t) = L_\omega \) has fundamental solutions

\[
G_{sL}(s, t) \quad \text{and} \quad G_{sR}(s, t)
\]

with supports contained in the sets

\[ |t/s| < \tilde{\lambda}_s \quad s < 0, \quad (s > 0) \]

(respectively), where \( \tilde{\lambda}_s \) is the maximal reciprocal propagation speed for \( L_\omega \).

Letting

\[
U_{sL} = G_{sL} * f_{sL}, \quad U_{sR} = G_{sR} * f_{sR},
\]

(with convolutions carried out in \( s - t \) space) we notice that these functions are \( C^\infty \) in \( s, t, \omega \) and satisfy

\[ L_\omega U_{sL} = f_{sL} \quad \text{(same with } s \text{ R } s'), \]

hence

\[ P(D_x, D_t) U_{sL}(\omega \cdot x, t) = f_{sL}(\omega \cdot x, t) \]

(and the same with \( s \text{ R } s' \)), with the supports of \( U_{sL} \) and \( U_{sR} \) both con-
tained in a neighborhood \( S''_\varepsilon \) of the set
\[
S\varepsilon: |t - \frac{1}{2} T_1| < \varepsilon_m |x \cdot \omega - m_\omega|.
\]
Moreover \( S''_\varepsilon \) approaches \( S\varepsilon_m \) as \( \varepsilon \to 0 \). A moment's reflection shows that for \( \varepsilon \) sufficiently small (which we now so choose), the set \( S''_\varepsilon \) is bounded away from the sets
\[
\{ |x \cdot \omega - m_\omega| < \frac{1}{4} d_\omega, \ t = 0 \},
\]
\[
\{ \quad , \ t = T_1 \},
\]
and hence from the sets
\[
(\ast)
\]
\[
\overline{D} \times (t = 0)
\]
and
\[
(\ast\ast)
\]
\[
\overline{D} \times (t = T_1),
\]
which are contained in the above sets (respectively). It follows that the supports of \( U_{\omega_L} \) and \( U_{\omega_R} \) both are bounded away from \( (\ast) \) and \( (\ast\ast) \), uniformly in \( \omega \). Hence, if we define
\[
U(t, x) = \int_\omega \left( U_{\omega_L}(x \cdot \omega, t) + U_{\omega_R}(x \cdot \omega, t) \right) d\omega,
\]
\( U \) will be a \( C^\infty \) solution to
\[
LU = \int_\omega f_\omega d\omega = f
\]
vanishing in neighborhood of \( (\ast) \) and \( (\ast\ast) \). Defining (as in the case \( n = 1 \))
\[
v = u \varphi - U
\]
we see that \( v \) satisfies all requirements of the theorem.

5. - The parabolic case \( n = 1 \).

To motivate our procedure, we rederive Frank Jones' result for the one dimensional heat equation along somewhat different lines. We wish to imitate the procedure followed in the hyperbolic case. We begin by
solving the traditional initial value problem, prescribing as initial data \( \delta(x) \),
thus arriving at the standard fundamental solution \( G_+(x, t) \), with support
in \( t > 0 \). Our next step would be to multiply \( G_+ \) by a cut-off function \( \varphi(t) \),
this time choosing \( \varphi(t) \) equal to one for \( |t| < \epsilon \) and equal to zero for \( |t| > 2\epsilon \).
To proceed with the analogy, we then must look for the analogue of \( G_L \)
and \( G_R \) with supports in neighborhoods of the negative and positive \( x \) axis
respectively. Such fundamental solutions do not exist as distributions, but
they do exist formally (see Ehrenpreis [1], p. 209)

\[
\sum_{i=0}^{\infty} \delta^{(i)}(t) \frac{x_+^{2i+1}}{(2i+1)!} 
\]

where \( \delta^{(i)} \) is the \( i \)-th derivative of \( \delta(t) \) and \( x_+ = \max(x, 0) \). This « fundamental solution » has support in \( (x > 0, t = 0) \) and may be interpreted as a
functional in a space of functions satisfying a Gevrey condition of index
\(< 2 \) in \( t \). A second « fundamental solution » having support on \( (x < 0, t = 0) \),
may be obtained by replacing \( x_+ \) by \( x_- = |x| - x_+ \). Let us hasten to add
that the existence of these generalized fundamental solutions is nothing
but a reformulation of the observation of Holmgren [5] to the effect that
the Cauchy problem for the heat equation in one space dimension can be
solved if Cauchy data in an appropriate Gevrey class is prescribed on the
\( t \) axis.

**Digression on the spaces \( \gamma^\theta \).** From the above discussion it becomes clear
that we should choose the cut-off function \( \varphi(t) \) in an appropriate Gevrey
class. Instead we choose to work with a variation of the \( \gamma^\theta \) classes. (For
these spaces and their properties see Hörmander [6], p. 146). We say \( \varphi(t) \in \in C^\infty_0(t; |t| < 1) \) belongs to \( \gamma^\theta_1 \equiv \gamma^\theta_1(t) \) if for every \( \theta > 0 \) there is a constant \( C_\theta \)
such that

\[
|D^j_\theta \varphi(t)| < C_\theta \theta^j, \quad j = 1, 2, 3, \ldots,
\]

We define \( \gamma^\delta_1 = \gamma^\delta_1(x, t) \) as the space of continuous functions \( f(x, t) \) belonging
for each \( x \) to \( \gamma^\delta_1(t) \) with bounds uniform in \( x \), and \( \gamma^\delta_2 = \gamma^\delta_2(x, t) \) the space of
continuous functions belonging for each \( x \) to \( \gamma^\delta_1(t) \) with bounds uniform
in \( x \) on bounded subsets of \( x \)-space. We shall always pick \( 1 < \delta < 2 \).

We now come back to the parabolic equation

\[
L u = A u - u, = 0,
\]

where \( A \) is a one dimensional differential operator with constant coefficients
of order \( m > 1 \). (The procedure with some modifications, also works with
variable coefficients depending on $x$ only). We assume that the highest
order term in $A$ is of even order and has positive coefficients. Under these
conditions $L$ has a fundamental solution $G_+$ with support in $t > 0$ which is
analytic for $t > 0$. We now proceed with the construction of formal fun-
damental solutions $G_L$, $G_R$ analogous to the ones exhibited for the heat
equation. Denoting $\partial/\partial t$ by $T$, we set

$$G_R = \sum_{i=0}^{\infty} T^i \delta(t) a^{-i-1}_R(x)$$

(with $G_L$ defined similarly), where $a^{-i-1}_R(x)$ ($a^{-i-1}_L(x)$) is the unique solu-
tion to $A^{i+1}a = \delta(x)$ with support on $x > 0$ ($x < 0$). A simple computation
verifies that these are indeed formal fundamental solutions.

As a convenience we shall from now on always choose $\varepsilon$ so that $0 < \varepsilon < \frac{1}{2}$. Having chosen the cut-off function $\varphi(t) \in \chi_0$ with $\varphi(t) = 0$ for $|t| > 2\varepsilon$ and $= 1$ for $|t| < \varepsilon$, we choose $\Psi(x) \in C^\infty$ equal to one for $x < -1$ and zero for
$x > 1$, set

$$f_L = \Psi(x) L(G_+ \cdot \varphi(t)),$$

$$f_R = (1 - \Psi(x)) L(G_+ \cdot \varphi(t)),$$

and notice that these functions belong to $\gamma_\varphi^\dagger$ (together with all their $x$ deri-
vatives) and have their supports in the sets (respectively)

$$\{ \varepsilon < t < 2\varepsilon, x < 1 \},$$

$$\{ \vdots, x > 1 \}.$$

We would like to define

$$U_L = G_L \ast f_L$$

and

$$U_R = G_R \ast f_R,$$

provided we can attribute a meaning to these expressions, and then proceed
as in the hyperbolic case. Formally,

$$U_R = \sum_{i=0}^{\infty} \int_0^{\frac{x+1}{2\varepsilon}} \frac{\partial^i}{\partial t^i} f_R(t, x-y) a^{-i-1}_L(y) \, dy,$$

$$U_L = \sum_{i=0}^{\infty} \int_0^{\frac{x-1}{2\varepsilon}} \frac{\partial^i}{\partial t^i} f_L(t, (x-y)) a^{-i-1}_L(y) \, dy.$$
For $y$ in a bounded interval $I$, we have the estimate

$$|a_{R}^{i-1}(y)| < K^i_1 \frac{y^{n^i}}{(m^i)!} \quad (y > 0)$$

(similarly for $a_L$ and $y < 0$). For $f_{R,L}$ we have the estimate

$$\left| \frac{\partial^i}{\partial t^i} f_{R,L} \right| < C_{a,t} \theta^i(t)^{m^i} , \quad (0 < \theta)$$

with a similar estimate holding for any finite number of $x$ derivatives of $f$ replacing $f$.

An application of Stirling's formula now insures the absolute and uniform convergence of the series defining $U_{R,L}$ in any finite interval $I$. Furthermore, the series may be differentiated with respect to $x$ or $t$ any number of times without affecting the convergence, thus justifying the hitherto formal relations

$$L(G_{a,R} * f_{a,R}) = f_{a,R},$$

Setting $v = qG_+ - U_+ - U_-$ we see that $Lv = 0$ for $t > 0$, $v$ agrees with $G_+$ for $t < \varepsilon$ and vanishes for $t > 2\varepsilon$, hence is our desired fundamental solution. Let us note that an alternative to the use of $G_+$ and $G_R$ would have been to apply Theorem 5.73 of Hörmander [6]. However then the construction would not generalize to variable coefficients (for $n = 1$).

6. – The parabolic case $n > 1$.

If $Lu = A(D_x)u - u$, we assume that $L$ is parabolic in the sense that $\text{Re} \, A_n(\xi) < -c|\xi|^m$, $A_n$ being the principal part of $A$. Under these conditions there exists a fundamental solution $G(x, t)$ with support in $t > 0$, analytic in $x$ and $t$ for $t > 0$. We shall use the following

**Estimates for the standard fundamental solution:**

$$\text{Max}_{0 < t < \xi < t_1} |D^i_t G(x, t)| < CK^i j! h(x),$$

where $h(x) \to 0$ faster than any negative power of $|x|$ as $|x| \to \infty$, and where $C$ and $K$ may depend on $t_0$ and $t_1$. $D^k_x D^i_t G$ may be estimated similarly with $C$ replaced by $C_k$. To see this we examine the proof (see for ex. [3])
or \([4]\) of the estimate

\[
|G(x, t)| < \text{const} \exp \left[ -\alpha |x|^h \right] \quad (\alpha > 0)
\]

valid for \(0 < t_0 < t < t_1\), and some \(h > 1\). We notice that the same estimate holds for complex \(t\) contained in a disk

\[
|t - t_0| < \delta \quad \text{with} \quad t_0 < t_1 < t_2,
\]

and \(\delta\) sufficiently small, depending on \(t_0, t_1\) but not on \(t_2\). Standard estimates for derivatives of complex analytic function then give the first estimate. The same argument can be applied to \(D_x^hG\), yielding different constants.

We now proceed as before.

Using Stirling's formula we see that \(f(x, t) = L(\varphi(t)|G)\) satisfies an estimate

\[
\max_{0 < t_0 < t < t_1} |D_x^1 D_s^h(\cdot)| < C_{\theta, h} \theta \delta^{-h} h(x)
\]

for \(\theta > 0\), with \(h(x) \to 0\) faster than any negative power of \(|x|\).

Using the estimates for \(f_\omega(s, t)\) and its derivatives stated in the first part of section 4, we see that \(f_\omega(s, t)\) and each \(s\) derivative belongs to \(\gamma_\omega(s, t)\), uniformly in \(\omega\). The same may be said for

\[
f_{\omega L} = \Psi(s) f_\omega(s, t) \quad \text{and} \quad f_{\omega R} = (1 - \Psi(s)) f_\omega(s, t)
\]

where \(\Psi(\cdot)\) is the same \(C^\infty\) cut-off function as in the case \(n = 1\).

The operator

\[
L_\omega = A_\omega(D_s) - \frac{\partial}{\partial t},
\]

defined (as in section 4) by

\[
LU(\omega \cdot x, t) = L_\omega U(s, t) \quad (\omega \cdot x = s)
\]

is again parabolic, uniformly with respect to \(\omega\), and varies smoothly with \(\omega\). We form

\[
U_{\omega L} = G_{\omega L} * f_{\omega L} \quad \text{and} \quad U_{\omega R} = G_{\omega R} * f_{\omega R}
\]

as in the case \(n = 1\). The series involved will converge absolutely and uniformly on bounded \(s\) intervals and uniformly in \(\omega\), and the same holds
for the differentiated series. We use here the fact that because of the para-
bolicity assumption, the operator $A_{\omega}(D_x)$ has its highest order coefficient bounded away from zero, uniformly in $\omega$, from which the estimate for $a_{\omega,b}^{-1}(\tau)$ (analogous to the one in section 5) can be made uniform in $\omega$.

The desired fundamental solution is then

$$v = \varphi(t)G - \int_\omega (U_{\omega L} + U_{\omega R}) \, d\omega.$$  

We have thus proved

**Theorem 3.** Let $L = A(D_x) - \partial/\partial t$ be a parabolic operator (as defined earlier in this section) in $\mathbb{R}^n \times (-\infty < t < \infty)$. Then given $\epsilon > 0$ there exists a fundamental solution with support in the strip $0 < t < \epsilon$, which is $C^\infty$ away from the origin.

The following theorem and corollary were proved by Frank Jones for the heat equation [7]. As he remarks, the proof remains valid for more general parabolic equations having fundamental solutions with support in $0 < t < \epsilon$. Applying his proof to our operator $L$ we obtain

**Theorem 4.** Let $L = P(D_x) - \partial/\partial t$ be parabolic in the sense of this section. Let $f$ belong to $\mathcal{D}'(\mathbb{R}^{n+1})$ and suppose $\text{supp } f \subset \mathbb{R}^n \times [a, b)$. Then there exists $u$ in $\mathcal{D}'(\mathbb{R}^{n+1})$ such that $Lu = f$ and $\text{supp } u \subset \mathbb{R}^n \times [a, b]$.

**Corollary.** Let $g$ be a continuous function on $\mathbb{R}^n$ and $\epsilon > 0$. Then there exists a continuous function $u$ on $\mathbb{R}^n \times [0, \infty)$ such that

\begin{align*}
Lu &= 0 \quad \text{on } \mathbb{R}^n \times (0, \infty), \\
u(x, 0) &= g(x) \quad \text{for all } x \in \mathbb{R}^n, \\
u(x, t) &= 0 \quad \text{for } x \in \mathbb{R}^n \text{ and } t > \epsilon.
\end{align*}

**BIBLIOGRAPHY**

