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Some critical point theorems and applications to semilinear elliptic partial differential equations

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Some Critical Point Theorems and Applications to Semilinear Elliptic Partial Differential Equations (*).

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dedicated to Jean Leray

Introduction.

Let E be a real Banach space and I a continuously differentiable map from E to $\boldsymbol{R}$, i.e. $I \in C^1(E, \boldsymbol{R})$. We say I satisfies the Palais-Smale condition (PS) if any sequence (u_m) such that $I(u_m)$ is bounded and $I'(u_m) \to 0$ is precompact. In [1] and [2], by imposing various qualitative conditions on I near 0 and ∞, Ambrosetti and the author obtained several results concerning the existence of critical points of I and applied them to semilinear elliptic boundary value problems to get existence theorems in that setting.

The purpose of this paper is to extend the theory of [1] and [2]. To be more precise, let $B_r = \{x \in E \big| \|x\| < r\}$. It was shown in [1, Theorem 2.1] that if $I(0) = 0$ and

1) There are constants $\varrho, \alpha > 0$ such that $I > 0$ in $B_\varrho \setminus \{0\}$ and $I \geqslant \alpha$ on ∂B_ϱ, and

2) There is an $e \in E$, $e \neq 0$ such that $I(e) = 0$,

then I has a critical value $c \geqslant \alpha$.

It was further shown in [1, Theorem 2.21], [2, Theorem 3.37] that if I is even and 2) is replaced by the requirement that I be negative at infinity

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in an appropriate sense (see condition (I_3) of § 1), then 1) can be weakened to hold in $B_\varrho \cap \hat{E}$ where $\hat{E}$ is a subspace of E of finite codimension. Moreover for this case I has an unbounded sequence of positive critical values.

Our main results here show how one can still obtain critical points of I under weakened versions of 1) without requiring that I be even. The abstract critical point theorems will be given in § 1 and some applications to semilinear elliptic partial differential equations will be carried out in § 2.

1. – The critical point theorems.

In this section we shall extend the results of [1] and [2] mentioned in the Introduction. Some additional variants will be presented for the special case of $E = \mathbf{R}^n$. Lastly a case where we only require 1) for B_ϱ intersected with a finite dimensional subspace of E will be treated. As in [1], [2], our arguments are based on minimax characterizations of critical values of I.

Our main result is

THEOREM 1.1. *Let E be a real Banach space and let $I \in C^1(E, \mathbf{R})$ and satisfy* (PS). *Suppose that $E = E_k \oplus \hat{E}$ where E_k is k dimensional and I satisfies*

(I_1) $I|_{E_k} \leqslant 0$,

(I_2) *There are constants $\varrho, \alpha > 0$ such that $I \geqslant 0$ in $B_\varrho \cap \hat{E}$ and $I \geqslant \alpha$ on $\partial B_\varrho \cap \hat{E}$.*

(I_3) *For each finite dimensional subspace $\tilde{E}$ of E, there is a constant $R = R(\tilde{E})$ such that $I \leqslant 0$ on $\tilde{E} \setminus B_{R(\tilde{E})}$.*

Then I has a positive critical value, c, characterized by

$$c = \inf_{h \in \Gamma} \max_{u \in \bar{B}_r \cap E_{k+1}} I(h(u)) \tag{1.2}$$

where $E_{k+1} \equiv E_k \oplus \operatorname{span}\{\varphi\}$ for some fixed $\varphi \in \hat{E} \setminus \{0\}$, $r = R(E_{k+1})$, and

$$\Gamma = \{h \in C(\bar{B}_r \cap E_{k+1}, E) | h(u) = u \quad \text{if } I(u) \leqslant 0\}\,.$$

A finite dimensional version of Theorem 1.1 was given in [5] and it motivated this paper. Two preliminary results are required for the proof of Theorem 1.1. The first is a standard lemma from the calculus of variations. Let $A_s = \{u \in E | I(u) \leqslant s\}$ and $K_s = \{u \in E | I(u) = s \text{ and } I'(u) = 0\}$.

LEMMA 1.3. *Let $I \in C^1(E, \boldsymbol{R})$ and satisfy* (PS). *Let $c \in \boldsymbol{R}$, $\mathcal{O}$ be any neighborhood of K_c, and $\bar{\varepsilon} > 0$. Then there is an $\varepsilon \in (0, \bar{\varepsilon})$ and an $\eta \in C([0,1]\times E, E)$ such that*

1) $\eta(t, x) = x$ *if* $I(x) \notin [c - \bar{\varepsilon}, c + \bar{\varepsilon}]$,
2) $\eta(1, A_{c+\varepsilon}\setminus\mathcal{O}) \subset A_{c-\varepsilon}$,
3) *If* $K_c = \emptyset$, $\eta(1, A_{c+\varepsilon}) \subset A_{c-\varepsilon}$.

See e.g. [2] or [3] for a proof of this lemma. Next we need a topological lemma. For $j \leqslant m$, let $\boldsymbol{R}^j = \{x = (x_1, \ldots, x_m) | x_i = 0, j + 1 \leqslant i \leqslant m\}$ and $(\boldsymbol{R}^j)^\perp$ the orthogonal complement of $\boldsymbol{R}^j$ in $\boldsymbol{R}^m$.

LEMMA 1.4. *Let $b_R = \boldsymbol{R}^{k+1} \cap \bar{B}_R$ and let $b_R^+ = \{x \in \bar{B}_R | x_{k+1} \geqslant 0\}$. If $g \in C(b_R^+, \boldsymbol{R}^m)$, $m > k$, $\varrho < R$, and there is a homotopy $G \in C([0,1]\times \partial b_R^+, \boldsymbol{R}^m\setminus(\partial B_\varrho \cap (\boldsymbol{R}^k)^\perp)$ such that $G(0, x) = x$ and $G(1, x) = g(x)$, then $g(b_R^+) \cap \cap\, \partial B_\varrho \cap (\boldsymbol{R}^k)^\perp \neq \emptyset$.*

PROOF. A proof of Lemma 1.4 due to E. Fadell using intersection theory [4, p. 197] can be found in [5, Lemma A-2]. For the convenience of the reader, we include it here. The problem is normalized by taking $R = 1$ and $\varrho < 1$. Let $D = \bar{B}_\varrho \cap (\boldsymbol{R}^k)^\perp$, $\hat{D} = (\partial B_\varrho) \cap (\boldsymbol{R}^k)^\perp$, $b^+ = b_1^+$, and $\partial b^+ = \{x \in b^+ | x_{k+1} = 0 \text{ or } |x| = 1\}$. The intersection pairing:

$$H_{m+k}(D, \hat{D}) \times H_k(\partial b^+) \to \boldsymbol{Z}$$

yields $+1$ as intersection number for appropriately chosen generators in the homology groups above. Furthermore naturality yields the diagram

$$\begin{tikzcd} H_{m+k}(D, \hat{D}) \times H_k(\partial b^+) \arrow[r] \arrow[d, "id \times j_*"] & \boldsymbol{Z} \\ H_{m+k}(D, \hat{D}) \times H_k(\boldsymbol{R}^m\setminus\hat{D}) \arrow[ur] & \end{tikzcd}$$

where $j\colon \partial b^+ \subset \boldsymbol{R}^m\setminus\hat{D}$ is inclusion. If $g \in C(b^+, \boldsymbol{R}^m\setminus\hat{D})$ and G is as in the statement of the lemma, then j_* would be the trivial homomorphism and the intersection number would be 0, a contradiction.

These preliminaries being completed, we can now give the

PROOF OF THEOREM 1.1. Assume for the moment that $c \geqslant \alpha$. If c is not a critical value of I, we can invoke Lemma 1.3 with $\bar{\varepsilon} = \alpha/2$. Let $h \in \Gamma$ be such that

$$\max_{u \in \bar{B}_r \cap E_{k+1}} I(h(u)) \leqslant c + \varepsilon\ . \tag{1.5}$$

Since $\eta(1, h) \in C(\bar{B}_r \cap E_{k+1}, E)$ and if $I(u) \leqslant 0$, then $\eta(1, h(u)) = \eta(1, u) = u$ by 1) of Lemma 1.3 and our choice of $\bar{\varepsilon}$, it follows that $\eta(1, h) \in \Gamma$. But then by (1.5) and 3) of Lemma 1.3,

$$\max_{u \in \bar{B}_r \cap E_{k+1}} I\big(\eta(1, h(u))\big) \leqslant c - \varepsilon \tag{1.6}$$

contrary to the definition of c.

To complete the proof, we must show $c \geqslant \alpha$. Using (I_2), it suffices to show $h(\bar{B}_r \cap E_{k+1}) \cap \partial B_\varrho \cap \hat{E} \neq \emptyset$ for all $h \in \Gamma$. Let $K = h(\bar{B}_r \cap E_{k+1})$. Since K is compact, by an approximation lemma of Leray-Schauder [9], for all $\varepsilon > 0$, there exists a finite dimensional subspace $F_\varepsilon \subset E$ and a mapping $i_\varepsilon \in C(K, F_\varepsilon)$ such that $\|i_\varepsilon(u) - u\| \leqslant \varepsilon$ for all $u \in K$. We can assume $F_\varepsilon \supset E_{k+1}$. Let $h_\varepsilon = i_\varepsilon \circ h$. Then $h_\varepsilon \in C(\bar{B}_r \cap E_{k+1}, F_\varepsilon)$ and $h_\varepsilon(u) \to h(u)$ as $\varepsilon \to 0$ uniformly for $u \in \bar{B}_r \cap E_{k+1}$. Observe that if $u \in (\partial B_r \cap E_{k+1}) \cup E_k$, by (I_1) and (I_3) we have

$$\|h_\varepsilon(u) - h(u)\| = \|h_\varepsilon(u) - u\| \leqslant \varepsilon . \tag{1.7}$$

Since $E = E_k \oplus \hat{E}$, $u \in E$ implies $u = v + w$, where $v \in E_k$, $w \in \hat{E}$. We can assume $\|u\|_E = (\|v\|^2_{E_k} + \|w\|^2_{\hat{E}})^{\frac{1}{2}}$. Thus if $u \in (\partial B_r \cap E_{k+1}) \cup E_k$, it follows that

$$\|u - (\partial B_\varrho \cap \hat{E})\| \geqslant \beta \equiv \min(r - \varrho, \varrho) . \tag{1.8}$$

Choosing $\varepsilon < \beta$, identifying F_ε with $\boldsymbol{R}^m$, E_k, E_{k+1} with $\boldsymbol{R}^k$, $\boldsymbol{R}^{k+1}$, and defining $G(t, u) = th_\varepsilon(u) + (1 - t)u$, it follows from (1.7)-(1.8) that G satisfies the hypotheses of Lemma 1.4. Hence $h_\varepsilon(\bar{B}_r \cap E_{k+1}) \cap \partial B_\varrho \cap \hat{E} \neq \emptyset$. Consequently there exists $u_\varepsilon \in \bar{B}_r \cap E_{k+1}$ such that $h_\varepsilon(u_\varepsilon) \in \partial B_\varrho \cap \hat{E}$. As $\varepsilon \to 0$, along a subsequence we have $u_\varepsilon \to u \in \bar{B}_r \cap E_{k+1}$ and

$$\|h_\varepsilon(u_\varepsilon) - h(u)\| \leqslant \|h_\varepsilon(u_\varepsilon) - h(u_\varepsilon)\| + \|h(u_\varepsilon) - h(u)\| \to 0$$

as $\varepsilon \to 0$. Therefore $h(u) \in \partial B_\varrho \cap \hat{E}$ and the proof is complete.

REMARK 1.9. If E is infinite dimensional, in general I will not be bounded from above in contrast to the finite dimensional case where this is a consequence of (I_3). By (I_2), I then has a positive maximum $\bar{c}$ in B_r. In general $c < \bar{c}$. However if $c = \bar{c}$, I possesses infinitely many critical points. Indeed (1.2) then shows I achieves its maximum on $h(B_r \cap E_{k+1})$ for each $h \in \Gamma$ and this implies I has infinitely many distinct critical points corresponding to $c = \bar{c}$. Our next result gives another characterization of critical values of I in the finite case and a more precise description of the degenerate situation.

THEOREM 1.10. *Let $I \in C^1(\boldsymbol{R}^m, \boldsymbol{R})$ and satisfy (I_1)-(I_3). Then I possesses at least two positive critical values characterized by*

$$\bar{c} = \max_{u \in B_r} I(u)\,, \qquad r = r(\boldsymbol{R}^m)$$

and

$$b = \sup_{\substack{A \in M \\ \mathrm{cat}_M A = 2}} \min_{u \in A} I(u)\,, \tag{1.11}$$

where $M = \bar{B}_r \cap (\boldsymbol{R}^m \setminus \boldsymbol{R}^k)$. If $b = \bar{c}$, $\mathrm{cat}_M K_b = 2$.

Before proving Theorem 1.10, a few remarks are in order. The notation $\mathrm{cat}_M A$ refers to the Ljusternik-Schnirelman category of the subset A of M. In (1.11) as admissible A we only consider $A \subset M$ with A closed in $\boldsymbol{R}^m$. By definition $\mathrm{cat}_M A = 1$ if A is contractible to a point in M and $\mathrm{cat}_M A = j$ if j is the least integer such that A can be covered by j closed sets A_i with $\mathrm{cat}_M A_i = 1$, $1 \leqslant i \leqslant j$. It is clear that any admissible $A \subset M$ is homotopic to a subset of the unit sphere S^{m-k} in $(\boldsymbol{R}^k)^\perp$. Moreover $\mathrm{cat}_M S^{m-k} = 2$ since if $\mathrm{cat}_M S^{m-k} = 1$, any homotopy of S^{m-k} to a point in M would induce a homotopy of S^{m-k} to a point in S^{m-k}. However as is well known (see e.g. [6]), $\mathrm{cat}_{S^{m-k}} S^{m-k} = 2$ so no such homotopy can exist.

PROOF OF THEOREM 1.10. Define

$$c_i = \sup_{\substack{A \subset M \\ \mathrm{cat}_M A \geqslant i}} \min_{u \in A} I(u)\,, \qquad i = 1,2\,. \tag{1.12}$$

Then $c_1 = \bar{c}$ since we can take $A = \{x\}$ for any $x \in M$ and $c_2 = b \leqslant \bar{c}$. Note further that by (I_2), $I > 0$ on ϱS^{m-k} and therefore $b > 0$. (I_3) implies I satisfies (PS) on the set where $I > 0$. A slightly strengthened version of Lemma 1.3 and a standard argument (similar to the first paragraph of the proof of Theorem 1.1) shows that b is a critical value of I and if $\bar{c} = b$, $\mathrm{cat}_M K_b = 2$. (See e.g. [6], [7], or [2]).

REMARK 1.13. We believe that b as defined in (1.11) equals c defined in (1.2).

An examination of the proof of Theorem 1.10 shows that the argument in fact gives the following result which interchanges the finite dimensional and finite codimensional hypotheses (I_1) and (I_2) of Theorem 1.1.

THEOREM 1.14. *Let E be a real Banach space, $I \in C^1(E, \boldsymbol{R})$ and satisfy* (PS).

Suppose that $E = E_k \oplus \hat{E}$ *where* $k \geqslant 1$, E_k *is* k*-dimensional, and* I *satisfies*

(I_4) $I|_{\hat{E}} \leqslant 0$,

(I_5) *There are constants* ϱ, $\alpha > 0$ *such that* $I \geqslant 0$ *in* $B_\varrho \cap E_k$ *and* $I \geqslant \alpha$ *on* $\partial B_\varrho \cap E_k$,

(I_6) I *is bounded from above.*

Then I *possesses at least two positive critical values characterized by*

$$b_i = \sup_{\substack{A \in M \\ \operatorname{cat}_M A \geqslant i}} \min_{u \in A} I(u) \qquad i = 1,2 \tag{1.15}$$

where A *is compact in* E *and* $M = E \setminus \hat{E}$. *Moreover if* $b_1 = b_2 = d$, $\operatorname{cat}_M K_d \geqslant 2$.

PROOF. Immediate from that of Theorem 1.10 and the remark preceding it.

2. – Applications to partial differential equations.

In this section we shall show how Theorems 1.1 and 1.14 can be employed to obtain existence theorems for semilinear elliptic boundary value problems. Consider

$$\begin{cases} Lu \equiv -\sum_{i,j=1}^{n} \left(a_{ij}(x) u_{x_i}\right)_{x_j} + c(x)u = a(x)u + p(x, u), & x \in \Omega \\ u = 0, \; x \in \partial\Omega \end{cases} \tag{2.1}$$

where Ω is a bounded domain in $\boldsymbol{R}^n$ with a smooth boundary, L is uniformly elliptic in $\bar{\Omega}$ with smooth coefficients, $c \geqslant 0$ in Ω, a is smooth and positive in $\bar{\Omega}$, and p is smooth and satisfies

$$(p_1) \qquad |p(x, z)| \leqslant a_1 + a_2 |z|^s, \qquad s < \frac{n+2}{n-2}, \qquad n > 2,$$

$$(p_2) \qquad p(x, z) = o(|z|) \quad \text{at } z = 0.$$

There are constants $M > 0$ and $\theta \in (0, \frac{1}{2})$ such that

$$(p_3) \qquad 0 < P(x, z) = \int_0^z p(x, t)\, dt \leqslant \theta z p(x, z) \qquad \text{for } |z| \geqslant M.$$

If $n \leqslant 2$, (p_1) can be considerably improved. See e.g. § 3 of [1].

Set

$$I(u) = \int_\Omega \left[\frac{1}{2}\left(\sum_{i,j=1}^{n} a_{ij}(x) u_{x_i} u_{x_j} + c(x) u^2 - a(x) u^2\right) - P(x, u)\right] dx .$$

Formally critical points of I in $E = W_0^{1,2}(\Omega)$ are weak solutions of (2.1). The smoothness of L, a, p, and (p_1) and standard regularity results then imply such weak solutions are smooth functions. Thus we focus our attention on finding critical points of I in E.

Consider the linear eigenvalue problem

$$\text{(2.3)} \qquad \begin{cases} Lv = \lambda a v , & x \in \Omega \\ v \;= 0 , & x \in \partial\Omega . \end{cases}$$

As is well known, (2.3) possesses an unbounded sequence of eigenvalues $0 < \lambda_1 < \lambda_2 \leqslant \ldots \leqslant \lambda_m \to \infty$ as $m \to \infty$ with corresponding eigenfunctions $v_1, \ldots, v_m, \ldots$. It was shown in [1] that if p satisfies (p_1)-(p_3) and $\lambda_1 > 1$, (2.1) possesses a positive solution (i.e. $u > 0$ in Ω) and a negative solution. If $\lambda_1 \leqslant 1$, the argument of [1] fails unless $p(x, z)$ is odd in z. We will show:

THEOREM 2.4. *If p satisfies (p_1)-(p_3) and*

$$(p_4) \qquad zp(x, z) \geqslant 0 \quad \textit{for } x \in \Omega,\ z \in \mathbf{R},$$

then (2.1) *possesses a nontrivial solution.*

PROOF. Because of the result just mentioned, we can assume $\lambda_1 \leqslant 1$. In fact suppose $\lambda_k \leqslant 1 < \lambda_{k+1}$. Let $E_k = \operatorname{span}\{v_1, \ldots, v_k\}$ and $\hat{E} = E_k^{\perp}$, the orthogonal complement of E_k. By (p_4), $I \leqslant 0$ on E_k. It was further shown in [1] that (p_1)-(p_3) imply that $I \in C^1(E, \mathbf{R})$ and satisfies (I_2), (I_3) and (PS). Hence Theorem 2.4 follows immediately from Theorem 1.1.

REMARK 2.5. As was noted above for $\lambda_1 > 1$, (2.1) has a positive and a negative solution. In contrast we next show:

COROLLARY 2.6. *Under the hypotheses of Theorem* 2.4, *if $\lambda_1 \leqslant 1$,* (2.1) *does not have a positive* (*or negative*) *solution $u(x)$ unless $\lambda_1 = 1$, $u(x)$ is a multiple of $v_1(x)$ and $p(x, u(x)) \equiv 0$.*

PROOF. Suppose u is a positive solution of (2.1) with $\lambda_1 \leqslant 1$ Then.

$$\text{(2.7)} \qquad \int_\Omega v_1 Lu\, dx = \int_\Omega v_1\big(au + p(x, u)\big)\, dx = \int_\Omega uLv_1\, dx = \lambda_1 \int_\Omega auv_1\, dx .$$

Since we can assume $v_1 > 0$ in Ω, we have

$$0 \geqslant (\lambda_1 - 1) \int_\Omega a u v_1 \, dx = \int_\Omega v_1 p(x, u) \, dx \geqslant 0 \tag{2.8}$$

which is impossibile unless $p(x, z) \equiv 0$ for $0 \leqslant z \leqslant \max u(x)$ and $\lambda_1 = 1$ in which case $u \equiv \beta v_1$.

Next we illustrate how Theorem 1.14 can be used in similar situations. Consider (2.1) again where p satisfies (p_2) and e.g.

$$(p_4) \qquad 0 \geqslant z^{-1} p(x, z) \to -\infty \quad \text{as } |z| \to \infty .$$

No growth conditions are needed for this case since (p_4) implies that $az + p$ can be redefined to be independent of z for large $|z|$ so the modified nonlinearity is uniformly bounded. Moreover solutions of the modified equation are still solutions of (2.1). (See e.g. Theorem 3.4 of [2]). It is easily verified that if $\lambda_k < 1 \leqslant \lambda_{k+1}$, E_k, $\hat{E}$ are as in Theorem 2.4 and

$$J(u) = -\int_\Omega \left[\frac{1}{2}\left(\sum_{i,j=1}^n a_{ij}(x) u_{x_i} u_{x_j} + c(x) u^2\right) - \bar{P}(x, u)\right] dx$$

where $\bar{P}$ is the primative of the modification of $az + p$, then J satisfies the hypotheses of Theorem 1.14.

Since stronger results can be obtained for this problem using methods based on Leray-Schauder degree theory [8] we will not carry out the details here. However if L were replaced by a higher order divergence structure elliptic operator one could obtain results using Theorem 1.14 where degree theoretic methods would fail.

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