CHARLES J. AMICK

Steady solutions of the Navier-Stokes equations in unbounded channels and pipes

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 4, no 3 (1977), p. 473-513

<http://www.numdam.org/item?id=ASNSP_1977_4_4_3_473_0>
Steady Solutions of the Navier-Stokes Equations in Unbounded Channels and Pipes (*).

CHARLES J. AMICK(**)

dedicated to Jean Leray

Summary. – This paper is concerned with the steady flow of a viscous incompressible fluid in channels and pipes (in two and three dimensions respectively) which are cylindrical outside some compact set K. (See Figure 1. In this paper, « cylinder » is used to mean « strip » for the two-dimensional case.) The existence of a weak (or generalized) solution to the steady Navier-Stokes equations is shown for all Reynolds numbers $R < R_0$ (or equivalently, for all values of the kinematic viscosity $v > a$) where $R_0$ does not depend on that part of the domain within K but only on the cylindrical parts of the domain upstream and downstream. Moreover, $R_0$ is determined by a variational problem formulated on an infinite cylinder, and can be computed without difficulty for some cross-sections; indeed, the critical value $R_0$ (typically in the range 100 to 300) is familiar in the nonlinear theory of stability of parallel flows in the infinite cylinder in question.

1. – Introduction.

Since Leray’s fundamental paper [1] in 1933, the extensive work on steady solutions of the Navier-Stokes equations has centered on flow in two types of domain: interior and exterior domains in $\mathbb{R}^N$ (where $N = 2$ or 3 always). The two cases correspond to flow (a) inside a bounded domain $\Omega$, and (b) in the complement of a bounded set; in both cases, $\partial \Omega$, the boundary of $\Omega$, is compact. The present work concerns a class of domains of a third type distinguished from the other two by non-compact boundaries.

(*) Research supported by a National Science Foundation Graduate Fellowship (U.S.A.).

(**) Department of Applied Mathematics and Theoretical Physics, University of Cambridge.

Pervenuto alla Redazione il 1° Giugno 1976.
Let $\Omega \subset \mathbb{R}^n$ be a domain (an open, connected set). The steady flow of a viscous incompressible fluid with density $\rho = 1$, kinematic viscosity $\nu$, velocity $u = (u_1, \ldots, u_n)$, pressure $p$ and subject to an external force $f$ satisfies the steady Navier-Stokes equations:

\begin{align}
(1.1) & \quad -\nu \Delta u + (u \cdot \nabla)u = f - \nabla p, \\
(1.2) & \quad \text{div } u = 0
\end{align}

in $\Omega$.

In addition, we have the boundary condition

\begin{equation}
(1.3) \quad u = g \quad \text{on } \partial \Omega,
\end{equation}

where $g$ is required to satisfy $\int_{\partial \Omega} (g \cdot n) = 0$ when $\Omega$ is bounded because of (1.2).

We assume throughout this paper that the force $f$ is derivable from a scalar potential; that is, $f = -\nabla P$. A sufficient condition for this is for $\Omega$ to be simply-connected and $\nabla \times f = \text{curl } f = 0$ in $\Omega$. We now write $p$ for the effective pressure, previously $p + P$.

If $\Omega$ is bounded, the problem of solving (1.1) to (1.3) for $(u, p)$ is said to be of Type 1. If $\Omega$ is bounded, and the velocity is required to approach a given constant vector at infinity, the problem is said to be of Type 2.

The problems of Type 1 and 2 have been examined extensively in recent years (see [2], [3], and [4]) and the existence of solutions has been proved for all $\nu > 0$ and for suitably restricted data $f$, $g$ and $\partial \Omega$. An exception is the problem of Type 2 for $N = 2$, for which an additional restriction to sufficiently large values of $\nu$ is required. The existence proofs make crucial use of the fact that $\partial \Omega$ is compact in these two cases. Type 3 problems are those for which $\partial \Omega$ is not compact.

An example of such a domain $\Omega$ is a cylinder (when $N = 3$, the cross-section is not necessarily circular) which is of the form $\mathbb{R} \times A$, where $A$ is either an open interval $(-d, d)$, $d \in (0, \infty)$, for $N = 2$ or a simply-connected bounded domain in the plane for $N = 3$. For any such domain, there is a relatively simple solution, called Poiseuille flow, representing a velocity field parallel to the axis and the same in every cross-section. Let $N = 3$, let the axis of the cylinder be the $x_1$ axis, and set $g = 0$ (corresponding to a fixed pipe). Then there is a solution of (1.1) to (1.3) of the form $u = (u_1, 0, 0)$ and $p = -C x_1$, provided that

\begin{align}
(1.4) & \quad -\nu \left( \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right) u_1 = C \quad \text{in } A, \\
(1.5) & \quad u_1 = 0 \quad \text{on } \partial A.
\end{align}
The constant \( C \) is to be such that the flux condition

\[
\int_A u_1 \, dx_2 \, dx_3 = M > 0
\]

is satisfied.

A unique solution of (1.4) to (1.6) exists for all \( \nu > 0 \), under mild conditions on \( \partial A \). The solution \((u, p)\) is the Poiseuille flow, and we define the Reynolds number to be \( R = M/(\nu l) \) where \( l = (|A|/\pi)^{1/2} \) and \( |S| \) denotes the area of a measurable set \( S \subset \mathbb{R}^2 \).

For \( N = 2 \), we define

\[ R = M/\nu = \int_{-d}^{d} u_1 \, dx_2 / \nu. \]

For a general domain \( \Omega \), the Reynolds number \( R \) is of the form \( \bar{u} \bar{L}/\nu \) where \( \bar{u} \) is a constant representative of the velocity field \( u \) satisfying (1.1) to (1.3), \( \bar{L} \) is a characteristic length depending on the geometry of \( \Omega \), and \( \nu \) is the kinematic viscosity.

Another widely-known example of a problem of Type 3 is Jeffery-Hamel flow in the plane. Here the domain is \( \Omega = \{(r, \theta) : r > 0, \theta \in (-\alpha, \alpha)\} \) where \((r, \theta)\) are plane polar coordinates, and the solutions \( u = u(\theta; R, \alpha) \) are given in terms of Jacobian elliptic functions.

The final example of a problem of Type 3 concerns a symmetrical channel \( \Omega \subset \mathbb{R}^2 \) with slowly curving walls and such that the product of local channel half-width and local wall curvature is bounded by a small parameter \( \varepsilon > 0 \). Fraenkel showed in [5] and [6] that under certain restrictions on \( R \) and \( \alpha \), a formal approximation in powers of \( \varepsilon \) to the stream function \( \psi \) (where \( \psi = (u_1, u_2, 0) = \nabla \times (0, 0, \psi) \)) is in fact a strict asymptotic expansion (for \( \varepsilon \to 0 \)) of an exact solution to the steady Navier-Stokes equations.

The problem in this paper concerns steady viscous incompressible flow in domains of the following type.

**DEFINITION 1.1.** A domain \( \Omega \subset \mathbb{R}^2 \) (\( N = 2 \) or 3) will be called admissible (Figure 1) if \( \partial \Omega \) is of class \( C^\infty \), \( \Omega \) is simply-connected and \( \Omega \) is the union of three disjoint subsets as follows (note that \( \Omega_3 \) is not open).

1. \( \Omega_1 = (-\infty, 0) \times A_1 \), where \( A_1 = (-d, d), \) \( d \in (0, \infty) \), for the case of a channel (\( N = 2 \)), or \( A_1 \) is a simply-connected bounded domain in the plane, with \( \partial A_1 \) of class \( C^\infty \), for a pipe (\( N = 3 \)).

2. In a different coordinate system, \( \Omega_3 = (0, \infty) \times A_3 \), where \( A_3 \) has the same properties as \( A_1 \). (However, \( A_3 \) need not equal \( A_1 \).)

3. \( \Omega_3 = \Omega \setminus (\Omega_1 \cup \Omega_2) \) is bounded.
Throughout this paper, superscripts will denote labels and not exponents unless the contrary is explicitly stated.

Let the flux $M > 0$ through an admissible domain $\Omega$ be prescribed; and let $q^1$ and $q^3$ be the Poiseuille velocities for $\Omega_1$ and $\Omega_3$, respectively, corresponding to flux $M$. We seek a solution $(u, p)$ of the steady Navier-Stokes equations

\begin{align}
-\nu \Delta u + (u \cdot \nabla) u &= -\nabla p, \\
\text{div } u &= 0
\end{align}

in $\Omega$, such that

\begin{equation}
 u = 0 \quad \text{on } \partial \Omega,
\end{equation}

and

\begin{equation}
 u \to q^j \text{ as } |x| \to \infty \quad \text{in } \Omega_j \ (j = 1 \text{ or } 3).
\end{equation}

Figure 1. – Notation for an admissible domain $\Omega \subset \mathbb{R}^N \ (N = 2 \text{ or } 3)$:

$\Omega_1 = \{x \in \Omega : x_1 < 0\}, \quad \Omega_2 = \{x \in \Omega : x_1' > 0\}, \quad \Omega_3 = \Omega \setminus (\Omega_1 \cup \Omega_2).

Note: for $N = 3$, $Ox_1$ and $Ox_1'$ are not necessarily coplanar.

This problem was proposed (I believe) by Leray to Ladyzhenskaya, who in [7] attempted an existence proof under no restrictions on the viscosity $\nu$. The problem is also mentioned by Finn in a review paper ([3], p. 150).
We shall seek a solution of (1.7) to (1.10) of the form
\[ u = q + w, \]
where the velocity field \( q \) is to be such that

Thus \( q \), which will be constructed a priori, is an « extended Poiseuille velocity field » that satisfies the boundary conditions. It follows that \( w \) is to be such that \( q + w \) satisfies (1.7) and

The paper is organized as follows.
In section 2, we give some notations, definitions, and preliminary lemmas; in section 3, the existence of a weak solution \( u \) of the problem (1.7) to (1.10) is proved for all values of the viscosity \( \nu > \sigma \), where \( \sigma \) does not depend on \( \Omega \). In terms of the (non-dimensional) Reynolds number, the condition \( \nu > \sigma \) becomes \( R < R_0 \), and some numerical values of the critical Reynolds number \( R_0 \) are given. In addition, we prove that a weak solution exists in certain domains which asymptotically approach cylinders as \( |x| \to \infty \). Section 4 consists of theorems, concerned with the constant \( \sigma \), which are needed in section 3.

If the admissible domain is symmetric about some axis, then many of the results for general admissible domains are improved. These results appear as corollaries to the main results.
2. - Preliminaries.

Let \( x = (x_1, \ldots, x_n) \) denote points in \( \mathbb{R}^n \), and use the standard inner product \( x \cdot y = \sum_i x_i y_i \). Let \( \mathbb{R}^+ \subset \mathbb{R} \) be given by \( \mathbb{R}^+ = \{ x : x > 0 \} \) and let \( \mathbb{R}^- \) be defined analogously. All integrals in this paper are in the sense of Lebesgue. Recall that the domain \( \Omega \subset \mathbb{R}^d \) (where \( N = 2 \) or \( 3 \) always) is a union of three disjoint subsets (Figure 1)

\[
\begin{align*}
\Omega_1 &= \{ x \in \Omega : x_1 < 0 \}, \\
\Omega_2 &= \{ x \in \Omega : x_1 > 0 \}, \\
\Omega_3 &= \Omega \setminus (\Omega_1 \cup \Omega_2),
\end{align*}
\]

where \( (x_1, \ldots, x_N) \) and \( (x'_1, \ldots, x'_N) \) are distinct coordinate systems such that \( O x_1 \) and \( O x'_1 \) are axes of the cylindrical domains \( \Omega_1 \) and \( \Omega_2 \), respectively.

2.1. Function spaces.

Let \( U \) be an arbitrary domain in \( \mathbb{R}^n \). Let \( \alpha = (\alpha_1, \ldots, \alpha_n) \), with each \( \alpha_i \) a non-negative integer, be a multi-index of order \( |\alpha| = \alpha_1 + \ldots + \alpha_n \) and let

\[
D_i = \partial / \partial x_i, \quad D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}.
\]

Write \( V \subset U \) when \( V \) is compact and \( V \subset U \). The support \( \text{supp} \ v \) of a function \( v : U \to \mathbb{R}^n \) is the closure of \( \{ x \in U : v(x) \neq 0 \} \). Thus, \( v \) is said to have compact support in \( U \) if \( \text{supp} \ v \subset \subset U \).

The set of functions \( C_0^\alpha(U \to \mathbb{R}^n) \) denotes those functions defined on \( U \) with image in \( \mathbb{R}^n \) and having all (partial) derivatives continuous. The set \( C_0^\alpha(\overline{U} \to \mathbb{R}^n) \) consists of those functions in \( C_0^\alpha(U \to \mathbb{R}^n) \) such that all derivatives can be extended to be bounded and continuous on \( \overline{U} \). We introduce two sets of functions commonly called test functions:

\[
C_0^\alpha(U \to \mathbb{R}^n) = \{ \varphi \in C_0^\alpha(U \to \mathbb{R}^n) : \text{supp} \ \varphi \subset \subset U \},
\]

\[
J(U \to \mathbb{R}^n) = C_0^\alpha(\overline{U} \to \mathbb{R}^n) = \{ \varphi \in C_0^\alpha(U \to \mathbb{R}^n) : \text{div} \ \varphi = 0 \ \text{in} \ U \}.
\]

(The \( * \text{sol} \) superscript denotes solenoidal vector fields.)

The following norm is used to define various Sobolev spaces:

\[
|v|_{W^{\alpha,p}_{l+}} = \left( \sum_{|\beta| \leq l} \int |D^\beta \varphi|^p \right)^{1/p}
\]
for non-negative integers $j$ and $p \in [1, \infty)$. Denote by $W^j_p(U \to \mathbb{R}^n)$ the completion of
\[ \{ v \in C^0(U \to \mathbb{R}^n) : \| v \|_{W^j_p} < \infty \} \]
in the norm (2.2). Similarly, define $\hat{W}^j_p$ and $\hat{W}^{j,\text{sol}}_p$ as the completion in (2.2) of $C^0_\delta$ and $J$, respectively. $W^j_p$, $\hat{W}^j_p$, and $\hat{W}^{j,\text{sol}}_p$ are Banach spaces (they are Hilbert spaces for $p = 2$ with the obvious inner product) and $\hat{W}^{j,\text{sol}}_p \subset \hat{W}^j_p \subset W^j_p$.

Some of the properties of $W^j_p$ and $\hat{W}^j_p$ are given in [9] and [11].

For normed spaces $A$ and $B$, we write $A \hookrightarrow B$ when the identity map $f \mapsto f$ is a bounded mapping from $A$ into $B$. The space $A$ is said to be imbedded in $B$.

2.2. The space $H(Q \to \mathbb{R}^p)$.

As stated in the Introduction, we seek a solution $(u, p)$ of the steady Navier-Stokes equations (1.7) to (1.10) of the form $u = q + w$ where $q$ is a known function satisfying (1.11) (the construction of $q$ appears in section 3.1) and $w$ is to satisfy (1.12). A natural setting for $w$ is as an element of the Hilbert-Sobolev space $H(Q \to \mathbb{R}^p)$ ($N = 2$ or $3$).

For an arbitrary domain $U \subset \mathbb{R}^n$, $H(U \to \mathbb{R}^n)$ is the completion of $J(U \to \mathbb{R}^n)$ in the norm implied by the inner product
\[ \langle f, g \rangle_H = \int_U \nabla f \cdot \nabla g = \sum_{i=1}^n \int_U (D_i f)(D_i g). \]

The norm for $H$ is
\[ \| f \|_H = \langle f, f \rangle_H^{1/2} \]
and will be referred to as the Dirichlet norm. In addition, for a Lebesque measurable set $V \subset U$ define
\[ \| f \|_V = \left( \int_V |\nabla f|^2 \right)^{1/2}. \]

Similarly, we define $E(U \to \mathbb{R}^n)$ to be the completion of $C^0_\delta$ in the Dirichlet norm; vector fields in $E$ need not be solenoidal.

We now give some properties of $H$.

(2.5) (a) If $U$ is such that $E(U) \hookrightarrow L_d(U)$, then $E$ is equivalent to $\hat{W}^1_2$ and $H$ to $\hat{W}^{1,\text{sol}}_2$.

(b) $\int_V (f \, \text{div} \, g) = 0$, $\forall f \in L_d(U)$ and $g \in H(U)$. 


(c) Let \( \partial U \) be of class \( C^\infty \) ([18], pp. 9-10). We can define in the usual manner a trace operator \( T_{\partial U} : H(U) \to L_{a,loc}(\partial U), \) where \( T_{\partial U} \) is a bounded linear map and \( T_{\partial U}(\varphi) = \varphi \big|_{\partial U} = 0 \ \forall \varphi \in J(U) \) since \( \text{supp} \varphi \subset U. \) Hence \( T_{\partial U} \) is the zero map and
\[
\int_{\partial U} |f| \, ds = 0 \quad \forall f \in H(U),
\]
where the integral is defined in terms of the local coordinates of \( \partial U \) ([11], pp. 231-232).

In fact, for the domains \( \Omega \) in this paper there exists a large class of unbounded manifolds \( \Gamma \) for which we can define a trace operator \( T_{\Gamma} : H(\Omega) \to L^2(\Sigma) \) and not merely into \( L_{a,loc}(\Gamma). \)

(d) Assume \( H(U) \hookrightarrow L^2(U) \) and let \( X \) be a cross-section of \( U. \) (We define a cross-section \( X \subset \mathbb{R}^{n-1} \) of a domain \( U \subset \mathbb{R}^n \) to be a bounded open set of the form \( X = U \cap P \) where \( P \) is an \((n-1)\)-dimensional hyperplane.) Then elements in \( H(U) \) carry no flux across \( X, \) i.e.,
\[
\int_X (f \cdot n) = 0 \quad \forall f \in H(U).
\]
A conventional estimate, beginning with integration along a normal to any point \( x \in X, \) shows that
\[
\int_X |f(x)|^2 \leq 2\|f\|_{L^2(U)} \|f\|_{W^1(U)} \quad \forall f \in H(U).
\]
Since \( X \) is bounded, we have \( f, (f \cdot n) \in L^2(X) \) and \( |f \cdot n|_{L^2(X)} < \text{const} \|f\|_{W^1(U)}. \) We may define a trace operator \( T_X : H(U) \to L^2(X) \) where \( T_X \) is a bounded linear map and \( T_X(\varphi) = (\varphi \cdot \eta)|_X \ \forall \varphi \in J(U). \) We now claim that \( T_X \) is identically zero, and it suffices to prove that \( T_X(\varphi) = 0 \) for an arbitrary \( \varphi \in J(U). \)

Since \( \varphi \in J(U) \subset J(\mathbb{R}^n), \) we can apply the divergence theorem to a half-ball \( B \) bounded by the hyperplane \( P \) and a hemisphere \( \Gamma \) so large that \( \text{supp} \varphi \cap \Gamma = 0. \) Then
\[
0 = \int_B \text{div} \varphi = \int\int_X (\varphi \cdot n) + \int\int_{\Gamma} (\varphi \cdot n) = \int_X (\varphi \cdot n).
\]

(e) We shall need the following result in sections 3 and 4 (Lions [12], pp. 67-68; Heywood [13]). Assume that \( \partial U \) is of class \( C^1, \) then
\[
\{ \varphi \in H(U \to \mathbb{R}^n) : \text{supp} \varphi \text{ is bounded} \} = \{ \varphi \in W^1_2(U \to \mathbb{R}^n) : \text{supp} \varphi \text{ is bounded, div} \varphi = 0 \in U \}.
\]
We now state two lemmas concerning the domain $U$ of functions in $H(U) \subset E(U)$. The results are of critical importance for admissible domains.

**Lemma 2.1.** (a) Let $L > 0$. Then

$$\int_0^L (v^2) \leq \left( \frac{L}{\pi} \right) \int_0^L \left( \frac{\partial v}{\partial x} \right)^2 = \left( \frac{L}{\pi} \right)^2 \|v\|_2^2 \quad \forall v \in E((0, L) \to \mathbb{R}).$$

(b) Let $U \subset \mathbb{R}^n$ be a domain lying between parallel planes (i.e. $(n-1)$-dimensional hyperplanes) a distance $L$ apart. Then

$$\int_U |v|^2 \leq \left( \frac{L}{\pi} \right)^2 \|v\|_2^2 \quad \forall v \in E(U \to \mathbb{R}^n).$$

(c) Let $U \subset \mathbb{R}^n$ be a domain with $U = \bigcup_{i=1}^m U_i$ $(m \geq 1)$ and such that each domain $U_i$ lies between parallel planes a distance $L$ apart. Then

$$\int_U |v|^2 \leq \text{const} \|v\|_2^2 \quad \forall v \in E(U \to \mathbb{R}^n),$$

where the constant depends only on the geometry of $U$.

**Proof.** It suffices to prove the lemma for all $v \in C_0^\infty$. Part (a) follows from a standard result of the calculus of variations [10] and (b) and (c) follow from (a).

The following form of certain Sobolev inequalities is due to Nirenberg ([19], p. 125).

**Lemma 2.2** Let a domain $U \subset \mathbb{R}^n$ be such that $E(U \subset \mathbb{R}^n) \hookrightarrow L^s(U \to \mathbb{R}^n)$. Then $E \hookrightarrow L^s$.

$$\int_U |v|^2 \leq \text{const} \|v\|_s^2 \quad \forall v \in E(U \to \mathbb{R}^n),$$

where the constant depends only on the geometry of $U$.

Since $H(U) \subset E(U)$, Lemma 2.1 also holds for functions $v \in H(U)$. If we apply Lemma 2.1(c) to an admissible domain $\Omega$, then it follows that $H(\Omega) \hookrightarrow L^2(\Omega)$ and Lemma 2.2 holds with $E(\Omega)$ replaced by $H(\Omega)$. 


We remind the reader that Lemmas 2.1 and 2.2 are applicable to an admissible domain $\Omega$ only because $\Omega_1$ and $\Omega_2$ can be individually bounded by parallel planes.

It is now clear why we seek a solution $u = q + w$ of the problem (1.7) to (1.10) with $w \in H(\Omega \to \mathbb{R}^n)$. Indeed, then $w$ satisfies (1.12) in a generalized sense; i.e. $w$ has finite Dirichlet norm $|w|_{\partial \Omega}$; (2.5)(b) implies $\text{div } w = 0$ almost everywhere in $\Omega$; (2.5)(c) gives $w = 0$ almost everywhere on $\partial \Omega$; and (2.5)(d) states that $w$ carries no flux. However, (1.12)(c) may be satisfied only in a generalized sense since elements in $H(\Omega)$ need not go to zero pointwise as $|x| \to \infty$ in $\Omega$.

2.3. Boundary-layer integrals.

For a domain $U \subset \mathbb{R}^n$ and $\varepsilon > 0$, we define $\varepsilon(x) = \text{dist}(x, \partial U)$ and $U_\varepsilon = \{ x \in U : \varepsilon(x) < \varepsilon \}$. If $\partial U$ is sufficiently smooth, then we can introduce $\varepsilon$ boundary-layer coordinates $(s, t)$ in $U_\varepsilon$ for small $\varepsilon$ ([18], p. 38). Here $s$ denotes a surface coordinate on $\partial U$, and $t$ distance from $\partial U$ along an inward normal.

**Lemma 2.3.** Let $U$ be a domain in $\mathbb{R}^n$ ($N = 2$ or $3$) with compact $\partial U$ of class $C^\infty$ or an admissible domain in $\mathbb{R}^n$. If $\varepsilon > 0$ is sufficiently small, then

(a) $\alpha \in C^\infty(\overline{U}_\varepsilon \to \mathbb{R})$, and

(b) every point $x_0 \in \partial U$ is the center of a ball $B_0 = B(x_0, r_0)$, with radius $r_0$ independent of $\varepsilon$, such that $x \mapsto (s, t)$ is a $C^\infty$ diffeomorphism from $B_0 \cap \overline{U}_\varepsilon$ to some compact subset of $\mathbb{R}^n$.

**Proof.** If $\partial U$ is compact, then the result is standard ([18], p. 38). The proof for an admissible domain is analogous and uses the fact that $U$ is cylindrical outside some bounded set.

**Lemma 2.4.** Let $v \in C^\infty([0, L] \to \mathbb{R})$ with $v(0) = 0$. Then

$$\int_0^L \left( \frac{v}{t} \right)^2 dt < 4 \int_0^L \left( \frac{dv}{dt} \right)^2 dt.$$

**Proof.** An integration by parts gives:

$$\int_0^L \left( \frac{v}{t} \right)^2 dt = -\frac{(v(L))^2}{L} + 2 \int_0^L \frac{v dv}{dt} dt,$$

and the lemma follows after an application of the Schwarz inequality.
LEMA 2.5. Let \( U \) be as in Lemma 2.3. If \( \varepsilon > 0 \) is sufficiently small, then
\[
\int_{\partial U} \frac{1}{\varepsilon} < (4 + O(\varepsilon)) \int_{\partial U} |\nabla \psi|^2 \quad \forall \psi \in \mathcal{E}(U \to \mathbb{R}^n).
\]

PROOF. If \( \partial U \) is compact, then the result is standard ([2], pp. 106-110) and makes use of Lemma 2.4. The proof for an admissible domain is analogous.

Particularly useful tools in the study of partial differential equations are mollifiers \( \mu \in C^\infty(\mathbb{R} \to [0, 1]) \) with \( \operatorname{supp}(d\mu/dt) \subset \mathbb{R} \). The following mollifier ([3], [8], [14]) \( \mu(t; \varepsilon) \) is important in section 3.

LEMMA 2.6. For every \( \varepsilon > 0 \), there exists a mollifier \( \mu(\cdot ; \varepsilon) \in C^\infty(\mathbb{R} \to [0, 1]) \) (see Figure 2a) with \( \operatorname{supp} \mu_t \subset (0, \varepsilon] \) and such that \( \mu(0; \varepsilon) = 1, \mu(\varepsilon; \varepsilon) = 0 \) and
\[
\mu(t; \varepsilon), \quad \left| \frac{d}{dt} \mu(t; \varepsilon) \right| < \frac{\varepsilon}{t} \quad \text{for} \quad t > 0.
\]

Figure 2. – (a) The mollifier \( \mu = \mu(\cdot ; \varepsilon) \in C^\infty(\mathbb{R} \to [0, 1]) \). (b) The function \( \tau \) used in the construction of \( \mu \).

PROOF. For any \( \alpha > 0 \) and \( \delta \in (0, \frac{1}{4}) \), let \( \tau(t) = \tau(t ; \alpha, \delta) \) be a \( C^\infty \) mollifier as in Figure 2b. The function \( \tau \) has the properties: (a) \( 0 < \tau(t) < 1/t \) everywhere, (b) \( \tau(t) = 1/t \) on \([2\alpha \delta, (1 - 2\delta)\alpha]\), (c) \( \tau(t) = 0 \) for \( t < \alpha \delta \) and \( t > (1 - \delta)\alpha \). Let \( T = \int_0^\pi \tau(s) \, ds \) and define
\[
\mu(t; \alpha, \delta) = 1 - \frac{1}{T} \int_0^t \tau(s) \, ds.
\]
Since $T > \int_{x=t}^{x=t+\delta} ds/s = \log \left( \frac{1 - 2\delta}{2\delta} \right)$, define $\delta$ by $\log \left( \frac{1 - 2\delta}{2\delta} \right) = \frac{1}{\varepsilon}$ so that $1/T < \varepsilon$ and $|\mu_s| = \tau(t)/T < \varepsilon/\varepsilon$. Choose $\alpha(\varepsilon) = \varepsilon/(1 - \delta)$ and let $\mu(t; \varepsilon) = \mu(t; \alpha(\varepsilon), \varepsilon)$. It follows that $\text{supp } \mu, \subset (0, \varepsilon]$ and if $t \in \text{supp } \mu_s$, then $\varepsilon/\varepsilon > 1 > \mu(t; \varepsilon)$.

If we combine the properties of $\mu$ with Lemma 2.5, then we obtain an estimate essentially due to Leray ([1], pp. 38-47), that is crucial in problems of Type 1 and 2 when one proves the existence of weak solutions for all $v > 0$. A similar inequality for admissible domains will be applied in Theorem 3.6 to the problem (1.7)-(1.10).

**Theorem 2.7.** Let $U$ be as in Lemma 2.3 and let $Q \in C^\infty(\overline{U} \to \mathbb{R}^n)$ with $\nabla Q = 0$ on $\partial U$. For $\varepsilon > 0$, define $g(x; \varepsilon) = \nabla \times \{p(x; \varepsilon) Q(x)\}$. If $\varepsilon$ is sufficiently small, then

$$
\int_{\overline{U}} |g \cdot (w \cdot \nabla) w| < \text{const } \varepsilon \int_{\overline{U}} |\nabla w|^2 \quad \forall w \in E(U \to \mathbb{R}^n),
$$

and the constant is independent of $\varepsilon$.

**Proof.** The properties of $\mu$ in Lemma 2.6 show that $\text{supp } g \cap U = U_s$ and $|g(x; \varepsilon)| < \text{const } \varepsilon/\varepsilon(x)$. Using the Schwarz inequality and then Lemma 2.5, we have

$$
\int_{\overline{U}} |g \cdot (w \cdot \nabla) w| < \text{const } \varepsilon \int_{\overline{U}} |w| |\nabla w| < \text{const } \varepsilon \int_{\overline{U}} |w| |\nabla w| < \text{const } \varepsilon \|w\|_{L_2(U)}^2 .
$$

3. - Existence of a weak solution.

Before giving the definition of a weak (or generalized) solution of the problem (1.7) to (1.10), we construct the velocity field $q$ satisfying (1.11).

3.1. **Construction of the extended Poiseuille velocity field $q$.**

Since $\partial \Omega$ is of class $C^\infty$ and the desired function $q$ is to coincide with the Poiseuille velocity field outside $\Omega_s$ (hence $q \in C^\infty(\overline{\Omega \setminus \Omega_s})$), it is reasonable to require that $q \in C^\infty(\overline{\Omega} \to \mathbb{R}^n)$.

We recall that, for given $M > 0$, $q$ is to satisfy:

(3.1) \hspace{1cm} (a) $q \in C^\infty(\overline{\Omega} \to \mathbb{R}^n)$ ($N = 2$ or 3);

(b) $\text{div } q = 0$ in $\Omega$, $q = 0$ on $\partial \Omega$; and

(c) $q = q^j$ in $\Omega_j$ ($j = 1$ or 3), where $q^j$ is the Poiseuille velocity field for the cylinder $\Omega_j$, and carries flux $M$. 

We wish to construct the extended Poiseuille velocity field in the form
\[ q = \nabla \times Q \] because then one can multiply the vector potential \( Q \) by a mollifier and retain solenoidality. As a first step in this construction, we shall need

**Lemma 3.1.** Let \( U \subset \mathbb{R}^N \) be a cylinder of the form \( \mathbb{R} \times A \) where \( A \) is an open interval \((-d, d)\), \( d \in (0, \infty) \), for \( N = 2 \), and \( A \) is a simply-connected bounded domain in the plane, with \( \partial A \) of class \( C^\infty \), for \( N = 3 \). Let \( v \in C^\infty(\overline{U} \to \mathbb{R}^N) \) with \( v = (v_1(x_2, x_3), 0, 0) \). Then there exists \( \psi \in C^\infty(\overline{U} \to \mathbb{R}^N) \) with \( \psi = (0, \psi_3(x_2, x_3), \psi_3(x_2, x_3)) \) such that \( \nabla \times \psi = v \).

**Proof.** For \( N = 2 \), set \( \psi_3 = 0 \) and define

\[ \psi_3(x_2) = \frac{1}{-d} \int_{-d}^{x_2} v_1(t) \, dt - \frac{1}{2} M. \]

For \( N = 3 \), \( \nabla \times \psi = v \) implies that we need \( D_2 \psi_3 - D_3 \psi_2 = v_1 \) in \( A \). Let \( \psi_3 \) be any solution of \( (D_2^2 + D_3^2) \psi_3 = -D_3 v_1 \) in \( A \), with \( \psi_3 \in C^\infty(\overline{A} \to \mathbb{R}) \) (standard theory shows this to be true if \( \psi_3 \) is chosen of class \( C^\infty \)). Define \( \psi_3 \in C^\infty(\overline{A} \to \mathbb{R}) \) by

\[ \psi_3(x_2, x_3) = \int_{(x_2, x_3)} \left\{ (v_1 + D_3 \psi_2) \, dt_2 - D_3 \psi_2 \, dt_3 \right\} \quad \forall (x_2, x_3) \in \overline{A}, \]

where \((p_2, p_3)\) is some arbitrary (fixed) point in \( \overline{A} \). It follows that \( \psi_3 \) is single-valued and of class \( C^\infty \) on \( \overline{A} \) since \( v_1 \) and \( \psi_3 \) are. If we let \( \psi = (0, \psi_2, \psi_3) \), then \( \nabla \times \psi = v \) in \( \overline{U} \), and in addition \( \text{div} \psi = 0 \) in \( \overline{U} \). The lemma is proved.

For \( N = 2 \), the Poiseuille velocity field in \( \mathbb{R} \times (-d, d) \) carrying flux \( M \) is given by \( (q_1(x_2), 0) \), where

\[ q_1(x_2) = \frac{3M}{4d} \left\{ 1 - \frac{x_2}{d} \right\}. \]

It follows that a corresponding vector potential \( \psi = (0, 0, \psi_3) \) (\( \psi_3 \) is often termed the stream function) is given by

\[ \psi_3(x_2) = \frac{3M}{4} \left\{ \frac{x_2}{d} - \frac{1}{3} \left( \frac{x_2}{d} \right)^3 \right\}. \]

For \( N = 3 \) and a circular cross-section \( A \) of radius \( a \), the Poiseuille velocity field in \( \mathbb{R} \times A \) is given by \( (q_1(x_2, x_3), 0, 0) \), where

\[ q_1(x_2, x_3) = \frac{2M}{\pi a^2} \left\{ 1 - \left( \frac{x_2}{a} \right)^2 - \left( \frac{x_3}{a} \right)^2 \right\}. \]
A corresponding vector potential \( \psi = (0, \psi_2, \psi_3) \) is given by

\[
\psi(x_1, x_2) = \frac{Mx_2}{2\pi a^2} \left\{ 2 - \left( \frac{x_2}{a} \right)^2 - \left( \frac{x_1}{a} \right)^2 \right\}, \quad \psi_3(x_1, x_2) = \frac{Mx_3}{2\pi a^2} \cdot \left\{ 2 - \left( \frac{x_2}{a} \right)^2 - \left( \frac{x_1}{a} \right)^2 \right\}.
\]

The following lemma is a step in the construction of \( q \) and is a slight generalization of a result due to Finn ([15], pp. 206-208); a detailed and somewhat different treatment is also given in [16].

**Lemma 3.2.** Let \( U \) be an open set in \( \mathbb{R}^n \) with \( \partial U \) of class \( C^\infty \) and such that \( \partial U \) consists of a finite number of compact components \( \partial U_i, i = 1, 2, \ldots, m \). If a given function \( g_0 \in C^\infty(\partial U \to \mathbb{R}^n) \) satisfies \( g_0 \cdot n = 0 \) on each boundary component \( \partial U_i \), then there exists a vector potential \( \psi \in C^\infty(\overline{U} \to \mathbb{R}^n) \) with \( \nabla \times \psi = g_0 \) on \( \partial U \).

We assume without loss of generality that the admissible domain \( \Omega \) is cylindrical for \( x_1 < 1 \) and \( x_1' > -1 \) (one may always translate the axes in Figure 1). For future reference, define

\[
\Omega_a = \{ x \in \Omega : x_1 < 1 \},
\]

and

\[
\Omega_c = \{ x \in \Omega : x_1' > -1 \}.
\]

The Poiseuille velocity fields \( q_1 \) and \( q_2 \) are defined in these respective domains. By Lemma 3.1, there exist vector potentials \( \psi_a \) and \( \psi_c \) such that

\[
q_1 = \nabla \times \psi_a \quad \text{in} \quad \Omega_a \quad (\supset \Omega_a) \quad \text{with} \quad \psi_a \in C^\infty(\overline{\Omega_a} \to \mathbb{R}^n),
\]

and

\[
q_2 = \nabla \times \psi_c \quad \text{in} \quad \Omega_c \quad (\supset \Omega_c) \quad \text{with} \quad \psi_c \in C^\infty(\overline{\Omega_c} \to \mathbb{R}^n).
\]

Let \( \Omega_4 \subset \Omega_1 \) be a domain with \( \partial \Omega_4 \) of class \( C^\infty \) and with \( \partial \Omega_4 = \partial \Omega_1 \) except for boundary points with \( x_1 \in (0, \frac{1}{2}) \) or \( x_1' \in (-\frac{1}{2}, 0) \); in these regions, \( \partial \Omega_4 \subset \Omega \) (see Figure 3). Define \( g_0 : \partial \Omega_4 \to \mathbb{R}^n \) by

\[
g_0(x) = \begin{cases} 
q_1(x) & \text{for } x \in \partial \Omega_4, \quad 0 < x_1 < \frac{1}{2}, \\
q_2(x) & \text{for } x \in \partial \Omega_4, \quad -\frac{1}{2} < x_1' < 0, \\
0 & \text{elsewhere on } \partial \Omega_4.
\end{cases}
\]
Since $q^1$ and $q^3$ are of class $C^\infty$ in $\overline{\Omega}_a$ and $\overline{\Omega}_3$, respectively, and $\partial \Omega_4$ is of class $C^\infty$, it follows that $g_0 \in C^\infty(\partial \Omega_4 \to \mathbb{R}^n)$. We also note that

$$\int_{\partial \Omega_4} (g_0 \cdot n) = 0.$$

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{The domain $\Omega_4$ used in the construction of the extended Poiseuille velocity field.}
\end{figure}

Hence, the domain $\Omega_4$ and the boundary function $g_0$ satisfy the hypotheses of Lemma 3.2, and so there exists a vector potential $\psi_5$ with

$$(3.4)(c) \quad \psi_5 \in C^\infty(\overline{\Omega}_4 \to \mathbb{R}^n) \quad \text{and} \quad \nabla \times \psi_5 = g_0 \quad \text{on} \ \partial \Omega_4.$$

**Theorem 3.3.** Let $\Omega$ be an admissible domain (in the sense of Definition 1.1). Then there exists a vector potential $Q \in C^\infty(\overline{\Omega} \to \mathbb{R}^n)$ ($N = 2$ or 3) such that

(a) $\nabla \times Q = 0$ on $\partial \Omega$, and

(b) $Q = \psi_a$ in $\Omega_1$ and $Q = \psi_c$ in $\Omega_3$, where $\psi_a$ and $\psi_c$ are as in (3.4).

Thus, $q = \nabla \times Q$ is an extended Poiseuille velocity field and satisfies (3.1).

**Proof.** We shall give the proof for $N = 3$ since that for $N = 2$ is analogous.
Let \( \lambda \in C^\infty(\mathbb{R} \rightarrow [0, 1]) \) be a mollifier with \( \lambda(x_1) = 1 \) for \( x_1 < \frac{1}{\varepsilon} \) and \( \lambda(x_1) = 0 \) for \( x_1 > 1 \). Then for \( \psi_a, \psi_b, \psi_c \) as in (3.4), define \( Q : \Omega \rightarrow \mathbb{R}^\nu \) by

\[
Q(x) = \begin{cases} 
\lambda(x_1)\psi_a + (1 - \lambda(x_1))\psi_b + \Psi & \text{for } x \in \overline{\Omega}_a, \\
\lambda(-x_1')\psi_c + (1 - \lambda(-x_1'))\psi_b + \Psi & \text{for } x \in \overline{\Omega}_c, \\
\psi_b(x) & \text{elsewhere in } \overline{\Omega},
\end{cases}
\]

where \( \Psi \) is to be determined by the condition \( \nabla \times Q = 0 \) on \( \partial \Omega \). (For the case \( \nu = 2 \), the function \( \Psi \) in (3.5) may be taken as identically zero.) Thus, \( \Psi = (\psi_1, 0, 0) \) is to be such that

\[
\begin{cases} 
\frac{\partial \psi_1}{\partial x_3} - \lambda'(x_1)(\psi_a - \psi_b) = 0, \\
- \frac{\partial \psi_1}{\partial x_2} + \lambda'(x_1)(\psi_c - \psi_b) = 0
\end{cases}
\]

for \( \frac{1}{\varepsilon} < x_1 < 1 \), \( (x_2, x_3) \in \partial \Omega_1 \),

and similarly for \( -1 < x_1' < -\frac{1}{\varepsilon} \), \( (x_2', x_3') \in \partial \Omega_3 \).

We construct \( \psi_1 \) as follows. Let \( (s, t) \) be « boundary-layer coordinates »: \( s \) denotes distance along \( \partial \Omega_1 \), measured from a line in \( [\frac{1}{\varepsilon}, 1] \times \partial \Omega_1 \) parallel to \( Ox_1 \), and in the direction that makes \( \partial \Omega_1 \) positively oriented; moreover, \( s \) is constant on each normal to \( \partial \Omega_1 \), while \( t \) denotes distance along the inward normal to \( \partial \Omega_1 \). By Lemma 2.3, the map \( (x_2, x_3) \mapsto (s, t) \) is one-to-one and of class \( C^\infty \) for sufficiently small positive values of \( t \), say for \( 0 < t < t_0 \).

Let \( \mu \in C^\infty(\mathbb{R} \rightarrow [0, 1]) \) be a mollifier with \( \mu(t) = 1 \) for \( t < t_0/2 \) and \( \mu(t) = 0 \) for \( t > t_0 \). We define

\[
\psi_1(x_1; s, t) = \mu(t)\left\{ g(x_1, s) + tf(x_1, s) \right\},
\]

where

\[
\begin{align*}
(3.7) & & f(x_1, s) = \lambda'(x_1)\left\{ \left( 0, -\frac{\partial x_2}{\partial s}, \frac{\partial x_3}{\partial s} \right) \cdot (\psi_a - \psi_b) \right\}_{\partial A_1}, \\
(3.8)(a) & & g(x_1, s) = \lambda'(x_1)\left( \psi_a - \psi_b \right) \cdot \left( 0, \frac{\partial x_2}{\partial s}, \frac{\partial x_3}{\partial s} \right) \bigg|_{\partial A_1} ds'.
\end{align*}
\]

Note that \( g \) is single-valued since

\[
\int_{\partial A_1}(\psi_a - \psi_b) \cdot (0, dx_2, dx_3) = \int_{A_1}\left\{ (D_2\psi_a - D_3\psi_a) - (D_2\psi_b - D_3\psi_b) \right\} dx_2 dx_3 = \int_{A_1}(q_1^* - q_2^*) dx_2 dx_3 = M - M = 0,
\]

where \( q^* = \nabla \times \psi_b \).
It follows immediately from (3.7) that \( \varphi_1 \) satisfies (3.6). Equation (3.8) gives: \( \Omega_a \cap \text{supp} \varphi_1 \subset (\text{supp} \lambda) \times A_1 \subset [\frac{1}{2}, 1] \times A_1 \). Hence, \( \Psi = 0 \) in \( \Omega_a \) and from (3.5) we have \( Q = \varphi_1 \) in \( \Omega_a \).

An analogous argument holds in \( \Omega_e \), and the theorem is proved.

For \( N=2 \) and certain admissible domains \( \Omega_1 \), we can give an explicit representation of \( Q \). Assume that \( f \in C^\infty(R \to R^3) \), \( \varphi_+, \varphi_- \in C^\infty(R \to R) \), and that the map \( \gamma: R \times [-1, 1] \to \Omega \) is a \( C^\infty \) homeomorphism given by

\[
x = \gamma(s, t) = f(s) + \frac{1}{2}n(t(\varphi_+ + \varphi_-) + (\varphi_+ - \varphi_-)) ,
\]

where \( n = (-f'_2(s), f'_1(s))/|\nabla f| \) is the unit normal to the curve \( x = f(s) \), and \( R \times \{1\}, R \times \{-1\} \) are mapped onto the « upper » and « lower » components of \( \partial \Omega \), respectively. The curve \( \{f(s) : s \in R\} \) is a generalized axis for \( \Omega \), and \( \varphi_+ \) is the distance along the normal from this axis to the « upper » boundary and similarly for \( \varphi_- \).

A vector potential satisfying Theorem 3.3 is given by \( Q = (0, 0, Q_3) \), where

\[
Q_3(x_1, x_2) = \frac{3}{4} \mathcal{M} \left[ t(x_1, x_2) - \frac{1}{2} t(x_1, x_2)^3 \right].
\]

3.2. The weak solution.

**Definition 3.1.** Let \( q \) satisfy (3.1). The function \( u = q + w \) is a weak (or generalized) solution of the problem (1.7) to (1.10) if \( w \in H(\Omega \to R^3) \) (\( N=2 \) or 3) and

\[
\nu \int_{\Omega} (\nabla \varphi : \nabla u) + \{\varphi, u, w\} = 0 \quad \forall \varphi \in J(\Omega)
\]

(\( J(\Omega) = C_0^{\infty}(\Omega \to R^3) \)), or, equivalently,

\[
\nu \langle \varphi, w \rangle_H + \{\varphi, q + w, w\} + \{\varphi, w, q\} = -\nu \int_{\Omega} (\nabla \varphi : \nabla q) - \{\varphi, q, q\}
\]

\( \forall \varphi \in J(\Omega) \),

where

\[
\{\varphi, \psi, \chi\} = \int_{\Omega} \varphi \cdot (\psi \cdot \nabla) \chi.
\]

Since we are interested in \( w \in H(\Omega) \), we shall work almost entirely with (3.11). Note that if \( (u, p) \) is a classical solution (i.e. has sufficiently many derivatives) of the problem (1.7) to (1.10), then, upon dot-multiplying (1.7) by \( \varphi \in J(\Omega) \) and integrating over \( \Omega \), we recover (3.11), since

\[
-\int_{\Omega} (\varphi \cdot \nabla p) = \int_{\Omega} (p \text{ div } \varphi) = 0.
\]
Accordingly, if \((u, p)\) is a classical solution with \(\int_{\Omega} |\nabla(u - q)|^2 < \infty\), then \(u - q = w\in H\) satisfies (3.11). The converse, that a weak solution is classical, is known for problems of Type 1 and 2 with smooth data, and will be shown for the present case in a later paper.

We now give some properties of the triple product \(\{\varphi, \psi, \chi\}\) which will be used throughout this section.

(a) If \(\varphi, \psi, \chi \in H(\Omega)\), then

\[
(3.12)(a) \quad |\{\varphi, \psi, \chi\}| \leq \text{const} |\varphi|_H |\psi|_H |\chi|_H
\]

by the Schwarz inequality and the imbedding \(H(\Omega) \hookrightarrow L_4(\Omega)\) of Lemma 2.2. Integrating by parts gives

\[
(3.12)(b) \quad \{\varphi, \psi, \chi\} = -\{\chi, \psi, \varphi\}.
\]

(b) If \(\psi = q\) and \(\varphi, \chi \in H\) or \(\chi = q\) and \(\varphi, \psi \in H\), then (3.12)(b) remains valid, and an analogous version of (3.12)(a) holds.

We first show that \(\Omega_1\) and \(\Omega_3\) do not contribute to the right-hand side of (3.11), which can therefore be bounded, and then extend (3.11) to all \(\varphi \in H(\Omega)\).

**Lemma 3.4.** There exists an element \(r \in H(\Omega \to \mathbb{R}^n)\) such that \(u = q + w\) is a weak solution if and only if

\[
(3.13) \quad \nu \langle \varphi, w \rangle_H + \{\varphi, q + w, w\} + \{\varphi, w, q\} = \langle \varphi, r \rangle_H \quad \forall \varphi \in H(\Omega).
\]

**Proof.** We shall prove that the expression

\[
-\nu \int_{\Omega} (\nabla \varphi : \nabla q) - \{\varphi, q, q\}
\]

defines a continuous linear functional on \(J\) which may be extended by continuity to \(H\).

Let \(\varphi \in J\), then

\[
(3.14) \quad \int_{\Omega} (\nabla \varphi : \nabla q) = -\int_{\Omega} (\varphi \cdot \Delta q).
\]

Since \(q\) is the Poiseuille velocity field in \(\Omega_1\), it follows by (1.4) that \(\Delta q = (\alpha_1, 0, 0)\) in \(\Omega_4\), where \(\alpha_1\) is a constant. The function \(\varphi\) carries no
fluct, and so, by (2.5)(d),
\[\int_{\Omega_1} (\varphi \cdot \Delta \varphi) = C_1 \int_{-\infty}^{0} \int_{X_1} \varphi_1 = 0,\]
and similarly for \(\Omega_2\). Use of this result in (3.14) gives
\[\int_{\partial} (\nabla \varphi \cdot \nabla q) = -\int_{\Omega_1} (\varphi \cdot \Delta \varphi) \leq \text{const } |\varphi|_{\partial} \leq \text{const } |\varphi|_{H},\]
since \(H \to L_2\) by Lemma 2.1(c), and the constant is independent of \(\varphi\).

If \(x \in \Omega_1\), then \(q(x) = (q_1(x_1, x_3), 0, 0)\) and so \((q \cdot \nabla) q = 0\) (similarly in \(\Omega_2\)). Hence,
\[(3.16) \quad \{\varphi, q, q\} = \int_{\Omega_1} (q \cdot \nabla) q \leq \text{const } |\varphi|_{H},\]
and the constant is independent of \(\varphi\).

Equations (3.15) and (3.16) show that the right-hand side of (3.11) is a continuous linear functional on \(J\); we extend the functional to \(H\) by continuity. Therefore, the Riesz representation theorem ensures existence of a unique element \(r \in H\) such that
\[-r \int_{\partial} (\nabla \varphi \cdot \nabla q) - \{\varphi, q, q\} = \langle \varphi, r \rangle_{H} \quad \forall \varphi \in H(\Omega).\]

If \(u = q + w\) is a weak solution, then it follows easily by (3.12) that the individual terms of the left-hand side of (3.11) define continuous linear functionals on \(J\), which we then extend by continuity to \(H\), and so (3.13) is satisfied.

Conversely, if (3.13) holds, then by restricting \(q\) to \(J\), it follows that \(w\) satisfies (3.11) and so \(u = q + w\) is a weak solution. The lemma is proved.

We consider an expanding sequence of bounded domains \(U_m\) such that \(U_m \to \Omega\) as \(m \to \infty\) and \(\partial U_m\) is of class \(C^\infty\) for \(m = 1, 2, \ldots\). Denote the surfaces (for \(N = 3\)) or arcs (for \(N = 2\)) \(\partial U_m \cap \Omega_i\) by \(I_{m, j}^1\), \(j = 1, 2\) or \(3\) (Figure 4). Since \(\Omega_1\) is cylindrical, we assume that the \(I_{m, j}^1\) are identical for \(m = 1, 2, \ldots\) in the sense that \(I_{m, j}^1\) is a translation parallel to \(Ox_1\) of \(I_1^1\) (similarly \(I_{m, j}^2\) is the translation parallel to \(Ox_1\) of \(I_1^2\)).

For reasons to be explained presently, we now construct velocity fields \(g_m \in C^\infty(\overline{U}_m \to \mathbb{R}^N)\) with \(g_m = q = \nabla \times Q\) on \(\partial U_m\). Let
\[a_m(x) = \text{dist } (x, \partial U_m);\]
and let $\mu(t; \varepsilon)$ be the mollifier in Lemma 2.6. Let $\varepsilon$ be sufficiently small in the sense of Lemma 2.3 for the domain $U_1$. Then, by the translation property of $\Gamma^0_m$, the same $\varepsilon$ serves for all $U_m$. Now define $G_m \in C^\infty(\overline{U}_m \to \mathbb{R}^n)$ by

$$G_m(x; \varepsilon) = \mu(x_m(x); \varepsilon)Q(x),$$

and let

$$g^m(x; \varepsilon) = \nabla \times G_m.$$

The function $g^m$ depends on $m$ near $\Gamma^1_m$, and $\Gamma^3_m$, but is independent of $m$ elsewhere on $\overline{U}_m$ for $m > k$; its support is in a layer of width $\varepsilon$ adjacent to $\partial U_m$.

![Figure 4. - The expanding sequence of domains $\{U_m\}$. $\Gamma^1_m = \Omega_1 \cap \partial U_m$ is a translation of $\Gamma^1_1$.](image)

In each domain $U_m$, we seek a solution $(u^m, p^m)$ of the steady Navier-Stokes equations (1.1) to (1.3) with $u^m = g^m$ on $\partial U_m$. Since $U_m$ is bounded, this is a problem of Type 1 and the following result is known (see Finn [3]).

**Theorem 3.5.** For every $\nu > 0$, there exists a solution

$$(u^m, p^m) \in C^\infty(\overline{U}_m \to \mathbb{R}^n) \times C^\infty(\overline{U}_m \to \mathbb{R})$$

of the steady Navier-Stokes equations (1.1) to (1.3) such that $u^m = g^m = q$ on $\partial U_m$.

Define $w^m = u^m - q$. Then $w^m = 0$ on $\partial U_m$ and $\text{div} w^m = 0$ in $U_m$, and it follows by (2.5)(e) that $w^m \in H(U_m \to \mathbb{R}^n) \subset H(\Omega \to \mathbb{R}^n)$ (set $w^m = 0$ out-
side $U_m$) and satisfies (cf. (3.11))

\begin{equation}
\nu \langle \varphi, w^m \rangle_H + \{ \varphi, q + w^m, w^m \} + \{ \varphi, w^m, q \} = \langle \varphi, r \rangle_H \quad \forall \varphi \in H(U_m),
\end{equation}

where $r \in H(\Omega)$ is as in Lemma 3.4.

If we can show that the sequence $\{w^m\}$ is bounded in $H(\Omega)$ independently of $m$, then, since $H$ is a Hilbert space, a suitable subsequence of $\{w^m\}$ will converge weakly in $H$ to an element $w \in H$. We shall then prove that this $w$ satisfies (3.13), so that $u = q + w$ is the weak solution of our problem.

**Remarks on the velocity fields $q$ and $g^m$.**

(a) A function $g$ closely related to $g^m$ is needed in Theorem 3.6 for the crucial estimate (3.25). In addition, certain properties of the function $g^m$ and $q$ show why the methods for problems of Type 1 and 2 fail here.

(b) The proof of Theorem 3.5 depends on a representation $u^m = g^m + v^m$, $v^m \in H(U_m)$, and on Theorem 2.7, i.e.,

\begin{equation}
|\langle g^m(\cdot, \varepsilon), \varphi, \varphi \rangle| \leq \text{const} \|\varphi\|^2_{H(U_m)} \quad \forall \varphi \in H(U_m),
\end{equation}

where $\varepsilon > 0$ is sufficiently small and the constant is independent of $\varepsilon$ and $m$. For a problem of Type 1 or 2, it is the arbitrary small parameter $\varepsilon$ in (3.19)(a) which is essential in the proof of the existence of a solution $(u^m, p^m)$ for all $\nu > 0$ (see [3], pp. 131-132).

(c) It follows that $v^m \in H(U_m) \subset H(\Omega)$ (set $v^m = 0$ outside $U_m$) satisfies (cf. (3.11))

\begin{equation}
\nu \langle \varphi, v^m \rangle_H + \{ \varphi, g^m + v^m, v^m \} + \{ \varphi, v^m, g^m \} = -\nu \int_{\Omega_m} (\nabla \varphi : \nabla g^m) - \{ \varphi, g^m, g^m \} \quad \forall \varphi \in H(U_m).
\end{equation}

However, the usual procedure for bounding $\{v^m\}$ fails here. For, setting $\varphi = v^m$ in (3.19)(b) gives, in view of (3.12) and (3.19)(a),

\begin{align*}
\nu \|v^m\|^2_{H_m} &= \{ g^m, v^m, v^m \} - \nu \int_{\Omega_m} (\nabla v^m : \nabla g^m) - \{ v^m, g^m, g^m \} \\
&\leq \text{const} \|v^m\|^2_{H_m} - \nu \int_{\Omega_m} (\nabla v^m : \nabla g^m) - \{ v^m, g^m, g^m \},
\end{align*}

where the constant is independent of $\varepsilon$ and $m$, but the non-compactness of $\partial \Omega$ makes it appear impossible to bound the last two terms independently of $m$. 
(d) The function $q$ satisfying (3.1) is inferior to $g^m$ in that it has only a feebler analogue of (3.19)(a) which gives

$$|\{q, \varphi, \varphi\}| < c|\varphi|_{H}^{2}, \quad \forall \varphi \in H(U_m),$$

where the constant $c$ is independent of $m$. On the other hand, Lemma 3.4 shows $q$ is superior to $g^m$ in that the term $-\nu(\nabla \varphi : \nabla q) - \{\varphi, q, q\}$ can be bounded by $\text{const} |\varphi|_{H}$, and the constant is independent of $m$. Moreover, the condition $u \to q$ in (3.10) forces us to consider $u - q$ and motivates Definition 3.1.

**Theorem 3.6.** Let $S_j$ be the cylinder $\mathbb{R} \times A_j$ of which $\Omega_j$ forms a part ($j = 1$ or 3). Let $M > 0$ and define

$$\sigma_j = \sigma_{\{S_j, M\}} = \sup_{\varphi \in H(S_j)} \frac{\{q', \varphi, \varphi\}_{S_j}}{|\varphi|_{H(S_j)}}, \quad (|\varphi|_{H(S_j)} \neq 0)$$

where

$$\{q', \varphi, \varphi\}_{S_j} = \int_{S_j} q'(\varphi \cdot \nabla)\varphi,$$

and $q'$ is the Poiseuille velocity field in $S_j$ carrying flux $M > 0$. Let $\sigma = \sigma_{\{S_1, S_3, M\}} = \max (\sigma_1, \sigma_3)$.

If $\nu > \sigma$ and $w^m$ satisfies (3.18), then $w^m$ is bounded in $H$ independently of $m$.

**Remarks.** We emphasize that the constant $\sigma$ in Theorem 3.6 is independent of $S_2$ and is determined only by the cross-section $A_j$ of $\Omega_j$. The constants $\sigma_j$ are familiar in the nonlinear theory of hydrodynamic stability. If $N = 2$, then a calculation shows that $\sigma_1 = \sigma_3$.

**Proof of Theorem 3.6.** We shall give the proof for the case $N = 3$ since that for $N = 2$ is analogous.

The choice $\varphi = w^m$ in (3.18) gives, in view of (3.12)(b),

$$\nu|w^m|^2_{H} = \{q, w^m, w^m\} + \langle w^m, r \rangle_{H}.$$

Although $q \notin H(U_m \to \mathbb{R}^n)$ (because elements in $H$ carry no flux by (2.5)(d)) while $q$ carries flux $M$, we shall construct a function $s \in H(U_m)$ for $m$ sufficiently large such that $\{s, w^m, w^m\}$ is an approximation to $\{q, w^m, w^m\}$ in a certain sense. This is important since the choice $\varphi = s$ in (3.18) shows that $\{s, w^m, w^m\}$ can increase only linearly with $|w^m|_{H}$. 


Let $\varepsilon$ be as in the definition (3.17) of $g^m$. Let $a(x) = \text{dist}(x, \partial \Omega)$ and let
\[
g(x; \varepsilon) = \nabla \times G(x; \varepsilon) \quad \text{where} \quad G(x; \varepsilon) = \mu(a(x); \varepsilon) Q(x).
\]

Now let $\delta > 0$, assume that $m$ is sufficiently large and define the function $s$ by (i)
\[
s(x; \varepsilon, \delta) = \begin{cases}
q(x) - g(x) & \text{in } \Omega_1, \\
\nabla \times \left[ (Q(0, x_2, x_3) - G(0, x_2, x_3)) \{1 - \theta(\delta x_1)^2\} \right] & \text{in } \Omega_1, \\
\nabla \times \left[ (Q(0, x'_2, x'_3) - G(0, x'_2, x'_3)) \{1 - \theta(-\delta x'_1)^2\} \right] & \text{in } \Omega_2,
\end{cases}
\]
where $\theta \in C^0(\mathbb{R} \to [0, 1])$ is a mollifier such that $\theta(t) = 1$ for $t < -1$ and $\theta(t) = 0$ for $t > 0$. Thus $s = 0$ for $x_1 < -1/\delta$ in $\Omega_1$ and $x'_1 > 1/\delta$ in $\Omega_2$; in due course we shall choose $\varepsilon$ and $\delta$ to be small positive constants independent of $m$. Therefore, we can take $m$ so large that $\text{supp } s \subset U_m$ and then $s \in J(U_m) \subset H(U_m)$.

(i) The contribution of $\Omega_1$ to the difference of the triple products is
\[
\{q, w^m, w^m\}_{\Omega_1} - \{s, w^m, w^m\}_{\Omega_1} = \{g, w^m, w^m\}_{\Omega_1} < \int_{\Omega_1} |g \cdot (w^m \cdot \nabla) w^m| \\
< \int_{\Omega_1} |g \cdot (w^m \cdot \nabla) w^m|
\]
because $\text{supp } g \cap \Omega_1 \subset \Omega_1$. Since $Q \in C^0(\overline{\Omega} \to \mathbb{R}^3)$ and $\nabla \times Q = 0$ on $\partial \Omega$, we have, by Theorem 2.7,
\[
(3.23) \quad \{q, w^m, w^m\}_{\Omega_1} - \{s, w^m, w^m\}_{\Omega_1} < \text{const } \varepsilon \|w^m\|^2_{L_1} < \text{const } \varepsilon \|w^m\|^2_{L_1}
\]
and the constant is independent of $\varepsilon$ and $m$.

(ii) Let $\theta_0(x)$ stand for $\theta(\delta x)$ in $\Omega_1$ and for $\theta(-\delta x')$ in $\Omega_2$. It follows from (3.22) that the components of $q$ and $s$ are related in $\Omega_1$ by
\[
q_1 - s_1 = q_1 \theta_0^3 + q_1(1 - \theta_0^3); \quad q_2 = s_2 = 0; \quad |s_1|, |s_2| < \text{const } \delta,
\]

(1) In (3.22) and throughout this proof, $\theta(\cdot)^3 = \{\theta(\cdot)\}^3$, the superscript denoting a square.
so that
\[
\{q, w^m, w^m\}_{\Omega} - \{s, w^m, w^m\}_{\Omega} - \{q\delta y^2, w^m, w^m\}_{\Omega} =
\int_{\Omega} \{(1 - \theta^2) (g, 0, 0) \cdot (w^m \cdot \nabla) w^m\} - \int_{\Omega} \{(0, s, s) \cdot (w^m \cdot \nabla) w^m\}.
\]

Estimating the first integral on the right as in (3.23) and the second by means of the Schwarz inequality and Lemma 2.1(c), we obtain
\[
(3.24) \quad \{q, w^m, w^m\}_{\Omega} - \{s, w^m, w^m\}_{\Omega} - \{q\delta y^2, w^m, w^m\}_{\Omega} < \text{const} (e + \delta)|w^m|_{H^1},
\]
and the constant is independent of \(e, \delta, \) and \(m\). A similar result holds in \(\Omega_s\):

(iii) From (3.23) and (3.24), we have the estimate
\[
(3.25) \quad \{q, w^m, w^m\} < \{s, w^m, w^m\} + \{q\delta y^2, w^m, w^m\}_{\Omega_s} + \text{const} (e + \delta)|w^m|_{H^1}.
\]

The choice \(q = s\) in (3.18) yields, since \(s\) has compact support,
\[
(3.26) \quad \{s, w^m, w^m\} = -\langle s, w^m \rangle_{H^1} - \{s, q, w^m\} - \{s, w^m, q\} + \langle s, r \rangle_{H^1}
\]
\[
< k_0(e, \delta) + k_1(e, \delta)|w^m|_{H^1}
\]
for certain (large) functions \(k_0\) and \(k_1\) independent of \(w^m\) and \(m\). Use of (3.25) and of (3.26) in (3.21) gives
\[
(3.27) \quad \langle v|w^m|_{H^1} < \{s, w^m, w^m\} + \{q\delta y^2, w^m, w^m\}_{\Omega_s} + \text{const} (e + \delta)|w^m|_{H^1} + |r|_{H^1}|w^m|_{H^1}
\]
\[
< \{q\delta y^2, w^m, w^m\}_{\Omega_s} + k_0(e, \delta) + k_1(e, \delta)|w^m|_{H^1} + C(e + \delta)|w^m|_{H^1} + |r|_{H^1}|w^m|_{H^1};
\]
where the constant \(C\) is independent of \(e, \delta, w^m,\) and \(m\).

Define
\[
(3.28) \quad \Gamma(\delta) = \sup_{v \in \mathcal{H}(D)} \frac{\{q\delta y^2, v, v\}_{\Omega_s}}{|v|^2_{H^1(D)}} \quad (|v|_{H^1(D)} \neq 0);
\]
we refer to Theorem 4.3 for the proof that \(\lim_{\delta \to 0^+} \Gamma(\delta) = \sigma = \sigma(S_1, S_2; \mathcal{M})\)
(where \(\sigma\) is described in the statement of the present theorem).

Now let \(v > \sigma\) and choose \(e\) and \(\delta\) so small that \(C(\epsilon + \delta) < \frac{1}{2}(v - \sigma)\)
and \(\Gamma(\delta) - \sigma < \frac{1}{2}(v - \sigma),\) then from (3.27) we obtain
\[
\frac{1}{2}(v - \sigma)|w^m|_{H^1} < k_0(e, \delta) + \{k_1(e, \delta) + |r|_{H^1}|w^m|_{H^1},
\]
and so \(w^m\) is bounded in \(H\) independently of \(m\). The theorem is proved.
The following standard lemma ([9], pp. 84-85) is needed in Theorem 3.8 to ensure that if \( w \) is the weak limit in \( H \) of a certain subsequence of \( \{w^m\} \), then \( w \) satisfies (3.13).

**Lemma 3.7.** Let \( U \subset \mathbb{R}^n \) be a bounded domain with \( \partial U \) of class \( C^1 \), then the imbedding \( W_2^1(U \to \mathbb{R}^n) \hookrightarrow L_s(U \to \mathbb{R}^n) \) is compact

\[
\text{for } 1 < s < \infty, \quad \text{if } N = 2, \text{ and } \\
\text{for } 1 < s < 6, \quad \text{if } N = 3.
\]

**Theorem 3.8.** If \( \nu > \sigma \) (\( \sigma \) as in the statement of Theorem 3.6), then there exists a weak solution \( u = q + w \) of the problem (1.7) to (1.10), i.e., there exists \( w \in H(\Omega \to \mathbb{R}^n) \) satisfying

\[
\dot{w} + \nabla \cdot (\nabla w + w w) = f \quad \text{in } \Omega,
\]

where \( q \) satisfies (3.1) and \( r \in H(\Omega) \) is as in Lemma 3.4.

**Proof.** Since the sequence \( \{w^m\} \) is bounded in the Hilbert space \( H(\Omega) \), it contains a weakly convergent subsequence; say

\[
w^m \rightharpoonup w \quad \text{weakly in } H(\Omega) \text{ as } i \to \infty.
\]

It suffices to prove that \( w \) satisfies (3.13) for all \( \varphi \in \mathcal{J} \) since \( \mathcal{J} \) is dense in \( H \).

Let \( \varphi \in \mathcal{J} \) and choose \( k \) such that \( \text{supp} \varphi \subset U_k \). Then for all \( m_i > k \), \( \varphi \in H(U_{m_i}) \) and \( w^m \) satisfies (3.18), i.e.

\[
\nu \langle \varphi, w^m \rangle_H + \{\varphi, q + w^m, w^m \} + \{\varphi, w^m, q \} = \langle \varphi, r \rangle_H.
\]

Since \( w^m \rightharpoonup w \) weakly in \( H \) and \( H \hookrightarrow L^4_0 \),

\[
\nu \langle \varphi, w^m \rangle_H + \{\varphi, q, w^m \} + \{\varphi, w^m, q \} \to \nu \langle \varphi, w \rangle_H + \{\varphi, q, w \} + \{\varphi, w, q \}
\]
as \( i \to \infty \) because, for fixed \( \varphi \in H(\Omega) \), the terms on the left-hand side are bounded linear functionals on \( H(\Omega) \) with argument \( w^m \).

Now consider \( \{\varphi, w^m, w^m\} \). By Lemma 3.7, \( W_2^1(U_{m_i}) \) is imbedded compactly in \( L_4(U_{m_i}) \), so that \( w^m \) converges strongly in \( L_4(U_{m_i}) \) and, by use of (3.12)(a), we have

\[
|\{\varphi, w^m, w^m\} - \{\varphi, w, w\}| = |\{\varphi, w^m - w, w^m\} - \{w^m - w, w, \varphi\}| \lesssim \text{const } |w^m - w|_{L_4(U_{m_i})} \to 0 \quad \text{as } i \to \infty
\]
because of the strong convergence in \( L_4(U_{m_i}) \).
REMARKS

(a) The constants \( \sigma_j \) (\( j = 1 \) or 3) in Theorem 3.6 occur in the nonlinear theory of hydrodynamic stability. Let \( S_j = \mathbb{R} \times A_j \) be an admissible cylindrical domain. If \( \nu > \sigma_j \), then the Poiseuille velocity field \( q' \) is the unique solution of the steady Navier-Stokes equations for the domain \( S_j \), among all functions \( u \) such that \( u - q' \in H(S_j) \). If \( \nu > \sigma_j \) and \( v(x, t) \) is a solution of the time-dependent Navier-Stokes equations in \( S_j \), with initial velocity \( v(x, 0) \) such that \( v(x, 0) - q'(x) \in H(S_j) \), then

\[
\lim_{t \to \infty} \| v(\cdot, t) - q' \|_H = 0
\]

(Serrin [17]), provided that \( v \) exists for all time.

(b) The restrictions on \( \Omega \) can be relaxed to allow a finite number of smooth bounded bodies \( V_i \) to be in the interior of \( \Omega \). In this case, the coordinate systems are chosen so that \( V_i \cap \Omega_j = \emptyset \) for \( j = 1 \) and 3 and \( i = 1, 2, \ldots \). Lemma 3.2 allows the construction in Theorem 3.3 of a velocity field \( q \) satisfying (3.1). The existence theorems 3.6 and 3.8 also hold after slight modification in their proofs.

(c) Instead of an admissible domain \( \Omega \) with one cylindrical domain \( \Omega_1 \) « upstream » and one \( \Omega_3 \) « downstream », we could have \( j \) disjoint cylinders \( \{ U_i \}_{i=1}^j \) upstream, where \( U_i = \mathbb{R} \times A_i \) in some coordinate system, and \( k \) disjoint cylinders \( \{ U_i \}_{i=j+1}^{j+k} \) downstream. As usual, we assume that \( \Omega \setminus \bigcup_{i=1}^{j+k} U_i \) is bounded. If \( q' \) is the Poiseuille velocity field in \( U_i \) carrying flux \( M_i \) and \( \sum_{i=1}^j M_i = \sum_{i=j+1}^{j+k} M_i \), then we can seek a solution of (1.7) to (1.9) with \( u(x) \to q'(x) \) as \( |x| \to \infty \) in \( U_i \). A slight modification in the proof of the theorems of this section gives the existence of a weak solution \( u \) for all \( \nu > \max (\sigma_1, \ldots, \sigma_{j+k}) \) where \( \sigma_i \) is defined as in the statement of Theorem 3.6.

(d) The assumption, in the Introduction, that the external force \( f \) be derivable from a scalar potential \( P \) (that is, \( f = -\nabla P \)) can be relaxed. It suffices for \( \int_{\partial} (\varphi \cdot f) \) to define a bounded linear functional on \( H(\Omega) \); in this case, there exists by the Riesz representation theorem a unique element \( \tilde{f} \in H(\Omega) \) such that

\[
\int_{\partial} (\varphi \cdot f) = \langle \varphi, \tilde{f} \rangle_H \quad \forall \varphi \in H(\Omega),
\]

and \( \tilde{f} \) may then be absorbed into the term \( r \) of Lemma 3.4.
3.3. Existence of a weak solution for a symmetric admissible domain.

If the admissible domain $\Omega$ is symmetric about $Ox_1$, then the results of Theorem 3.8 can be improved.

We take cylindrical coordinates $(x_1, r, \theta)$ in $\Omega$ (the $\theta$ coordinate is omitted for $N = 2$). It is possible to show the existence of $Q_s = Q_s(x_1, r)$ such that $q_s = \nabla \times Q_s$ satisfies (3.1) and is symmetric about $Ox_1$. Let $H_s(\Omega \to \mathbb{R}^n)$ denote the completion of $J_s = \{Q \in J: Q$ is symmetric about $Ox_1\}$ in the Dirichlet norm.

**Definition 3.2.** Let $\Omega$ be a symmetric admissible domain and let $q_s = q_s(x_1, r)$ satisfy (3.1). The function $u_s = q_s + w_s$ is a symmetric weak solution of the problem (1.7) to (1.10) if $w_s \in H_s(\Omega \to \mathbb{R}^n)$ ($N = 2$ or 3) and

\[
\text{where } r_s \in H_s(\Omega) \text{ is as in Lemma 3.4.}
\]

A simple argument shows that if $u_s$ is a symmetric weak solution of the problem, then it is also a weak solution in the sense of Definition 3.1.

**Corollary 3.9.** Let $\Omega \subset \mathbb{R}^n$ be a symmetric admissible domain and let $S_i$ be as in the statement of Theorem 3.6. Define

\[
\sigma_{s,i} = \sup_{q \in H_s(s_i)} \frac{\langle q_i^t, \hat{q}, q_s \rangle_{S_j}}{\|q\|_{H_s(s_i)}} \quad (\|q\|_{H_s(s_i)} \neq 0),
\]

where $q_i^t$ is the symmetric Poiseuille velocity field in $S_i$ carrying flux $\mathcal{M} > 0$. Let $\sigma_s = \max(\sigma_{s,i}, \sigma_{s,i})$. If $\nu > \sigma_s$, then there exists a symmetric weak solution of the problem (1.7) to (1.10).

**Remarks.** We note that the cross-sections of $S_i$ for $N = 3$ are circular, and a simple calculation shows that $\sigma_{s,i} = \sigma_{s,i}$ for $N = 2$.

**Proof of Corollary 3.9.** The corollary follows easily by certain observations.

(i) Lemma 3.4 holds with $u$ replaced by $u_s$, $q$ by $q_s$, $w$ by $w_s$, $H$ by $H_s$, and $r \in H$ by $r_s \in H_s$.

(ii) If the increasing bounded domains $U_m$ are chosen to be symmetric about $Ox_1$, then the functions $u^m$ in Theorem 3.5 can be chosen to be symmetric about $Ox_1$.

(iii) The functions $w^m = u^m - q_s$ are symmetric and the proof of Theorem 3.6 holds with the obvious change of notation.
(iv) As in Theorem 3.6, we leave to Corollary 4.4 the proof that
\[ \lim_{\delta \to 0^+} I_\delta(\sigma) = \sigma, \]
where
\[ I_\delta(\sigma) = \sup_{v \in H_0^1(\Omega)} \frac{\langle a_0^2, v, v \rangle_{\Omega_1 \cup \Omega_3}}{|v|_{H_0^1(\Omega)}} \quad (|v|_{H_0^1(\Omega)} \neq 0). \]

(v) Theorem 3.8 holds with the obvious change of notation.

3.4. The critical Reynolds number.

**Definition 3.3.** Let \( \Omega \) be an admissible domain in \( \mathbb{R}^N \) (\( N = 2 \) or \( 3 \)) and let \( M, v > 0 \). We define the Reynolds number \( R = R(\Omega, M, v) \) by

\[
R = \begin{cases} 
\frac{M}{v} & \text{for } N = 2, \\
\frac{M}{v} \left( \frac{|A_1|^{-1} |A_3|^{-1}}{\sigma_1} \right) = (R_1, R_3) & \text{for } N = 3,
\end{cases}
\]

where \( |A_i| \) is the two-dimensional measure of the cross-section \( A_i \) of \( S_i \). For \( N = 3 \), we write \( R' < R \) if and only if \( R'_1 < R_1 \) and \( R'_3 < R_3 \).

The following theorem follows immediately from Theorems 3.6 and 3.8.

**Theorem 3.10.** Let \( \Omega \subset \mathbb{R}^N \) be an admissible domain and let \( M > 0 \). Let \( \sigma_1, \sigma_3 \) and \( \sigma \) be as in Theorem 3.6 and define

\[
R_0 = \begin{cases} 
\frac{M}{\sigma} = \frac{M}{\sigma_1} = \frac{M}{\sigma_3} & \text{for } N = 2, \\
\frac{M}{v} \left( \frac{|A_1|^{-1} |A_3|^{-1}}{\sigma_1} \right) & \text{for } N = 3.
\end{cases}
\]

If \( R < R_0 \), then there exists a weak solution \( u \) of the problem (1.7) to (1.10).

In the case that \( \Omega \) is a symmetric admissible domain, a symmetric weak solution exists for all \( R < R_{0s} \), where \( R_{0s} \) is as in Theorem 3.10 with the replacement of \( \sigma, \sigma_1, \) and \( \sigma_3 \) by \( \sigma_s, \sigma_{1s}, \) and \( \sigma_{3s}, \) respectively.

We now give some numerical values of \( R_0 \) and \( R_{0s} \), for various types of admissible domains; these values follow immediately from the calculation of \( \sigma \), which will be discussed in a forthcoming paper. The values of \( R_0 \) and \( R_{0s} \) are known for the case \( N = 2 \) and for certain simple geometries of \( \Omega \) for \( N = 3 \).

The cross-sections \( A_i \) of \( \Omega_i \) (\( i = 1 \) or \( 3 \)) are not necessarily equal in any of the following domains.
(a) $N=2$ and $\Omega$ is an admissible domain:

$$R_0 = 116.5.$$  

(b) $N=2$ and $\Omega$ is a symmetric admissible domain:

$$R_{o,s} = 194.6.$$  

(c) $N=3$ and $\Omega$ is an admissible domain such that $\Omega_1$ and $\Omega_3$ have circular cross-sections:

$$R_0 = (127.9, 127.9).$$  

(d) $N=3$ and $\Omega$ is a symmetric admissible domain:

$$R_{o,s} = (282.6, 282.6).$$

3.5. Asymptotically cylindrical domains.

In this section, we prove the existence of a weak solution of (1.7) to (1.10) for a domain $\Omega \subset \mathbb{R}^N$ which asymptotically approaches a cylinder $S_j$ as $|x| \to \infty$ in $\Omega_j$ ($j = 1$ or 3). We shall show that if the asymptotic approach is sufficiently fast, then there exists a weak solution for all $\nu > \sigma(S_1, S_2; M)$, where $\sigma$ is as in the statement of Theorem 3.6. For simplicity, we shall consider only the case $N=2$, since that for $N=3$ is more tedious although analogous.

We shall be concerned with asymptotically cylindrical domains $\Omega \subset \mathbb{R}^2$ with $\partial\Omega$ of class $C^\infty$ and of the form $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$ where

1. $\Omega_1 = \{(x_1, x_2): x_1 < 0, x_2 \in (-d_1 \varphi_-(x_1), d_1 \varphi_+(x_1))\}$ (where $d_1 > 0$),

where $\varphi_-, \varphi_+ \in C^\infty((-\infty, 0])$ and $\varphi_-(x_1), \varphi_+(x_1) \to 1$ as $x_1 \to -\infty$.

2. $\Omega_2 = \{(x_1', x_2')': x_1' > 0, x_2' \in (-d_2 \varphi_-(x_1'), d_2 \varphi_+(x_1'))\}$ (where $d_2 > 0$),

where $\varphi_-, \varphi_+ \in C^\infty([0, \infty))$ and $\varphi_+(x_1'), \varphi_-(x_1') \to 1$ as $x_1' \to \infty$.

3. $\Omega_3 = \Omega \setminus (\Omega_1 \cup \Omega_2)$ is bounded.

We assume that the boundary functions $(\varphi_-, \varphi_+, \psi_-, \psi_+)$ are positive-valued. The domain $\Omega_1$ approaches the cylinder $S_1 = \mathbb{R} \times (-d_1, d_1)$ as $x_1 \to -\infty$ in $\Omega_1$ and similarly $\Omega_3$ approaches $S_3 = \mathbb{R} \times (-d_3, d_3)$.  

As in section 3.2, we seek a weak solution \( u \) of (1.7) to (1.10) in the form \( u = q + w \) where \( q \) is to satisfy

\[
(3.31) \quad \begin{align*}
(a) & \ q \in \mathcal{C}^\infty(\overline{\Omega} \rightarrow \mathbb{R}^d); \\
(b) & \ \text{div} \ q = 0 \text{ in } \Omega, \ q = 0 \text{ on } \partial \Omega; \text{ and} \\
(c) & \ q(x) \to q'(x) \text{ uniformly as } |x| \to \infty \text{ in } \Omega, \text{ where } q' \text{ is the Poiseuille velocity field in the cylinder } S_i \text{ carrying flux } M > 0.
\end{align*}
\]

The function \( u = q + w \), where \( w \in H(\Omega \rightarrow \mathbb{R}^d) \), is to satisfy (Definition 3.1)

\[
(3.11) \quad \nu \langle \varphi, w \rangle_u + \{ \varphi, q + w, w \} + \{ \varphi, w, q \} =
\]

\[= -\nu \int_\Omega (\nabla \varphi : \nabla q) - \{ \varphi, q, q \} \quad \forall \varphi \in \mathcal{J}(\Omega \rightarrow \mathbb{R}^d).
\]

The construction of a function \( q \) satisfying (3.31) is analogous to that in section 3.1. We can define \( Q_p \) such that \( \nabla \times (0, 0, Q_p) \) satisfies (3.31) in \( \overline{\Omega}_j \cup \overline{\Omega}_s \). Let \( M > 0 \) and define

\[
(3.32) \quad Q_p(x) = \frac{3}{4} M \left[ t(x) - \frac{1}{2} \{ t(x) \}^2 \right] \quad \text{for } x \in \overline{\Omega}_j \cup \overline{\Omega}_s,
\]

where

\[
t(x) = \begin{cases}
\{ 2x_3/d_1 - (\varphi_+ - \varphi_-)(x_1) \}/(\varphi_+ + \varphi_-)(x_1) & \text{in } \overline{\Omega}_j, \\
\{ 2x_3/d_1 - (\varphi_+ - \varphi_-)(x_1) \}/(\varphi_+ + \varphi_-)(x_1') & \text{in } \overline{\Omega}_s.
\end{cases}
\]

It follows that for \( x \in \overline{\Omega}_j \), we have

\[
q(x) = (D_2 Q_p(x), - D_1 Q_p(x)) = \frac{3}{4} M \left[ 1 - \{ t(x) \}^2 \right] (D_2 t(x), - D_1 t(x)),
\]

where \( D_2 t = 2/[d_1(\varphi_+ + \varphi_-)] \) and

\[
D_1 t = \left\{ -\frac{2x_3}{d_1} D_1(\varphi_+ + \varphi_-) + 2\varphi_+ D_1 \varphi_- - 2\varphi_- D_1 \varphi_+ \right\}/(\varphi_+ + \varphi_-)^2.
\]

Since the Poiseuille velocity field \( q^1 \) in \( S_i = \mathbb{R} \times (-d_1, d_1) \) is given by

\[
q^1(x) = \frac{3M}{4d_1} \left[ 1 - \left( \frac{x_3}{d_1} \right)^2 \right] (1, 0),
\]
it follows that $q(x) \to q^1(x)$ uniformly as $|x| \to \infty$ in $Q_1$ if

$$P(1)(a) \quad D_1 q_+(x_1), \quad D_1 q_-(x_1) \to 0 \quad \text{as} \quad x_1 \to -\infty.$$  

Similarly $q(x) \to q^3(x)$ uniformly as $|x| \to \infty$ in $Q_3$ if

$$P(1)(b) \quad D_1 q_+(x_1'), \quad D_1 q_-(x_1') \to 0 \quad \text{as} \quad x_1' \to \infty.$$  

Using the same arguments as in Theorems 3.1 and 3.3, we have the following result.

**THEOREM 3.11.** Let $Q \subset \mathbb{R}^2$ be an asymptotically cylindrical domain and assume that the boundary functions $q_+, \varphi_-, \psi_+, \psi_-$ satisfy $P(1)$. Then there exists $Q \in C^\infty(S^2 \to \mathbb{R})$ such that

$$q^1 = \nabla \times (0, 0, Q) \text{ satisfies } (3.31).$$

We now state conditions which ensure that $q$ defines a continuous linear functional on $J(Q)$ which can be extended by continuity to $H(Q)$. A calculation shows that this will be true if, in addition to $P(1)$, we have

$$P(2)(a) \quad (1 - q_+), \quad (1 - q_-), \quad D_1^2 q_+, \quad D_1^2 q_- \in L_2(\mathbb{R}^-);$$

$$P(2)(b) \quad (1 - q_+), \quad (1 - q_-), \quad D_1^2 q_+, \quad D_1^2 q_- \in L_2(\mathbb{R}^+).$$

We use this result and follow the arguments of Lemma 3.4 to obtain

**LEMMA 3.12.** Let $Q \subset \mathbb{R}^2$ be an asymptotically cylindrical domain and assume that the boundary functions $q_+, \varphi_-, \psi_+, \psi_-$ satisfy $P(1)$ and $P(2)$. Then there exists an element $v \in H(\Omega)$ such that $u = q + w$ is a weak solution of the problem (1.7) to (1.10) if and only if

$$\nu \langle q, w \rangle_H + \{q, q + w, w\} + \{q, w, q\} = \langle q, v \rangle_H \quad \forall v \in H(\Omega \to \mathbb{R}^2).$$
Before proceeding as in section 3.3, we need a preliminary result. For each positive integer \( n \), decompose \( \Omega \) by

\[
\begin{align*}
\Omega_1^n & = \{ x \in \Omega_1 : x_1 < -n \}, \\
\Omega_2^n & = \{ x \in \Omega_2 : x_1' > n \}, \\
\Omega_3^n & = \Omega \setminus (\Omega_1^n \cup \Omega_2^n).
\end{align*}
\]

Recall that the mollifier \( \theta \in C^\infty(\mathbb{R} \to [0, 1]) \) used in the proof of Theorem 3.6 satisfies \( \theta(t) = 1 \) for \( t < -1 \) and \( \theta(t) = 0 \) for \( t > 0 \). For \( n \) a positive integer and \( \delta > 0 \), define

\[
\theta_n^\delta(x) = \begin{cases} 
\theta(\delta(x_1 + n)) & \text{for } x \in \Omega_1^n, \\
\theta(-\delta(x_1' - n)) & \text{for } x \in \Omega_2^n.
\end{cases}
\]

**Lemma 3.13.** Let \( \Omega \subset \mathbb{R}^3 \) be an asymptotically cylindrical domain and assume that the boundary functions \( \varphi_+, \varphi_-, \varphi_+, \varphi_- \) satisfy \( P(1) \). For \( \delta > 0 \) and \( n \) a positive integer, let

\[
\Gamma_n(\delta) = \sup_{\varphi \in H(\Omega)} \frac{\{q(\theta_n^\delta)^*, \varphi, \varphi\}_{\Omega_1^n \cup \Omega_2^n}}{|\varphi|_{H(\Omega)}^2} \quad (|\varphi|_{H(\Omega)} \neq 0),
\]

where \( q \) satisfies (3.31). Let \( S_j = \mathbb{R} \times (-d_j, d_j) \) (\( j = 1 \) or 3) be the cylinder which \( \Omega_j \) approaches as \( |x| \to \infty \), and define

\[
\sigma_j = \sigma_j(S_j; M) = \sup_{\varphi \in H(S_j)} \frac{\{q^j, \varphi, \varphi\}_{S_j}}{|\varphi|_{H(S_j)}^2} \quad (|\varphi|_{H(S_j)} \neq 0),
\]

where \( q^j \) is the Poiseuille velocity field in \( S_j \) carrying flux \( M > 0 \). Then

\[
\lim_{n \to \infty} (\lim_{\delta \to 0} \Gamma_n(\delta)) = \sigma(S_1, S_3; M) = \max(\sigma_1, \sigma_3).
\]

**Proof.** Let \( \{ U^s \} \) be a non-decreasing sequence of admissible domains contained in \( \Omega \) with \( \lim U^s = \Omega \) and such that \( U^s = U_1^s \cup U_3^s \cup U_3^s \), where

\[
\begin{align*}
U_1^s & = (-\infty, n) \times (-d_1^s, d_1^s) \subset \Omega_1^s, \\
U_2^s & = (n, \infty) \times (-d_2^s, d_2^s) \subset \Omega_2^s, \\
U_3^s & = U^s \setminus (U_1^s \cup U_3^s).
\end{align*}
\]

Since \( \lim U^s = \Omega \), we have necessarily that \( \lim d_1^s = d_1 \) and \( \lim d_2^s = d_2 \), where \( d_1 \) and \( d_2 \) are as in the definition of the asymptotically cylindrical domain \( \Omega \).
Let $r^n$ denote the extended Poiseuille velocity field in $U^n$ satisfying (3.1).

By Theorem 4.3, we have

$$\lim_{\delta \to 0^+} \sup_{\varphi \in H(U^n)} \frac{\{r^n(\varphi^n)^*, \varphi, \varphi\} \alpha^n_{i,j} \cup \alpha^n_{i,j}}{\|\varphi\|_{H^n(U^n)}} = \sigma(R \times (-d^n_1, d^n_1), R \times (-d^n_2, d^n_2); M).$$

By construction $H(U^n) \subset H(\Omega)$ and $U^n_j \subset \Omega^n_j$ ($j = 1$ or 3), so that

$$\sup_{\varphi \in H(U^n)} \frac{\{r^n(\varphi^n)^*, \varphi, \varphi\} \alpha^n_{i,j} \cup \alpha^n_{i,j}}{\|\varphi\|_{H^n(U^n)}} < \sup_{\varphi \in H(\Omega)} \frac{\{r^n(\varphi^n)^*, \varphi, \varphi\} \alpha^n_{i,j} \cup \alpha^n_{i,j}}{\|\varphi\|_{H(\Omega)}}.$$

Letting $\delta \to 0^+$, we have

$$\sigma(R \times (-d^n_1, d^n_1), R \times (-d^n_2, d^n_2); M) < \lim_{\delta \to 0^+} \sup_{\varphi \in H(\Omega)} \frac{\{r^n(\varphi^n)^*, \varphi, \varphi\} \alpha^n_{i,j} \cup \alpha^n_{i,j}}{\|\varphi\|_{H(\Omega)}}.$$

If we use property P(1) of the boundary functions, then

$$\lim_{n \to \infty} \sup_{x \in \partial \alpha^n_{i,j} \cup \partial \alpha^n_{i,j}} |q(x) - r^n(x)| = 0,$$

and since one can show easily that

$$\lim_{n \to \infty} \sigma(R \times (-d^n_1, d^n_1), R \times (-d^n_2, d^n_2); M) = \sigma(S_1, S_2; M),$$

we have

$$\sigma(S_1, S_2; M) < \lim_{n \to \infty} \sigma(S_1, S_2; M),$$

A similar argument using a non-increasing sequence of admissible domains $\{V^n\}$ each containing $\Omega$ with $\lim V^n = \Omega$ gives

$$\lim_{n \to \infty} \sigma(S_1, S_2; M) < \sigma(S_1, S_2; M),$$

and the lemma is proved.

Consider a non-decreasing sequence of bounded domains $\{U_m\}$ analogous to that of Figure 4 such that each $U_m$ is contained in $\Omega$ and $\lim U_m = \Omega$. We assume that there exists a positive $\varepsilon_0$ independent of $m$ and sufficiently small in the sense of Lemma 2.3 for all $U_m$. This assumption is valid if we have the following condition on the boundary functions:

$$P(3) \quad \sup_{x_1 \in \partial} |D^2_x \varphi_+(x_1)|, \quad |D^2_x \varphi_-(x_1)| < \infty,$$

and

$$\sup_{x_1 \in \partial} |D^2_x \varphi_+(x_1)|, \quad |D^2_x \varphi_-(x_1)| < \infty.$$
Let \( w^m = u^m - q \) where \( u^m \) is the solution (cf. Theorem 3.5) of the steady Navier-Stokes equations (1.1) to (1.3) in \( U_m \) such that \( u^m = q \) on \( \partial U_m \). We have the following version of Theorems 3.6 and 3.8.

**Theorem 3.14.** Let \( M > 0 \). Let \( \Omega \subset \mathbb{R}^d \) be an asymptotically cylindrical domain and \( S_j = \mathbb{R} \times (-d_j, d_j) \) \((j = 1 \text{ or } 3)\) the cylinder which \( \Omega \) approaches as \( |x| \to \infty \) in \( \Omega \). Assume that the boundary functions satisfy \( P(1), P(2), \) and \( P(3) \). If \( v > \sigma(S_1, S_3; M) = \max \{ \sigma(S_1; M), \sigma(S_3; M) \} \) \( \sigma \) as in (3.20), then

(a) \( w^m \) is bounded in \( H(\Omega) \) independently of \( m \);

(b) there exists a weak solution \( u \) of the problem (1.7) to (1.10) of the form \( u = q + w \) where \( q \) satisfies (3.31) and \( w \in H(\Omega) \).

**Proof.** (a) Let \( v > 1 \). By Lemma 3.13, we can choose a positive integer \( k \) so that \( |\alpha - \lim_{\delta \to 0^+} \Gamma_\delta(\alpha)| < \frac{1}{4} (v - \sigma) \). The proof of Theorem 3.6 holds with minor changes and \( Q_2 \), \( \mathcal{O}_a \), and \( \Gamma(\delta) \) replaced by \( \Omega_1 \), \( \Omega_2 \), \( \theta' \), and \( \Gamma(\delta) \), respectively. The parameter \( \varepsilon \) must be restricted to the interval \((0, \varepsilon_\delta)\) by \( P(3) \).

(b) The proof of (b) is identical to that of Theorem 3.8.

4. - Comparison of the suprema \( \sigma \) and \( \Gamma(\delta) \).

In this section, we complete the proof of Theorem 3.6 by showing in Theorem 4.3 that \( \lim_{\delta \to 0^+} \Gamma(\delta) = \sigma \) where \( \Gamma(\delta) \) is defined in (3.28) and \( \sigma \) in (3.20).

As a corollary, for symmetrical admissible domains, we complete the proof of Corollary 3.9 by proving that \( \Gamma'((\delta) \to \sigma \) as \( \delta \to 0^+ \).

Our plan is to shift the \( \theta_0 \) from \( q \) to \( v \) in (3.28); unfortunately, \( \theta_0 v \) is not necessarily solenoidal and so we shall need the following lemma, which shows that \( \theta_0 v \) is close to a solenoidal function in a certain sense.

Throughout this section, \( \Omega \) will be an admissible domain in \( \mathbb{R}^N \) \((N = 2 \text{ or } 3)\) unless the contrary is stated.

**Lemma 4.1.** Denote by \( S_1 \) the cylinder \( \mathbb{R} \times A_1 \) of which \( \Omega_1 \) forms a part.

Let \( E = E(S_1 \to \mathbb{R}^n) \) denote as before the completion of \( C_{0}^m(S_1 \to \mathbb{R}^n) \) in the Dirichlet norm and let \( H = H(S_1 \to \mathbb{R}^n) \). Given any \( v \in H(\Omega \to \mathbb{R}^n) \) define \( f \in E \) by

\[
f(x) = \theta(\delta x_1) v(x)
\]
where the mollifier \( \theta \) is as in (3.22). Under the orthogonal decomposition
\[
f = g + h, \quad g \in H^1, \ h \in H,
\]
we have

(a) \( \text{div} \ f = \text{div} \ g \in E(S_1 \rightarrow \mathbb{R}), \)

(b) \( |g|_x \leq C\delta \|v\|_{\mathcal{A}_1}, \)

where the constant \( C \) depends only on the cross-section \( A_1 \) of \( \Omega_1 \).

**Proof.** We shall give the proof for the case \( N = 3 \) since that for \( N = 2 \)
is analogous.

(a) Since \( \text{div} \ v = 0 \) in \( \Omega \) (in a generalized sense), we have
\[
\text{div} \ f = \text{div} \ g = \partial \theta'(\delta x_1) v_1(x) \in E(S_1 \rightarrow \mathbb{R}) \quad \left( \theta'(t) = \frac{\partial \theta(t)}{\partial t} \right)
\]
because \( v \in H(\Omega \rightarrow \mathbb{R}^n) \).

(b) It suffices to prove (b) for \( v \in J(\Omega \rightarrow \mathbb{R}^n) \), and this will be done
in three steps.

(i) Suppose that we can construct \( g^0 \in E \) with bounded support
such that

\[
\begin{align*}
(4.1)(a) \quad & \text{div} \ g^0 = \text{div} \ f \quad \text{in} \ S_1, \\
(4.1)(b) \quad & g^0 = 0 \quad \text{on} \ \partial S_1,
\end{align*}
\]

and

\[ (4.2) \quad |g^0|_x \leq C\delta \|v\|_{\mathcal{A}_1}. \]

Now \( g - g^0 = f - h - g^0 \in H \) because \( h \in H \), and because \( \text{div}(f - g^0) = 0 \) in \( S_1 \),
\( (f - g^0) = 0 \) on \( \partial S_1 \), and \( \text{supp}(f - g^0) \) is bounded, so that it follows by (2.5)(e)
that \( (f - g^0) \in H \). Hence \( g_0 = g + (g^0 - g) \) with \( g \in H^1 \) and \( (g^0 - g) \in H \), so that
\[
|g^0|_x = |g_2^0|_x + |g^0 - g|_x \quad \text{implies} \quad |g|_x \leq |g^0|_x.
\]

(ii) We seek \( g^0 = (0, g_2^0, g_3^0) \) in the form

\[ (4.3) \quad g^0(x) = \delta \theta'(\delta x_1) \{\nabla' \tau + \nabla' \times (\psi, 0, 0)\} \]

where \( \nabla' = (0, D_2, D_3) \). Recall that \( \text{supp} \ \theta'(\delta x_1) \subset [-1/\delta, 0] \).
We choose \( \tau \) to be the solution of

\[
(4.4)(a) \quad (D_2^\alpha + D_3^\beta) \tau = v_1 \quad \text{in} \quad (-1/\delta, 0) \times A_1,
\]

\[
(4.4)(b) \quad \tau = 0 \quad \text{on} \quad (-1/\delta, 0) \times \partial A_1,
\]

and then construct \( \psi \) to make \( g^\delta = 0 \) on \( \partial A_1 \); equation (4.4)(a) ensures that (4.1)(a) is satisfied.

Now (4.4) is the familiar Dirichlet problem, and \( v_1(x_1, \cdot, \cdot) \in E(A_1 \rightarrow \mathbb{R}) \) for each fixed \( x_1 \in (-1/\delta, 0) \). Abbreviating this statement to \( v_1 \in E(A_1) \), we know from standard results that \( \tau \in \mathcal{W}^2_\delta(A_1) \) and

\[
(4.5)(a) \quad |\tau|_{\mathcal{W}^2_\delta(A_1)} \leq \text{const} \left| v_1 \right|_{\mathcal{E}(A_1)}.
\]

Moreover, (4.4) holds with \( \tau \) replaced by \( D_1 \tau \) and \( v_1 \) by \( D_1 v_1 \), and since \( D_1 v_1 \in L_2(A_1) \), we have \( D_1 \tau \in \mathcal{W}^2_\delta(A_1) \) and

\[
(4.5)(b) \quad |D_1 \tau|_{\mathcal{W}^2_\delta(A_1)} \leq \text{const} |D_1 v_1|_{\mathcal{E}(A_1)}.
\]

Integrating these estimates with respect to \( x_1 \), we find that

\[
(4.6) \quad |\partial \theta^\prime \nabla' \tau|_g \leq \text{const} \delta \| v \|_{A_1}.
\]

(iii) To construct \( \psi \), we need results concerning the trace of \( \tau \) and its derivatives on \( \partial A_1 \). The trace maps \( \mathcal{W}^2_\delta(A_1) \rightarrow \mathcal{W}^1_\delta(\partial A_1) \) and \( \mathcal{W}^2_\delta(A_1) \rightarrow \mathcal{W}^1_\delta(\partial A_1) \) are bounded (Treves [11], p. 237), so that

\[
(4.7)(a) \quad |\tau|_{\mathcal{W}^1_\delta(\partial A_1)} \leq \text{const} |\tau|_{\mathcal{W}^2_\delta(A_1)} \leq \text{const} \left| v_1 \right|_{\mathcal{E}(A_1)},
\]

\[
(4.7)(b) \quad |D_1 \tau|_{\mathcal{W}^1_\delta(\partial A_1)} \leq \text{const} |D_1 \tau|_{\mathcal{W}^2_\delta(A_1)} \leq \text{const} |D_1 v_1|_{\mathcal{E}(A_1)},
\]

and the constant depends only on \( A_1 \).

Since \( \partial A_1 \) is of class \( C^\infty \), we define boundary-layer coordinates \((s, t)\) as in Theorem 3.3. Recall that the map \((x_2, x_3) \mapsto (s, t)\) is one-to-one and \( C^\infty \) for sufficiently small positive values of \( t \), say for \( 0 < t < t_0^\prime \). In order that \( g^\delta \) vanish on \((-1/\delta, 0) \times \partial A_1 \), we demand in view of (4.3) that

\[
\begin{align*}
\tau_s + \psi_t = 0 & \quad \text{for } t = 0, -1/\delta < x_1 < 0, \\
\tau_t - \psi_s = 0 & \quad \text{for } t \neq 0.
\end{align*}
\]
Using the fact that \( \tau \in W^2_0(\partial A_1) \), define \( F(x_1, \cdot) \in W^2_0(\partial A_1) \) by

\[
F(x_1, s) = \frac{\partial \tau}{\partial t} |_{t=0}.
\]

For \( t_0 \) as above, let \( \mu \in C^\infty(\mathbb{R} \to [0, 1]) \) be a mollifier with \( \mu(t) = 1 \) for \( t < t_0/2 \) and \( \mu(t) = 0 \) for \( t > t_0 \). Define

\[
\psi(x_1, s, t) = \mu(t) \int_0^t F(x_1, s') \, ds'.
\]

The function \( \psi \) is well-defined because, by the zero-flux property of \( v \) noted in (2.5)(d),

\[
\int_{\partial A_1} \frac{\partial \tau}{\partial t} = -\int_{A_1} (D_1 + D_2) \tau = -\int_{A_1} v_1 = 0.
\]

It remains to verify that

\[
|\partial \psi/\partial x' \times (\psi, 0, 0)|_x \leq \operatorname{const} \|v\|_{D_1}.
\]

The relevant second derivatives of \( \psi \) are given by

\[
\psi_{ss} = \tau_{ss} |_{t=0} \mu(t), \quad \psi_{ss} = \tau_{ss} |_{t=0} \mu'(t),
\]

\[
\psi_{st} = \mu(t) \int_0^t F(x_1, s') \, ds', \quad \psi_{tx_1} = \mu'(t) \int_0^t D_1 F(x_1, s') \, ds',
\]

\[
\psi_{sx_1} = \mu(t) \tau_{tx_1} |_{t=0}.
\]

By (4.7), we have

\[
|\psi|_{W^2_0(A_1)} \leq \operatorname{const} |\tau|_{W^2_0(A_1)} \leq \operatorname{const} |v_1|_{H_1(A_1)},
\]

\[
|D_1 \psi|_{W^2_0(A_1)} \leq \operatorname{const} |D_1 \tau|_{W^2_0(A_1)} \leq \operatorname{const} |D_1 v_1|_{H_1(A_1)}.
\]

and the constant depends only on \( A_1 \).

Integrating these estimates with respect to \( x_1 \), we obtain (4.10) and the lemma is proved.

A corresponding result holds with \( \Omega_1 \) replaced by \( \Omega_2 \) in Lemma 4.1.

The lemma has a simple analogue if the admissible domain \( \Omega \) is symmetrical about \( O x_1 \). Let \( E_1 \) and \( H_1 \) denote the axisymmetric functions in \( E(S_1 \to \mathbb{R}^n) \) and \( H(S_1 \to \mathbb{R}^n) \), respectively.
COROLLARY 4.2. Let $\Omega$ be axisymmetric and denote by $S_1$ the cylinder $\mathbb{R} \times A_1$ of which $\Omega_1$ forms a part. Given any $v \in H_1(\Omega \to \mathbb{R}^n)$, define $f \in E_\ast$ by

$$f(x) = \theta(\delta x) v(x).$$

Under the orthogonal decomposition

$$f = g + h, \quad g \in H_1^+, \quad h \in H_1,$$

we have

(a) $\text{div} \ f = \text{div} \ g \in E_\ast(S_1 \to \mathbb{R}),$

(b) $|g|_\ast < \text{const} \delta \|v\|_{L^2}.$

We are now able to complete the proof of Theorem 3.6.

THEOREM 4.3. Let

$$(\text{as in (3.28)}), \quad \text{and let } \sigma_1, \sigma_2, \text{ and } \sigma \text{ be as in Theorem 3.6. (Recall that } \theta_0^2 = (\theta_0)^2). \quad \text{Then}$$

$$\lim_{\delta \to 0^+} \Gamma(\delta) = \sigma.$$

PROOF. The theorem will be proved in several steps. We shall consider the case $N=3$ since that for $N=2$ is analogous.

(i) First we show that $\Gamma(\delta) > \sigma = \max(\sigma_1, \sigma_2)$. Given $\varphi \in J(S_1 \to \mathbb{R}^n)$, we can form a corresponding function $v_\varphi \in J((- \infty, -1/\delta) \times A_1) \subset J(\Omega_1)$ by translation parallel to $Ox_1$ (thus $v_\varphi(x_1, x_2, x_3) = \varphi(x_1 + k, x_2, x_3)$ for some $k > 0$). Then $\theta_0^2 = q^1$ on $v_\varphi$, and

$$\Gamma(\delta) \geq \sup_{\varphi \in J(S_1)} \frac{\{q_\theta^2, v_\varphi, v_\varphi\}_{L^2}}{\|v_\varphi\|_{L^2}^2} = \sup_{\varphi \in J(S_1)} \frac{\{|q_1, \varphi, \varphi\}_{L^2}}{|\varphi|_{L^2}^2} = \sigma_1,$$

and similarly $\Gamma(\delta) > \sigma_2$.

(ii) Next, we show that $\Gamma(\delta) \leq \sigma + o(\delta)$. Define for $j = 1$ or 3:

$$\Gamma_j(\delta, v) = \frac{\{q_\theta^2, v, v\}_{L^2}}{\|v\|_{L^2}^2} \quad \text{and} \quad \Gamma_j(\delta) = \sup_{v \in E_\ast} \Gamma_j(\delta, v).$$
Since (for fixed $v \in J(\Omega)$)

\[
\frac{\langle q^2, v, v \rangle}{|v|^2_{H(\Omega)}} < \max \{\Gamma_1(\delta, v), \Gamma_3(\delta, v)\},
\]

it suffices to show that

\[
\Gamma_j(\delta) < \sigma_j + o(\delta), \quad j = 1 \text{ or } 3.
\]

An integration by parts gives (with superscript 2 denoting a square and not a label)

\[
\begin{align*}
\Gamma_1(\delta, v) &= \int_{\Omega_1} \{q^1_2 \theta^2_0 \phi_0 \phi_0' + v_2 D_2 v_1 + v_2 D_2 v_1\} / \|v\|_{H_1}^2 \\
&= -\int_{\Omega_1} \{q^1_2 \theta^2_0 \phi_0' \phi_0 + v_2 D_2 v_1 + v_2 D_2 v_1\} / \|v\|_{H_1}^2 \\
&= o(\delta) + \int_{\Omega_1} \{q^1_2 \theta^2_0 \phi_0 \phi_0' + v_2 D_2 v_1 + v_2 D_2 v_1\} / \|v\|_{H_1}^2
\end{align*}
\]

since $|v|_{H_1(\omega_0)} < \text{const} \|v\|_{H_1}$.

(iii) Consider the last integral in (4.11); our plan is to shift the $\theta^2_0$ from $q^1_2$ to the $v$ terms, and to show that the projection $h$ of $\theta_0 v$ into $H(S_1)$ is sufficiently close to $\theta_0 v$ for our purposes.

Define $f \in E(\Omega_1) \subset H(S_1) = E$ by $f(x) = \theta_0(x_1)v(x)$. Then (4.11) becomes

\[
\begin{align*}
\Gamma_1(\delta, v) &= o(\delta) + \int_{\Omega_1} \{q^1_2 f_2 D_2 f_1 + f_2 D_2 f_1\} / \|v\|_{H_1}^2 \\
&= o(\delta) + \{q^1_2 f_2 D_2 f_1 + f_2 D_2 f_1\} / \|v\|_{H_1}^2
\end{align*}
\]

because $q^1_2 f_2 D_2 f_1$ integrates to zero.

One can show that

\[
\|f\|_{H_1}^2 = \int_{\Omega_1} (\theta^2_0|\nabla v|^2) + o(\delta) \|v\|_{H_1}^2,
\]

\[
< (1 + o(\delta)) \|v\|_{H_1}^2,
\]

since $\theta(t) \in [0, 1]$ for all $t$.

We write $f = g + h$, where $h \in H(S_1)$ and $g \in H^1(S_1)$; by Lemma 4.1, we have

\[
|g|_{H^1} < \text{const} \|v\|_{H_1}.
\]
Moreover,
\[ \| h \|_{s_i} = |h|_s < |f|_s < (1 + o(\delta)) \| v \|_{\partial_i}. \]
Accordingly, setting \( f = g + h \),
\[
\Gamma_1(\delta, v) = o(\delta) + \left[ \left\{ q^1, g, g \right\}_{s_i} + \left\{ q^1, q, h \right\}_{s_i} + \left\{ q^1, h, h \right\}_{s_i} \right] / \| v \|_{\Omega_1}^2 \\
< o(\delta) + \left\{ q^1, h, h \right\}_{s_i} / \| v \|_{\Omega_1}^2 + \text{const} \left( |g|_s |h|_s + |g|_{\Omega_1}^2 / \| v \|_{\Omega_1}^2 \right) \\
< o(\delta) + \left\{ q^1, h, h \right\}_{s_i} / \| h \|_{s_i}^2.
\]
By taking the supremum of (4.13) over \( v \in J(\Omega) \), it follows that \( \Gamma_1(\delta) < \sigma_1 + o(\delta) \) and the theorem is proved.

We have the following corollary for an axisymmetric admissible domain.

**Corollary 4.4.** For a symmetrical admissible domain \( \Omega \), let
\[
\Gamma_4(\delta) = \sup_{v \in J(\Omega)} \frac{\{ q, g, g \}_{\Omega_1 \cup \Omega_2} + \{ q, h, h \}_{\Omega_1 \cup \Omega_2} + \{ q, h, h \}_{\Omega_1 \cup \Omega_2}}{\| v \|_{\Omega_1}, \| v \|_{\Omega_2}} \\
(\text{as in (3.30)}), \text{ and let } \sigma_{1,*}, \sigma_{2,*}, \text{ and } \sigma_* \text{ be as in Corollary 3.9. Then}
\[
\lim_{\delta \to 0^+} \Gamma_4(\delta) = \sigma_*.
\]

**Acknowledgement.** I am indebted to my research supervisor, Edward Fraenkel, for his immense help in the preparation of this paper.

**REFERENCES**


