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The Sieve of Eratosthenes-Legendre.

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By the sieve of Eratosthenes-Legendre we mean that described by Halberstam and Richert in their beautiful book [1], Chapter 1.

Let be given a finite sequence $\mathcal{A}$ of integers and a set $\mathcal{P}$ of primes. For each real number $z > 2$ let

$$P(z) = \prod_{p < z, p \in \mathcal{P}} p$$

and

$$S(\mathcal{A}, \mathcal{P}, z) = \sum_{a \in \mathcal{A}} 1,$$

the number of elements in the sequence $\mathcal{A}$ which are not divisible by any prime number $p < z$ from $\mathcal{P}$. The sieve method deals with estimates of $S(\mathcal{A}, \mathcal{P}, z)$ by linear forms in

$$|\mathcal{A}_d| = \sum_{a \in \mathcal{A}} 1,$$

(the number of elements in $\mathcal{A}$ which are divisible by $d|P(z)$)

$$\sum_{d|P(z)} \lambda_d^- |\mathcal{A}_d| < S(\mathcal{A}, \mathcal{P}, z) < \sum_{d|P(z)} \lambda_d^+ |\mathcal{A}_d|.$$

The multipliers $\lambda_d^+$, usually called weights, are real numbers satisfying the following conditions

$$\lambda_1^- = \lambda_1^+ = 1,$$

$$\sum_{d|D} \lambda_d^- < 0 < \sum_{d|D} \lambda_d^+ \text{ for all } D > 1, D|P(z).$$


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Next, for each squarefree integer \( d \) we choose \( \omega(d) \) so that \( (\omega(d)/d) X \), where \( X \) is a suitable number, approximates \( |\mathcal{A}_d| \), and we write the remainder as
\[
 r_d = |\mathcal{A}_d| - \frac{\omega(d)}{d} X.
\]
Inserting this into (2) we get
\[
 S(\mathcal{A}, \mathfrak{P}, z) < X \sum_{d \in \mathfrak{P}(\epsilon)} \lambda^+_d \frac{\omega(d)}{d} + \sum_{d \in \mathfrak{P}(\epsilon)} |\lambda^+_d r_d|,
\]
and
\[
 S(\mathcal{A}, \mathfrak{P}, z) > X \sum_{d \in \mathfrak{P}(\epsilon)} \lambda^-_d \frac{\omega(d)}{d} - \sum_{d \in \mathfrak{P}(\epsilon)} |\lambda^-_d r_d|.
\]
We want to make these estimates optimal. This requirement determines the parameter \( X \) and the function \( \omega(d) \) almost uniquely. It appears in practice that \( \omega(d) \) is multiplicative and for some \( \kappa > 0 \) satisfies the condition
\[
 -L < \sum_{\omega(p) \leq \kappa} \frac{\omega(p) - \kappa}{p} \log p < A_2,
\]
for all \( 2 < \omega < z \), where \( L \) and \( A_2 \) are some constants \( > 1 \). The parameter \( \kappa \) is called the "dimension" of the sieve.

The method of Eratosthenes-Legendre rests on the use of Möbius function \( \mu(d) \) as common value of the weights
\[
 \lambda^+_d = \lambda^-_d = \mu(d).
\]
It turns formula (2) into the Legendre identity
\[
 S(\mathcal{A}, \mathfrak{P}, z) = \sum_{d \in \mathfrak{P}(\epsilon)} \mu(d)|\mathcal{A}_d|
\]
and this identity usually leads to a bad result because, unless \( z \) is very small, the remainder sum
\[
 \sum_{d \in \mathfrak{P}(\epsilon)} |r_d|
\]
has too many terms. It was Viggo Brun who first showed how to construct the sieving weights \( \lambda^+_d \) more effectively. For details see [1], Chapter 1.

The aim of this paper is to show that the Eratosthenes-Legendre sieve yields an asymptotic formula for the sifting function \( S(\mathcal{A}, \mathfrak{P}, z) \) in the case of the dimension \( \kappa < \frac{1}{2} \) and the sequence \( \mathcal{A} \) with elements not too large:
\[
 x = \max_{a \in \mathcal{A}} |a| < A_2 X
\]
for some $A_4 > 1$. It is assumed that the remainders $r_d$ are also not too large:

\[(9) \quad |r_d| < A_4 o(d)\]

for some $A_4 > 1$.

Suppose that

\[(10) \quad 0 < \frac{o(p)}{p} < 1 - \frac{1}{A_4}\]

for all $p \in \mathcal{P}$ and put

\[ W(z) = \prod_{p \in \mathcal{P}(o)} \left( 1 - \frac{o(p)}{p} \right). \]

The result reads as follows:

**THEOREM.** Under the assumptions (7)-(10) we have

\[ S(\mathcal{A}, \mathcal{F}, z) = \frac{e^{\gamma y}}{I(1-\gamma)} W(z) X \left\{ f(s) + O\left( \frac{(s + 1) L}{1 - 2\kappa} (\log z)^{s - 1} \right) \right\}, \]

where $\gamma = 0.577 \ldots$ is the Euler constant, $s = \log x / \log z$. The function $f(s)$ is defined in Section 2. For $0 < s < 1$ we have

\[ f(s) = s^{-\kappa}. \]

The constant in the symbol $O$ depends only on $A_1, A_2, A_3$ and $A_4$.

In his thesis, Sullivan proved the asymptotic formula

\[ S(\mathcal{A}, \mathcal{F}, z) \sim W(z) X \]

under the condition

\[ \sum_{p < z} \frac{o(p)}{p} \log p = o(\log x) \]

instead of (7). His method is based on Halberstam-Richert’s Fundamental Lemma [2] (oral communication).

A comparison of our method with Brun’s method will be given elsewhere.

Keeping the notations introduced above we have

\[ S(\mathcal{A}, \mathcal{F}, z) = \sum_{d \in \mathcal{P}(o)} \mu(d) |\mathcal{A}_d| = \sum_{d \in \mathcal{P}(o), d < z} \mu(d) |\mathcal{A}_d| = X \sum_{d \leq x} \mu(d) \frac{o(d)}{d} + \sum_{d \leq x} \mu(d) s_d. \]
1. – An estimate of the remainder sum.

From (9) we get

\[ \sum_{d \leq x \atop d \mid P(x)} |r_d| \ll A_i \sum_{d \leq x} \omega(d). \]

With \( \omega \) we associate the generalized Mangoldt’s function \( \lambda \) as usual by Dirichlet’s convolution

\[ \omega(d) \log d = \sum_{n \mid d} \omega(n) \lambda(d/n). \]

Since \( \omega \) is multiplicative, the support of \( \lambda \) is contained in the set of powers of primes. It is easily seen that \( \lambda(p) = \omega(p) \log p \), so

\[ \sum_{d \leq x \atop d \mid P(x)} \omega(d) \log d = \sum_{n \leq x \atop n \mid P(x)} \omega(n) \sum_{m \leq x/n \atop m \mid P(x)} \lambda(m) \]

\[ < \sum_{n \leq x \atop n \mid P(x)} \omega(n) \sum_{p \leq x/n} \omega(p) \log p. \]

Using partial summation, from the upper bound (7) we get

\[ \sum_{p \leq x} \omega(p) \log p \ll x \]

and

\[ \sum_{p \leq x} \omega(p)/p < x \log \log x + O(1). \]

Hence

\[ \sum_{d \leq x \atop d \mid P(x)} \omega(d) \log d \ll x \sum_{d \leq x \atop d \mid P(x)} \frac{\omega(n)}{n} < x \prod_{p \leq x} \left( 1 + \frac{\omega(p)}{p} \right) < x \exp \left( \sum_{p \leq x} \omega(p)/p \right) \ll x (\log x)^{\omega-1}. \]

Using partial summation again we get the final result

\[ \sum_{d \mid P(x), d \leq x} \omega(d) \ll x (\log x)^{\omega-1} \]

and thus we have the same estimate for the remainder sum (12).
2. The function $f(s)$.

**Lemma 1.** Let $0 < x < \frac{1}{2}$ and $f(s)$ be the continuous solution of

\[
\begin{cases}
  f(s) = s^{-x} & \text{for } 0 < s < 1 \\
  sf'(s) = xf(s-1) - xf(s) & \text{for } s > 1.
\end{cases}
\]

Then, for $s \to \infty$ we have

\[f(s) = e^{-\pi y} \Gamma(1-x) + O(e^{-s}).\]

**Proof.** The derivative $f'(s)$ satisfies the equation

\[sf'(s) = -xf(u) du + O(e^{-t}),\]

so $f'(s) = O(e^{-s})$ and thus

\[f(s) = 1 + \int_1^s f'(u) du = c + O(e^{-s}).\]

It remains to calculate the constant $c$. To this end, let us consider the Laplace transform

\[L(s) = \int_0^\infty e^{-st} f(t) dt = \int_0^\infty e^{-st}(c + O(e^{-t})) dt = cs^{-1} + O(1).\]

It can easily be checked that

\[\frac{d}{ds} (sL(s)) = x(1 - e^{-s}) L(s)\]

and thus

\[sL(s) = c \exp \left( x \int_0^s (1 - e^{-t}) \frac{dt}{t} \right).\]
Now, we calculate in two ways the limit $\lim_{s \to \infty} s^{1-\xi} L(s)$. We have

$$
\lim_{s \to \infty} s^{1-\xi} L(s) = c \lim_{s \to \infty} s^{-\xi} \exp \left( \int_0^s (1-e^{-t}) \frac{dt}{t} \right) = ce^{\gamma}.
$$

On the other hand

$$
\lim_{s \to \infty} s^{1-\xi} L(s) = \lim_{s \to \infty} s^{1-\xi} \int_0^s e^{-st} f(t) dt
$$

$$
= \lim_{s \to \infty} \frac{1}{s} e^{-st} \int_0^1 e^{-tu} du
$$

$$
= \lim_{s \to \infty} \int_0^1 e^{-su} u^{-\xi} du
$$

$$
= \Gamma(1-\xi).
$$

This completes the proof of the Lemma.

**Corollary.** The function

$$
F(s) = \int_0^s f(u) du, \quad s > 0
$$

is of $C^1$-class and satisfies the equations

$$
F(s) = \frac{1}{1-\xi} s^{1-\xi}, \quad \text{for } 0 < s < 1,
$$

$$
sF'(s) = (1-\xi)F(s) + \xi F(s-1), \quad \text{for } s > 1,
$$

$$
\frac{d}{ds} \left( \frac{F(s)}{s^{1-\xi}} \right) = \frac{F(s-1)}{s^{2-\xi}}, \quad \text{for } s > 1.
$$

3. – An asymptotic formula for the main term.

Let us put

$$
g(d) = \mu(d) \frac{o(d)}{d}
$$
and define for all \( x > 1, \ z > 1 \)

\[
G(x, z) = \sum_{d \leq \sqrt{x} \atop d \in \mathcal{P}} g(d).
\]

To get an asymptotic formula for \( G(x, z) \) we apply the first step of the Levin-Fainleib's iteration method \([4]\). This method works effectively for sums of positive multiplicative functions, in particular for the one which appears in the Selberg sieve (see \([3]\)). Although our function \( g(d) \) changes sign, it turns out that the method still works in the case \( x < \frac{1}{2} \) considered here. Imitating \([3]\), we shall prove

**Proposition 1.** For \( x > 2, \ z > 2 \) we have

\[
G(x, z) = C f(s) (\log z)^{-\kappa} + O \left( \frac{(s+1)L}{1-2\kappa} (\log x)^{\kappa-1} \right),
\]

where

\[
s = \frac{\log x}{\log z}, \quad C = \frac{1}{\Gamma(1-x)} \prod_p \left( 1 - \frac{\omega(p)}{p} \right) \left( 1 - \frac{1}{p} \right)^{-\kappa}
\]

and the constant in the symbol \( O \) depends only on \( A_1, A_2 \).

Theorem will follow from

**Corollary.** For \( x > 2, \ z > 2 \) we have

\[
G(x, z) = \frac{e^{\kappa \gamma}}{\Gamma(1-x)} W(z) \left\{ f(s) + O \left( \frac{(s+1)L}{1-2\kappa} (\log z)^{\kappa-1} \right) \right\}.
\]

The constant implied in the symbol \( O \) depends only on \( A_1 \) and \( A_2 \).

To derive Corollary from Proposition 1 we have to show

\[
W(z) = e^{-\kappa \gamma} \Gamma(1-x) C (\log z)^{-\kappa} \left\{ 1 + O \left( \frac{L}{\log z} \right) \right\}.
\]

But this is a simple consequence of the Mertens formula

\[
\prod_{p \leq z} \left( 1 - \frac{1}{p} \right) = \frac{e^{-\gamma}}{\log z} \left( 1 + O \left( \frac{1}{\log z} \right) \right)
\]

and of the estimate

\[
\Gamma(1-x) C = \prod_{p \leq z} \left( 1 - \frac{\omega(p)}{p} \right) \left( 1 - \frac{1}{p} \right)^{-\kappa} \left\{ 1 + O \left( \frac{L}{\log z} \right) \right\}.
\]

For details see \([1]\), Lemma 5.3.
Before we prove Proposition 1 we shall show a few auxiliary lemmas. Let

$$T(x, z) = \int_1^x \frac{G(t, z)}{t} \, dt = \sum_{d \leq x \atop d \mid F(x)} g(d) \log \frac{x}{d}$$

for $x > 1$ and $z > 1$. For $0 < x < 1$ we put $T(x, z) = 0$.

**Lemma 2.** For $x > 2$ and $z > 2$ we have

$$G(x, z) \log x = (1 - \kappa) T(x, z) + \kappa T \left( \frac{x}{z}, z \right) + O(L \log^* x).$$

The constant in the symbol $O$ depends only on $A_1$ and $A_2$.

**Proof.** We start with the definition of generalized Mangoldt’s function $\chi$ associated with $g$:

$$g(d) \log d = \sum_{n \mid d} g(n) \chi(d/n).$$

Since $g$ is multiplicative, the support of $\chi$ is contained in the set of powers of primes. It is easily seen that $\chi(p) = - (\omega(p)/p) \log p$, so

$$\sum_{d \leq x \atop d \mid F(x)} g(d) \log d = \sum_{d \leq x \atop d \mid F(x)} g(d) \sum_{n < x/d \atop n \mid F(x)} \chi(n) = - \sum_{d \leq x \atop d \mid F(x)} g(d) \sum_{p < x/d \atop p \nmid d} \frac{\omega(p)}{p} \log p.$$

Since

$$\sum_{p < y} \frac{\omega(p)}{p} \log p = \kappa \log y + O(L), \quad \sum_{d < x \atop d \mid F(x)} |g(d)| < \prod_{p < x} \left(1 + \frac{\omega(p)}{p}\right) \ll (\log x)^\kappa,$$

we obtain

$$\sum_{d < x \atop d \mid F(x)} g(d) \log d = - \kappa \sum_{d < x \atop d \mid F(x)} g(d) \log \left( \min \left( z, \frac{x}{d} \right) \right) + O(L \log^* x).$$

If we add the sum $\sum_{d < x \atop d \mid F(x)} g(d) \log x/d$ to both sides, we arrive at (14).
**Lemma 3.** For \( x > y > 2 \) and \( x > 2 \) we have

\[
\frac{T(x, z)}{(\log x)^{1-x}} = \frac{T(y, z)}{(\log y)^{1-x}} + \int_y^z \frac{T(t, z)}{(\log t)^{1-x}} \, dt + O\left(\frac{L}{1 - 2x}\right)(\log y)^{2x-1}.
\]

The constant in the symbol \( O \) depends only on \( A_1 \) and \( A_2 \).

**Proof.** Let us write (14) with \( x \) replaced by \( t \) and divide throughout by \( t(\log t)^{2-x} \). Integrating with respect to \( t \) from \( y \) to \( x \), we obtain

\[
\int_y^x \frac{G(t, z)}{t(\log t)^{1-x}} \, dt = (1 - \kappa) \int_y^x \frac{T(t, z)}{t(\log t)^{1-x}} \, dt + \int_y^x \frac{T(t, z)}{t(\log t)^{1-x}} \, dt + O\left(\frac{L}{1 - 2x}\right)(\log y)^{2x-1}.
\]

If we integrate the identity

\[
\frac{\partial}{\partial t} \left( \frac{T(t, z)}{t(\log t)^{1-x}} \right) = \frac{G(t, z)}{t(\log t)^{1-x}} - (1 - \kappa) \frac{T(t, z)}{t(\log t)^{1-x}}
\]

and add the result to the formula above, we arrive at (14).

**Lemma 4.** For \( x > 2 \), we have

\[
T(x, z) = \frac{C}{1 - \kappa} (\log x)^{1-x} + O\left(\frac{L}{1 - 2x}\log^x x\right).
\]

The constant in the symbol \( O \) depends only on \( A_1 \) and \( A_2 \).

**Proof.** For \( x > t > 2 \) we have

\[
G(t, z) = G(t, t) = G(t),
\]

\[
T(t, z) = T(t, t) = T(t)
\]

and by Lemma 2

\[
r(t) = G(t) \log t - (1 - \kappa) T(t) \ll L \log^x t.
\]

If we divide (16) throughout by \( t(\log t)^{2-x} \) and integrate with respect to \( t \)
from 2 to \( x \), we obtain

\[
\int_2^x \frac{G(t)}{t^{(\log t)^{1-\kappa}}} dt - \left(1 - \kappa\right) \int_2^x \frac{T(t)}{t^{(\log t)^{1-\kappa}}} dt = \int_2^x \frac{r(t)}{t^{(\log t)^{1-\kappa}}} dt = c_1 + O\left(\frac{L}{1 - 2\kappa} (\log x)^{\kappa - 1}\right).
\]

But

\[
\frac{d}{dt} \left(\frac{T(t)}{t^{(\log t)^{1-\kappa}}}\right) = \frac{G(t)}{t^{(\log t)^{1-\kappa}}} - \left(1 - \kappa\right) \frac{T(t)}{t^{(\log t)^{1-\kappa}}},
\]

so integration by parts leads to

\[
\frac{T(x)}{(\log x)^{1-\kappa}} - \frac{T(2)}{(\log 2)^{1-\kappa}} = c_1 + O\left(\frac{L}{1 - 2\kappa} (\log x)^{\kappa - 1}\right)
\]

and finally

\[
T(x) = c_2 (\log x)^{1-\kappa} + O\left(\frac{L}{1 - 2\kappa} \log \kappa x\right).
\]

It remains to calculate the constant \( c_2 = c_1 + \log^x 2 \).

From (16) we get

(17) \quad G(x) = (1 - \kappa) c_4 (\log x)^{-\kappa} + O\left(\frac{L}{1 - 2\kappa} (\log x)^{\kappa - 1}\right)

for \( x >> 2 \). Hence, for \( s > 0 \) we have

\[
\prod_p \left(1 - \frac{\omega(p)}{p^{s+1}}\right) = \prod_{n=1}^{\infty} \frac{g(n)}{n^{s+1}} = s \int_1^\infty \frac{G(x)}{x^{s+1}} dx = s \int_1^2 + s \int_2^\infty = O(s) + (1 - \kappa) c_2 s.
\]

\[
\cdot \int_2^\infty \frac{\log^{-\kappa} x}{x^{s+1}} dx + O\left(\frac{L s^x}{1 - 2\kappa} \int_2^\infty \frac{\log x}{x^{s+1}} dx\right) = (1 - \kappa) c_2 s x^x \Gamma(1 - x) + O\left(\frac{L}{1 - 2\kappa} x^{1-x}\right).
\]

Since \( \lim_{s \to 0} s^x (s + 1) = 1 \), by the Euler product formula we obtain

\[
(1 - \kappa) c_4 \Gamma(1 - \kappa) = \lim_{s \to 0} s^x \prod_p \left(1 - \frac{\omega(p)}{p^{s+1}}\right) = \prod_p \left(1 - \frac{\omega(p)}{p}\right) (1 - \frac{1}{p})^{-\kappa}.
\]

This completes the proof of Lemma 4.
Now, we are ready to prove the general result

**PROPOSITION 2.** For \( x > 2 \) and \( z > 2 \) we have

\[
T(x, z) = CF(s)(\log x)^{1-x} + O\left(\frac{(s + 1)L}{1 - 2x} \log^x x\right).
\]

The constant in the symbol \( O \) depends only on \( A_1 \) and \( A_2 \).

**PROOF.** The proof will proceed by induction. Putting

(19)

\[
T(x, z) = CF(s)(\log x)^{1-x} + R(x, z)
\]

we have to prove

(20)

\[
R(x, z) \ll \frac{(s + 1)L}{1 - 2x} \log^x x,
\]

for all \( x > 2 \) and \( z > 2 \). This has already been proved in Lemma 4 for all \( z > x > 2 \). In this range \( R(x, z) = R(x, x) \).

If we introduce (19) into (15) we find out that the leading terms disappear throughout and we are left with a relation between the remainder terms only, namely

(21)

\[
\frac{R(x, z)}{(\log x)^{1-x}} = \frac{R(y, z)}{(\log y)^{1-x}} + \int \frac{R(t/z, z)}{(\log t)^{1-x}} d(\log \log t) + O\left(\frac{L}{1 - 2x} (\log y)^{x-1}\right)
\]

for all \( x > y > 2 \) and \( z > 2 \).

Let us assume for a moment that \( y = z \) and

(22)

\[
z < x < z^2.
\]

Accordingly, we can use (20) to the right-hand side of (21) and in the result we arrive again at (20) but now for the range (22). Therefore, we already have

(23)

\[
R(x, z) \ll \frac{BL}{1 - 2x} (s + 1) \log^x x,
\]

for all \( z > 2 \) and \( 2 < x < z^2 \). The constant \( B \) depends only on \( A_1 \) and \( A_2 \).

Now, we shall show by induction that if \( B \) is sufficiently large then (23) holds for all \( z > 2 \) and \( x > 2 \). For that, it is enough to prove the implication: if (23) holds for all \( x < x^u \) then it holds for \( x = x^{u+1} \), where \( u \) is real number \( > 1 \).
After putting
\[ y = z^u, \quad x = z^{u+1} \]
in (21), we get by the inductive assumption

\[
\frac{R(x, z)}{(\log x)^{1-\alpha}} < \frac{BL}{1-2\alpha} \left( u + 1 (\log x)^{3\alpha-1} + \frac{\log x}{\log z} \left( \log \frac{t}{2} \right)^\alpha d (\log \log t) \right) + \\
+ O \left( \frac{L}{1-2\alpha} (\log x)^{3\alpha-1} \right) < \frac{BL}{1-2\alpha} \left( u + 1 + \frac{u+1}{u} \right)^{1-\alpha} + \varepsilon (\log x)^{3\alpha-1}.
\]

If \( B \) is sufficiently large then \( \varepsilon < 1 - 1/\sqrt{2} \) and thus the term in the bracket is less than \( u + 2 \). This completes the proof of Proposition 2.

5. – Completion of the proof of Proposition 1.

Putting \( F'(s) = 0 \) for all \( s < 0 \) we find out from (14) and (18) that

\[
G(x, z) = C s^{-1} [(1-x) F(s) + x F(s-1)] (\log x)^{-\alpha} + \\
+ O \left( \frac{(s+1)}{1-2\alpha} (\log x)^{\alpha-1} + L (\log x)^{\alpha-1} \right) = CF'(s) (\log x)^{-\alpha} + \\
+ O \left( \frac{(s+1)}{1-2\alpha} (\log x)^{\alpha-1} \right)
\]

for all \( x > 2 \) and \( z > 2 \). This completes the proof of Proposition 1.

REFERENCES