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Completely monotone families of solutions of $n$-th order linear differential equations and infinitely divisible distributions


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Completely Monotone Families of Solutions of $n$-th Order Linear Differential Equations and Infinitely Divisible Distributions (*)

PHILIP HARTMAN (**)  

*dedicated to Hans Lewy*

1. – Introduction.

It was shown in [9] that if $x = I_{\mu}(t)$ is the unique solution of the modified Bessel differential equation

(1.1) \[ t^2 x'' + tx' - \left( t^2 + \mu^2 \right) x = 0 \]

satisfying $x \sim (t/2)^{\mu}/\Gamma(1 + \mu)$ as $t \to 0$, then $I_{\mu}(t)$ is a completely monotone function of $t$ for fixed $t > 0$; for the definitions of the solutions $I_{\mu}(t)$, $K_{\mu}(t)$ of (1.1), see [15], pp. 77-80. Thus, by the theorem of Hausdorff-Bernstein (cf. [16], p. 160), there exists a (unique) distribution function $W(r) = W(r, t, \infty)$ on $r \geq 0$ satisfying

\[
I_{\mu}(t) = I_{\mu}(t) \int_0^\infty \exp(-\tau \mu^2) W(\mu, t, \infty) \ d\tau \quad \text{for } \mu \geq 0,
\]

so that $W(0) = 0$, $W(\infty) = 1$, $W(r - 0) = W(r)$, and $W(\mu') \geq 0$. The results below will imply the following generalization:

**Theorem 1.0.** (a) $0 < t < r \leq \infty$. Then $I_{\mu}(t)/I_{\mu}(r)$ is a completely monotone function of $t = \mu^2 \geq 0$, so that there exists a (unique) distribution func-

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tion \( W(r) = W(r, t, \tau) \) on \( r \geq 0 \) satisfying

\[
I_\mu(t)/I_\mu(\tau) = [I_\theta(t)/I_\theta(\tau)] \int_0^\infty \exp(-r\mu^2) W(dr, t, \tau) \quad \text{for} \quad \mu \geq 0 ;
\]

the distribution function

(1.2) \( W(r) = W(r, t, \tau) \) is infinitely divisible

for \( 0 < t \leq \tau \leq \infty \);

(1.3) \( W(\cdot, \tau_1, \tau_n) = W(\cdot, \tau_1, \tau_3) \ast W(\cdot, \tau_2, \tau_3) \ast \cdots \ast W(\cdot, \tau_{n-1}, \tau_n) \)

for \( \tau_1 < \cdots < \tau_n \)

with \( \tau_1 > 0, \tau_n \leq \infty \);

(1.4) \( W(r, t, \tau) \) is non-decreasing in \( r \) and \( t \), and non-increasing in \( \tau \);

(1.5) \( W(r, t, \tau) \to 0 \) as \( t \to +0 \)

for \( 0 < \tau \leq \infty \);

(1.6) \( W(r, t, \tau) \to W(r, t, \infty) \) as \( \tau \to \infty \) for \( r \geq 0 \),

and \( t > 0 \); and

(1.7) \( W(r, t, \tau) \to \delta_\sigma(r) \) as \( t \uparrow \sigma, \tau \downarrow \sigma \)

with \( 0 < \sigma < \infty \),

(1.8) \( W(r, t, \tau) \to \delta_\sigma(r) \) as \( \tau > t \to \infty \),

where \( \delta_\sigma(r) \) is the unit distribution function \( i.e., \delta(0) = 0 \) and \( \delta(r) = 1 \) for \( r > 0 \).

(b) Also, \( 1/K_\mu(t) \) and \( K_\mu(\tau)/K_\mu(t) \) are completely monotone functions of \( \lambda = \mu^2 \geq 0 \) for \( 0 < t < \tau < \infty \), so that there exists a distribution function \( W(r) = W(r, t, \tau) \) on \( r > 0 \) satisfying

\[
1/K_\mu(t) = [1/K_\theta(t)] \int_0^\infty \exp(-r\mu^2) W(dr, t, \infty) \quad \text{for} \quad \mu > 0 ,
\]

\[
K_\mu(\tau)/K_\mu(t) = [K_\theta(\tau)/K_\theta(t)] \int_0^\infty \exp(-r\mu^2) W(dr, t, \tau) \quad \text{for} \quad \mu \geq 0 .
\]
The distribution function \( W(r) = W(r, t, \tau) \) satisfies (1.2), (1.3) with \( \tau_1 > 0, \tau_n < \infty \); (1.4), (1.5) for \( 0 < \tau \leq \infty \); (1.6); (1.7) with \( 0 < \sigma < \infty \); and (1.8).

The arguments of [9] can be used to show that if the definition of \( W(r, t, \tau) \) for \( r \geq 0, t > 0 \) is extended by putting \( W(r, t, \tau) = 0 \) for \( r < 0 \) and/or \( t = 0 \), then \( W(r, t, \tau) \) is continuous for \( -\infty < r < \infty \) and \( 0 \leq t < \tau(\leq \infty) \), of class \( C^\infty \) for \( -\infty < r < \infty \), \( 0 < t < \tau(\leq \infty) \), and satisfies the parabolic equation

\[
t^2 W_{tt} + t W_t - t^2 W = W_r,
\]

for fixed \( \tau \), on \(-\infty < r < \infty \) and \( 0 < t < \tau(\leq \infty) \).

The result of [9] concerning the complete monotony of \( I_\tau(t) \) as a function of \( \tau, \alpha \geq 0 \) (for fixed \( t > 0 \)) suggests the questions as to when a family of solutions \( x = X(t, \lambda) \) of a 1-parameter family of differential equations

\[
(1.9) \quad D^\alpha x + q_{n-1}(t)D^{n-1}x + ... + q_1(t)Dx - q(t, \lambda)x = 0, \quad D = d/dt,
\]

\((t, \lambda) \in T \times \Lambda, \) is a completely monotone function of \( \lambda \in \Lambda \) (for fixed \( t \in T \)) or when such a completely monotone family of solutions \( x = X(t, \lambda) \) exists, and when the analogue of Theorem 1.0 is valid.

**Notation.** – Unless otherwise specified, an « interval » can be bounded or unbounded, and closed or open or neither. \( T \) is a \( t \)-interval with endpoints \( \alpha \) and \( \beta \), \( -\infty \leq \alpha < \beta \leq \infty \) and \( T_0 = (\alpha, \beta) \) is its interior. \( \Lambda \) is a \( \lambda \)-interval and \( \Lambda_0 \) its interior.

We shall make some of the following assumptions from time to time.

\( (A1) \) \( q_1(t), ..., q_{n-1}(t) \in C^0(T) \) and \( q(t, \lambda) \in C^0(T \times \Lambda) \). For fixed \( \lambda \in \Lambda \), (1.9) is disconjugate on \( T \); i.e., no solution \( x(t) \neq 0 \) has more than \( n-1 \) zeros on \( T \).

\( (A2) \) \( \partial^m q/\partial \lambda^m \) exists, is continuous, and satisfies \((-1)^{m+1}\partial^m q/\partial \lambda^m \geq 0 \) on \( T \times \Lambda_0 \) for \( m = 1, 2, ... \) and a fixed \( k \) (in particular, \((-1)^{k+1}\partial q(t, \cdot)/\partial \lambda \) is completely monotone on \( \Lambda_0 \) for fixed \( t \in T \)).

It is immaterial in assumption (A1) whether or not zeros are counted with multiplicities. For a more general result, see [6]; and for a simple proof for the linear case at hand, see [13].

When (A1) holds, (1.9) has first, second, ..., \( n \)-th principal solutions at \( t = \beta \), say \( x = \xi_1(t, \lambda), ..., \xi_n(t, \lambda) \); i.e., solutions satisfying \( \xi_j(t, \lambda) \neq 0 \) for \( t \) near \( \beta \) and \( \xi_j(t, \lambda)/\xi_{j+1}(t, \lambda) \to 0 \) as \( t \to \beta \) for \( j = 1, ..., n \); [7], pp. 328-347 or pp. 353-355 or [11]. (In [7], pp. 353-355, the assumption of « disconjugacy » is relaxed to « non-oscillatory at \( t = \beta \) »). Furthermore, if \( \tau \in T_0 \),
then there exist unique principal solutions \( \eta_1(t, \tau, \lambda), \ldots, \eta_n(t, \tau, \lambda) \) at \( t = \beta \) such that \( x = \eta_j(t, \tau, \lambda) \) satisfies

\[
D^{i-1}x = 0 \quad \text{for} \quad 1 \leq i < j \quad \text{and} \quad D^{i-1}x = 1 \quad \text{at} \quad t = \tau;
\]

[7], Theorem 7.2n(vii), p. 332. The solution \( x = \eta_j(t, \tau, \lambda) \) is called the \( j \)-th special principal solution at \( t = \beta \), determined by \( \tau \). A first principal solution \( x = \xi_1(t, \lambda) \) is unique up to a non-zero constant factor and \( \xi_1(t, \lambda) \neq 0 \) for \( t \in \mathbb{T}^\circ \) ([7], Theorem 7.1n(i), pp. 331-332), so that \( \eta_j(t, \tau, \lambda) = \xi_1(t, \lambda)/\xi_1(\tau, \lambda) \). Sometimes it will be convenient to assume

\[
(\Delta 3_\alpha) \quad \text{Let} \quad \sigma = \alpha \quad \text{or} \quad \sigma = \beta. \quad \text{There exists a family of non-negative first principal solutions} \quad x = \xi_1(t, \lambda) \quad \text{at} \quad t = \beta \quad \text{satisfying}
\]

\[
(1.11) \quad \xi_1(t, \lambda)/\xi_1(t, \mu) \to 1 \quad \text{as} \quad t \to \sigma \quad \text{for} \quad \lambda, \mu \in \Lambda.
\]

This assumption is equivalent to the requirement that

\[
(1.12) \quad h(\lambda, \mu) \equiv \lim_{t \to \sigma} x_1(t, \lambda)/x_1(t, \mu) \text{ exists and} \neq 0 \quad \text{for} \quad \lambda, \mu \in \Lambda
\]

holds for arbitrary first principal solutions \( x_1(t, \lambda) \geq 0 \) of (1.9). For in this case, \( \xi_1(t, \lambda) = a_1(t, \lambda)/h(\lambda, \nu), \) for a fixed \( \nu \in \Lambda \), satisfies (1.11). The asymptotic integration theory of (1.9) gives simple sufficient conditions for the validity of (1.12), hence of (\( \Delta 3_\alpha \)). Also, if \( T = (\alpha, \beta] \) is closed and bounded on the right and \( x = \xi_1(t, \lambda) \) is the solution of (1.9) satisfying the initial conditions (1.10) with \( j = n \) and \( \tau = \beta \), then (1.11) holds; contrast part (b) of the following theorem with the result in the Appendix 2 below.

**THEOREM 1.1.** Assume (\( \Delta 1 \)) and (\( \Delta 2 \)) for a fixed \( k, 0 < k < n \). (a) Let \( \tau \in \mathbb{T}^\circ \) and \( x = \eta_{n-k}(t, \tau, \lambda) \) be the special \((n-k)\)-th principal solution at \( t = \beta \) determined by \( \tau \). Then \( D^i \eta_{n-k}(t, \tau, \lambda) \in C^0(\mathbb{T} \times \mathbb{T} \times \mathbb{A}^2) \) for \( i = 0, \ldots, n \), and \( \eta_{n-k}(t, \tau, \cdot) \) is completely monotone on \( \Lambda^\circ \), for fixed \( (t, \tau) \) with \( \alpha < \tau \leq t < \beta \). (b) If, in addition, \( n - k = 1, \, \mathbb{A}^2 \supset [0, \infty) \), and \( x = \xi_1(t, \lambda) \) is a first principal solution, then, for \( \alpha < t \leq \tau < \beta \), there exists a distribution function \( W(r) = W(r, t, \tau) \) on \( r \geq 0 \) such that

\[
(1.13) \quad \xi_1(\tau, \lambda)/\xi_1(t, \lambda) = \left[ \xi_1(\tau, 0)/\xi_1(t, 0) \right] \exp\left( -\lambda r \right) W(dr, t, \tau) \quad \text{for} \quad \lambda \geq 0.
\]

The distribution function \( W(r) = W(r, t, \tau) \) satisfies (1.2), (1.3) with \( \tau_1 > \alpha \) and \( \tau_n < \beta \); (1.4); and (1.7) with \( \alpha < \sigma < \beta \).
Remark 1. In Theorem 1.1(b), condition \( A \ni [0, \infty) \) can be replaced by \( A = (0, \infty) \) provided that \( \xi_i(t, \lambda) \), defined on \( T \times A = T \times (0, \infty) \), has a continuous extension to \( T \times [0, \infty) \); see [7], p. 335, or [5], p. 360, Corollary 6.6 and the comment following it. A similar remark applies to Theorem 1.2(b) and the last part of Lemma 2.2 below.

Remark 2. If \( (A2_k) \) holds, then \( (A2_{n-k}) \) holds after the change of variables \( t \rightarrow -t \). Thus, part (a) implies that if \( x = \xi_k(t, \tau, \lambda) \) is the \( k \)-th special principal solution of (1.9) at \( t = \alpha \) determined by \( \tau \), i.e., \( \sigma(t) = \xi_k(t, \tau, \lambda) \) satisfies the conditions \( D^{k-1}x = 0 \) for \( 1 \leq i < k \) and \( D^{k-1}x = (-1)^{k-1} \) at \( t = \tau \), then \( D^i \xi_k(t, \tau, \lambda) \in C^0(T \times T \times A^\delta) \) for \( i = 0, \ldots, n \), and \( \xi_k(t, \tau, \cdot) \) is completely monotone on \( A^\delta \), for fixed \( (t, \tau) \) with \( \alpha < t \leq \tau < \beta \).

Theorem 1.2. Assume \( (A1), (A2_{n-1}) \) and \( (A3, \alpha, \beta) \) with \( \sigma = \alpha \) [or \( \sigma = \beta \)].

(a) Then \( D^i \xi_i(t, \lambda) \in C^0(T \times A^\delta) \) for \( 0 \leq i \leq n \), and \( \xi_i(t, \cdot) \) [or \( 1/\xi_i(t, \cdot) \)] is completely monotone on \( A^\delta \) for fixed \( t \in T^\alpha \). (b) If, in addition, \( A^\delta \ni [0, \infty) \), then there exists a distribution function \( W(r) = W(r, \alpha, t) \) [or \( W(r) = W(r, \tau, \beta) \)]
on \( r \geq 0 \) such that

\[
\xi_i(t, \lambda) = \xi_i(t, 0) \int_0^\infty \exp(-\lambda r) W(dr, \alpha, r) \quad \text{for } \lambda \geq 0
\]

and \( t \in T_\alpha \) [or such that]

\[
1/\xi_i(t, \lambda) = [1/\xi_i(t, 0)] \int_0^\infty \exp(-\lambda r) W(dr, t, \beta) \quad \text{for } \lambda \geq 0
\]

and \( t \in T_\alpha \); also \( \nu_\alpha \) [or \( \nu_\beta \)] is admissible in (1.3); \( W(r, t, \tau) \rightarrow \delta_\alpha(r) \) as \( t < \tau \rightarrow \alpha \) [or \( \tau \rightarrow t \rightarrow \beta \)] and \( W(\cdot, \tau, \alpha) \) [or \( W(\cdot, t, \beta) \)] is infinitely divisible.

Remark 3. It will be clear from the proofs that Theorems 1.1 and 1.2 remain valid if (1.9) is replaced by an equation involving quasi-derivatives of the form

\[
\left\{ p_{n+1}^{-1} D p_n^{-1} D \ldots p_1^{-1} D + \sum_{j=1}^{K-1} q_j D^j \right\} x = 0,
\]

the operators \( D^i \) for \( i = 0, \ldots, n \) are replaced by \( L_i = D^i \) for \( 0 \leq i < K \) and \( L_{i+K} = p_{i+1} D \ldots p_1^{-1} D^K \) for \( 0 \leq i \leq M, \) and assumption (A1) is replaced by

\( (A1') \) \( K \geq 1, M \geq 0, \) and \( n = K + M; \) \( 0 < p_j = p_j(t) \in C^0(T) \) for \( j = 1, \ldots, M + 1; \) \( q_j = q_j(t) \in C^0(T) \) for \( j = 1, \ldots, K - 1; \) \( q = q(t, \lambda) \in C^0(T \times A \alpha); \) and (1.9') is disconjugate on \( T \) for fixed \( \lambda \in \Lambda. \)
As an application, different from that concerning $I_\mu(t)$, we might mention that the results above imply that the Legendre differential equation

$$
[(t^2-1)x']' - [v(v+1) + \mu^2/(t^2-1)]x = 0
$$

has a family of principal solutions $x = X(t, \mu, \nu)$ at $t = 1$ on $T = (1, \infty)$ such that $X(t, \mu, \nu)$ is a completely monotone function of $\lambda = \mu^2 \geq 0$ for fixed $t > 1, \nu \geq 0$. In fact, from Theorem 1.2, we obtain $X(t, \mu, \nu)$ in the form $c(\mu, \nu)P_{-\nu}(t)$, where $c(\mu, \nu)$ can be given explicitly in terms of the $\Gamma$-function and $P_{-\nu}$ is a Legendre function of the first kind; cf. [4], p. 122, (3) and (5).

Also, if $Q_{\nu}(t)$ is a Legendre function of the second kind, then $Q_{\nu}(t)$ is a principal solution at $t = \infty$ for fixed $\mu, \nu \geq 0$ and, by Theorem 1.2, $\Gamma(\nu+\mu+1) \cdot \exp(\pi i \mu)/Q_{\nu}(t)$ is a completely monotone function of $\lambda = \mu^2 > 0$ for fixed $t > 1, \nu \geq 0$; cf. [4], p. 122, (5). We could also formulate an analogue of Theorem 1.0.

In Section 2, we state the main Lemma 2.1 concerning the complete monotony of certain Green's functions with respect to $\lambda$ (for fixed $s, t$). Section 3 contains the proofs of Theorems 1.1 and 1.2. Section 4 gives somewhat different criteria for a family of solutions $X(t, \lambda)$ of second order differential equations to be completely monotone with respect to $\lambda$. Theorem 5.1 deals with a generalization of (1.1) in a neighborhood of a regular singular point. Finally, Section 6 contains interesting related observations about the $\Gamma$-function.

Appendix 1 concerns the situation when $m$ is restricted to a finite range in condition (A2). Appendix 2 deals with the Cauchy functions of a 1-parameter family of $n$-th order linear differential equations. Appendix 3 extends from $j = 1$ to $1 \leq j < n$ a known characterization of a $j$-th special principal solution of a disconjugate equation; cf. Theorem 7.1, [7], p. 330 for the case $j = 1$.

2. - On Green's functions.

Assume (A1) and fix $[a, b] \subset T$ and $k, 0 < k < n$. Write the left side of (1.9) as $L(\lambda)x$, where $L(\lambda) = D^n + q_{n-1}D^{n-1} + \ldots + q_1 D - q$ is a differential operator. Consider the boundary value problem on $[a, b]$ consisting of the differential equation $L(\lambda)x = 0$ and boundary conditions

\begin{align*}
(D^{i-1}x)(a) &= 0 & \text{for } i = 1, \ldots, n-k, \\
(D^{i-1}x)(b) &= 0 & \text{for } i = 1, \ldots, k.
\end{align*}
The fact that (1.9) is disconjugate on $T$, hence on $[a, b]$, implies the existence of a Green's function $G_{ab}(t, s, \lambda)$ defined on $[a, b] \times [a, b] \times \Lambda$ such that if $h(t) \in C^0[a, b]$ or $h(t) \in L^2(a, b)$, then

\begin{equation}
L(\lambda)x = h
\end{equation}

has a unique solution satisfying (2.1), (2.2), and $x$ is given by

\begin{equation}
x(t) = \int_a^b G_{ab}(t, s, \lambda)h(s)\, ds.
\end{equation}

Recall that $G_{ab}$ is uniquely determined by the conditions: (i) as a function of $t$, $x(t) = G_{ab}(t, s, \lambda)$ is a solution of (1.4) on $[a, s), (s, b]$ and satisfies (2.1), (2.2); (ii) $D^{i-1}x(t) = D^{i-1}G_{ab}(t, s, \lambda) \in C^0([a, b] \times [a, b])$ for fixed $\lambda \in \Lambda$ and $i = 1, \ldots, n-1$, and $D^{n-1}G_{ab}(s + 0, s, \lambda) - D^{n-1}G_{ab}(s - 0, s, \lambda) = 1$.

The main lemma is the following

**LEMMA 2.1.** Assume $(A1)$, $(A2_k)$ with fixed $k$, $0 < k < n$, and $[a, b] \subset T$.

Then $(-1)^k G_{ab}(t, s, \cdot)$ is completely monotone on $\Lambda$ for fixed $s, t \subset [a, b]$.

McKean [12] contains a similar result for self-adjoint (possibly singular) boundary value problems with $n = 2$, $q(t, \lambda) = \lambda + Q(t) \geq \lambda \geq 0$. We simplify his operator-theoretic proof.

**REMARK.** Lemma 2.1 implies that if $h(t) = h(t, \lambda) \in C^0([a, b] \times \Lambda)$ has the property that $h(t, \cdot)$ is completely monotone on $\Lambda$ for fixed $t \in [a, b]$, then the unique solution $x(t) = x(t, \lambda)$ of (2.1)-(2.3) given by (2.4) is completely monotone on $\Lambda$ for fixed $t \in [a, b]$.

The proofs of Lemma 2.1 and Theorem 1.1-1.2 can be modified to obtain the following:

**LEMMA 2.2.** Let $0 < p(t) \in C^0(T)$, $q(t, \lambda) \in C^0(T \times \Lambda)$ satisfy $(A2_1)$, and

\begin{equation}
(p(t)x')' - q(t, \lambda)x = 0
\end{equation}

disconjugate on $T$ for fixed $\lambda$. Let $x = Y(t, \lambda)$, $Z(t, \lambda)$ be non-negative (first) principal solutions of (2.5) at $t = \alpha, \beta$, and suppose that $Y, Z$ are linearly independent and normalized by $p(t)[ZY' - Z'Y] = 1$. Then $G(t, s, \lambda)$, defined as $Y(t, \lambda)Z(s, \lambda)$ or $Y(s, \lambda)Z(t, \lambda)$ according as $t \leq s$ or $t \geq s$, is a completely monotone function of $\lambda \in \Lambda^*$ for fixed $s, t \in T$. If either $s \leq \tau \leq t$ or $t \leq \tau \leq s$,
then \( G(t, s, \cdot)/G(r, \tau, \cdot) \) is completely monotone on \( A^0 \). If also \( A^0 \subset [0, \infty) \), then there exists a distribution function \( P(r) = P(r, t, s, \tau) \) on \( r \geq 0 \) satisfying

\[
G(t, s, \lambda)/G(r, \tau, \lambda) = [G(t, s, 0)/G(r, \tau, 0)] \int_0^\infty \exp(-r\lambda)P(dr, t, s, \tau)
\]

for \( \lambda \geq 0 \), and \( P(\cdot, t, s, \tau) \) is infinitely divisible. Note that

\[
G(t, s, \lambda)/G(r, \tau, \lambda) = [Y(s, \lambda)/Y(r, \lambda)]\cdot[Z(t, \lambda)/Z(\tau, \lambda)] \quad \text{if} \quad s \leq \tau \leq t.
\]

**Proof of Lemma 2.1.** It will be clear from the explicit construction of \( G_{ab}(t, s, \lambda) \) in the proof of Theorem 1.1 that \( \partial^{i+j+m}G_{ab}(t, s, \lambda)/\partial t^i\partial s^j\partial \lambda^m \) exists and is continuous on \([a, b] \times [a, b] \times \Lambda\) if \( m = 0 \) [or \( m > 0 \)] and either \( 0 \leq i, j \leq n - 2 \) or \( 0 \leq i, j \leq n \) and \( t \neq s \). It is known ([1], [10], [14]; cf. [2], p. 105) that \((-1)^kG_{ab}(t, s, \lambda) \geq 0\) on \([a, b] \times [a, b] \times \Lambda\).

If \( t \neq s \), then \( L(\lambda)G(\cdot, s, \lambda) = 0 \). Differentiation of this equation with respect to \( \lambda \in \Lambda^0 \) gives (2.3), where \( x = \partial G_{ab}(\cdot, s, \lambda)/\partial \lambda \) and \( h = G_{ab}(\cdot, s, \lambda) \cdot \partial g(\cdot, \lambda)/\partial \lambda \). As functions of \( t, h \) is continuous and \( x \in C^0[a, b] \) satisfies (2.1)- (2.2). Hence (2.4) holds, i.e.,

\[
(-1)^k \partial G_{ab}(t, s, \lambda)/\partial \lambda = \int_a^b \left[ (-1)^k G_{ab}(t, r, \lambda) \cdot [(-1)^k g_4(r, \lambda)] \cdot [(-1)^k G_{ab}(r, s, \lambda)] \right] dr \leq 0.
\]

Successive differentiations with respect to \( \lambda \in \Lambda_0 \) and an obvious induction give Lemma 2.1.

**3. – Proof of Theorem 1.1(a).**

Let \( x(t) = X(t, \tau, \lambda) \) be the solution of (1.4) satisfying the initial conditions (1.10) with \( j = n \), so that \( X(t, \tau, \lambda) \) is the Cauchy function for (1.9) for fixed \( \lambda \in \Lambda \). It is clear that (A1) implies that \( \partial^iX/\partial t^i\partial \tau^j\partial \lambda^m \in C^0(T \times T \times \Lambda) \) for \( j + k = i = 0, \ldots, n \). Also if \( (A_{2k}) \) holds, then \( \partial^i+j+mX/\partial t^i\partial \tau^j\partial \lambda^m \in C^0(T \times T \times \Lambda^0) \) for \( i, j = 0, \ldots, n \) and arbitrary \( m \).

Let \( x(t) = X(t, \alpha, \beta, \lambda), Z(t, \alpha, \beta, \lambda) \) be the solutions of (1.9) such that

\[
G_{ab}(t, s, \lambda) = X(t, a, s, \lambda) \quad \text{or} \quad Z(t, a, s, \lambda)
\]

according as \( a \leq t \leq s \) or \( s \leq t \leq b \). Thus, condition (ii), following (2.4) and
defining \( G_n \), implies that \( Z - Y = X(t, s, \lambda) \), while (i) implies that \( x(t) = Y \)
satisfies (2.1) and \( x(t) = Z \) satisfies (2.2). Since \( Y = Z - X, x = Z \) is deter-
mined by the boundary conditions

\[
\begin{align*}
[D^{i-1}Z]_{t=a} &= [D^{i-1}X]_{t=a} & 1 \leq i \leq n - k, \\
[D^{i-1}Z]_{t=b} &= 0 & 1 \leq i \leq k.
\end{align*}
\]

Similarly, \( x = Y \) is determined by

\[
\begin{align*}
[D^{i-1}Y]_{t=a} &= 0 & 1 \leq i \leq n - k, \\
[D^{i-1}Y]_{t=b} &= -[D^{i-1}X]_{t=b} & 1 \leq i \leq k.
\end{align*}
\]

Let \( x(t) = X(t, s, \tau, \lambda) \) be the unique solution of (1.9) satisfying the boundary conditions

\[
\begin{align*}
[D^{i-1}X']_{t=a} &= 0 & 1 \leq i \leq n - j, \\
[D^{i-1}X']_{t=b} &= 0 & 1 \leq i < j \text{ or } i = j.
\end{align*}
\]

Thus, by (3.2) and (3.3), we have

\[
\begin{align*}
Z(t, a, s, b, \lambda) &= \sum_{i=1}^{n-k} X^i(t, b, a, \lambda) \cdot D^{i-1}X(a, s, \lambda), \\
Y(t, a, s, b, \lambda) &= -\sum_{i=1}^{k} X^i(t, a, b, \lambda) \cdot D^{i-1}X(b, s, \lambda).
\end{align*}
\]

This makes it clear that \( Y, Z \) have smoothness properties similar to those of \( X \).

**Proposition 3.1.** Assume (A1) and (A2) with \( k \) fixed, \( 0 < k < n \). Then \((-1)^{k+1} X^k(t, a, s, \cdot), \) for \( a < t < s, \) and \( X^{n-k}(t, b, s, \cdot), \) for \( s < t < b, \) are completely monotone on \( \Lambda \).

**Proof.** The definition of the Cauchy function \( X(t, \tau, \lambda) \) implies that \( D^{i-1}X(t, \tau, \lambda) \sim (t - \tau)^{n-j} / (n-j)! \) as \( t \to \tau \) uniformly on compacts of \( \Lambda \). Hence, by (3.5) and (3.6),

\[
\begin{align*}
X^k(t, a, s, \lambda) &= -\lim_{b \to a} Y(t, a, s, b, \lambda)(n-k)!/(b-s)^{n-k}, \\
X^{n-k}(t, b, s, \lambda) &= \lim_{a \to b} (-1)^k Z(t, a, s, b, \lambda)k!/(s-a)^k.
\end{align*}
\]
Thus, Proposition 3.1 follows from Lemma 2.1 and (3.1); cf. [16], pp. 147 ff.,
or Appendix 1 below.

**Proposition 3.2.** Assume (A1) and let $\lambda \in \Lambda$ be fixed. Then
\[
\eta_i(t, s, \lambda) = \lim_{b \to \beta} X'(t, b, s, \lambda) \quad \text{for } j = 1, \ldots, n - 1.
\]
This limit is valid in $C^\alpha$ on arbitrary $(t, s)$-compacts of $T \times T^\alpha$.

**Remark.** This proposition and (3.5) imply, for example, that
\[
(-1)^k \sum_{i=1}^{n-k} \eta_i(t, s, \cdot) D^{i-1} X(a, s, \cdot)
\]
is completely monotone on $A^\alpha$ for fixed $(a, s, t)$, $a < s < t$ and $a, t \in T$.
Proposition 3.2 is contained in Appendix 3 below.

It is now easy to complete the proof of Theorem 1.1(a). Propositions 3.1, 3.2 imply that $\eta_{n-k}(t, s, \cdot)$ is completely monotone on $A^\alpha$ for fixed $(s, t)$, $a < s \leq t < \beta$. In particular, it is continuous and non-increasing with respect to $\lambda$ for fixed $(s, t)$. It therefore follows from Dini's theorem that
\[
\eta_{n-k}(t, s, \lambda) \to \eta_{n-k}(t, s, \lambda_0), \quad \text{as } \lambda \to \lambda_0 \in A^\alpha,
\]
uniformly on $(t, s)$-compacts of $A$ with $\alpha < s \leq t < \beta$, hence of $T \times T^\alpha$. Consequently, $\eta_{n-k}(t, s, \lambda) \in C^\alpha(T \times T^\alpha \times A^\alpha)$.

Hence, $D^i \eta_{n-k}(t, s, \lambda) \in C^\alpha(T \times T^\alpha \times A^\alpha)$ when $i = 0, \ldots, n$; for a solution of (1.9) is uniquely determined by its values at $n$ distinct $t$-values (i.e., if $x_1(t, \lambda), \ldots, x_n(t, \lambda)$ are linearly independent solutions of (1.9) and $t_1 < \ldots < t_n$, then $\det(x_{i}(t_j, \lambda)) \neq 0$). (For an analogous non-linear result, see the proof of Lemma 1.1, [8], pp. 207-210. In the linear case at hand, a proof need not use the disconjugacy of (1.4), cf. Lemma 7.1, [5], p. 479.)

**Proof of Theorem 1.1(b).** Since $\eta_1(t, r, \cdot) = \xi_1(\tau, \cdot) / \xi_1(t, \cdot)$ is completely monotone on $A = [0, \infty)$ for fixed $(t, \tau)$ with $\alpha < \tau \leq \beta$ by part (a), the existence of a unique distribution function $W(r) = W(r, t, \tau)$ satisfying (1.13) follows from the Hausdorff-Bernstein theorem. The relation (1.3) for $\tau_1 < \ldots < \tau_n$ with $\tau_1, \tau_n \in T$ is a consequence of (1.13) and the standard theorem on the products of Laplace-Steltjes transforms. The statement (1.4) follows from (1.3); in fact, if $\tau_1 < \sigma < \tau_2$, then (1.3) implies that
\[
W(r, \tau_1, \tau_2) = \int_0^r W(r - \varrho, \tau_1, \sigma) dW(\varrho, \tau_2) = \int_0^r W(r - \varrho, \sigma, \tau_2) dW(\varrho, \tau_1, \sigma)
\]
so that
\[
W(r, \tau_1, \tau_2) \leq W(r, \tau_1, \sigma) \quad \text{and} \quad W(r, \tau_1, \tau_2) \leq W(r, \sigma, \tau_2),
\]
which implies (1.4). The statement (1.7) is clear from (1.13) since
\[ \eta_1(\tau, t, \lambda) = \frac{\xi_1(\tau, \lambda)}{\xi_1(t, \lambda)} \to 1 \quad \text{as} \quad t, \tau \to \sigma \quad \text{uniformly on} \quad (\sigma, \lambda)-\text{compacts of} \quad T^o \times [0, \infty). \]
Furthermore, the monotony condition (1.4) and (1.7) imply, by virtue of Dini’s theorem, that (1.7) holds uniformly on compact \((r, \sigma)\)-subsets of \(\{r > 0\} \times T^o\). The infinite divisibility of \(W(\cdot, t, r)\), \(\alpha < t \leq \tau < \beta\), follows from (1.3) and the uniformity of (1.7); cf. [3], pp. 128-135.

**Proof of Theorem 1.2(a).** By Theorem 1.1, \(\eta_1(\tau, t, \lambda) = \frac{\xi_1(\tau, \lambda)}{\xi_1(t, \lambda)}\) is completely monotone on \(A^o\) if \(\alpha < t \leq \tau < \beta\). Thus if \((A3_\lambda)\) holds and \(\mu \in A\) is fixed, \(\xi_1(\tau, \cdot) = \lim_{\mu \to \alpha} \eta_1(\tau, t, \mu)\xi_1(t, \mu)\), as \(t \to \alpha\), is completely monotone on \(A^o\) for fixed \(t \in T^o\). Similarly, if \((A3_\mu)\) holds, then \(1/\xi_1(t, \cdot)\) is completely monotone on \(A^o\) for fixed \(t \in T^o\). The asserted smoothness properties of \(\xi_1(t, \lambda)\) follow as in the proof of Theorem 1.1(a).

**Proof of Theorem 1.2(b).** This is similar to the proof of Theorem 1.1(b) and will be omitted.

**4. Use of asymptotic estimates.**

It was mentioned in [9], although no details were given, that a proof that \(I_\mu(t)\) is a completely monotone function of \(\lambda = \mu^2 \geq 0\) could be based on a knowledge of the asymptotic behavior of \(I_\mu(t)\) near \(t = 0\) and \(t = \infty\), instead of using Green’s functions. A generalization of such a proof can be obtained from the following simple proposition.

**Theorem 4.1.** In the differential equation
\begin{equation}
(p(t)x')' - q(t, \lambda)x = 0,
\end{equation}
let \(0 < p(t) \in C^\infty(T)\), \(q(t, \lambda)\) satisfies assumption \((A_2)\) and \(q(t, \lambda) \geq 0\). Let \(x = X(t, \lambda)\) be a solution of (4.0) such that \(X, X' \in C^\infty(T \times A)\), and \(\partial^n X/\partial \lambda^n\) exists and is continuous on \(T \times A^o\) for \(n = 0, 1, \ldots\). Then \(X(t, \cdot)\) is completely monotone on \(A\) (for fixed \(t \in T\)) if and only if
\begin{equation}
0 \leq \limsup_{t \to \sigma} (-1)^n \partial^n X(t, \lambda)/\partial \lambda^n (\leq \infty) \quad \text{for} \quad n = 0, 1, \ldots
\end{equation}
holds for \(\sigma = \alpha, \beta\) and all \(\lambda \in A^o\).

In contrast to Theorems 1.1 and 1.2, it is not supposed that \(x = X(t, \lambda)\) is a principal solution. A variant of Theorem 4.1 is the following:
THEOREM 4.2. Let \( p(t), q(t, \lambda) \) be as in Theorem 4.1 and \( x = X(t, \lambda) \) a principal solution (4.0) at \( t = \beta \) such that \( \partial^n X/\partial \lambda^n \) exists and is continuous on \( T \times A^\circ \) for \( n = 0, 1, \ldots \) and that either

\[
\int_{t \to \beta} \frac{dt}{p(t)} = \infty \quad \text{or} \quad \lim_{t \to \beta} X(t, \lambda) = 0 \quad \text{for } \lambda \in A.
\]

Then \( X(t, \cdot) \) is completely monotone on \( A \) if and only if

\[
0 \leq \limsup_{t \to \infty} (-1)^n \frac{\partial^n X(t, \lambda)}{\partial \lambda^n} (\leq \infty) \quad \text{for } \lambda \in A^\circ, \quad n = 0, 1, \ldots .
\]

PROOF OF THEOREM 4.1. Since the necessity of (4.1a) and (4.1b) for \( \lambda \in A^\circ \) is clear, we only prove the sufficiency. Since \( q(t, \lambda) \geq 0 \), a standard maximum principle shows that a solution of (4.0), for fixed \( \lambda \), cannot have a negative minimum interior to \( T \). Thus (4.1a) for \( n = 0 \) and \( \sigma = \alpha, \beta \) imply the case \( n = 0 \) of

\[
(-1)^n X^{(n)} \geq 0 \quad \text{on } T \times A^\circ, \quad \text{where } X^{(n)} = \partial^n X/\partial \lambda^n.
\]

Let \( n - 1 \geq 0 \) and assume (4.4a) for \( k = 0, \ldots, n - 1 \). If \( x = X(t, \lambda) \) in (4.0), \( n \) differentiations with respect to \( \lambda \) on \( A^\circ \) give

\[
(p(t) y')' - q(t, \lambda) y = - F_n(t, \lambda),
\]

where

\[
y = (-1)^n X^{(n)}, \quad F_n = \sum_{k=0}^{n-1} C_{nk} (-1)^{n-k+1} g^{n-k} (-1)^k X^{(k)} \geq 0,
\]

\( g^{(m)} = \partial^m g/\partial \lambda^m \), and \( C_{nk} = n!/k!(n-k)! \). Another appeal to the maximum principle shows that, for a fixed \( \lambda \), \( y \) cannot have a negative minimum interior to \( T \). Hence (4.1a) for \( \sigma = \alpha, \beta \) implies (4.4a). This completes the proof.

PROOF OF THEOREM 4.2. Since one-half of Theorem 1.2 is trivial, we prove only the sufficiency of (4.3) for the complete monotony of \( X(t, \cdot) \).

We first consider the case

\[
X(t, \lambda) \to 0 \quad \text{as } t \to \beta \quad \text{for } \lambda \in A
\]

of (4.2). By induction, we shall prove (4.4a) and

\[
X^{(n)}(t, \cdot) \to 0 \quad \text{as } t \to \beta \quad \text{in } L_{\text{loc}}^1(A^\circ).
\]
Of course, (4.8n) is trivial if \( T = (\alpha, \beta] \) is closed on the right, by virtue of (4.7) and the continuity of \( X^{(\alpha)} \) on \( T \times A \).

The case \( n = 0 \) of (4.4n) follows as in Theorem 4.1. Also the case \( n = 0 \) of (4.8n) follows from (4.7) and Lebesgue's monotone convergence theorem. Assume (4.4n) and (4.8k) for \( k = 0, \ldots, n-1 \), and differentiate (4.0) with respect to \( \lambda \in A^0 \) to obtain (4.5), (4.6). Suppose, if possible, that

\[
y(\tau, \lambda_0) < 0 \quad \text{for some } (\tau, \lambda_0) \in T \times A^0.
\]

Note that if \( T = (\alpha, \beta] \) is closed on the right, then \( \tau \neq \beta \) by (4.7). Since \( y \) cannot have a negative minimum at an interior point of \( T \), it follows from (4.3) that \( y'(t, \lambda_0) \leq 0 \) and \( y(t, \lambda_0) \leq y(\tau, \lambda_0) < 0 \) for \( t \geq \tau \). By continuity, \( y(\tau, \lambda) < 0 \) and similarly \( y(t, \lambda) \leq y(\tau, \lambda) \leq y(\tau, \lambda_0)/2 < 0 \) for \( t \geq \tau \) and \( \lambda \) near \( \lambda_0 \).

If \( \lambda > \mu \), then we have

\[
(-1)^{n+1} \left\{ X(t, \lambda) - \sum_{k=0}^{n-1} X^{(k)}(t, \mu)(\lambda - \mu)^k/k! \right\} =
\int_0^\lambda (\lambda - \nu)^{n-1} y(t, \nu) \, d\nu/(n-1)!
\]

The right side is not less than \( - (\lambda - \mu)^n y(\tau, \lambda_0)/2n! > 0 \) for \( \lambda, \mu \) near \( \lambda_0 \) and \( t \geq \tau \). Integrate (4.10) with respect to \( \mu \) over an interval \([\varrho, \lambda]\) to obtain

\[
(-1)^{n+1} X(t, \lambda)(\lambda - \varrho) - \sum_{k=0}^{n-1} \int_\varrho^\lambda X^{(k)}(t, \mu)(\lambda - \mu)^k \, d\mu/k! =
\int_\varrho^\lambda (\lambda - \nu)^{n-1} (\nu - \varrho) y(t, \nu) \, d\nu/(n-1)!,
\]

which is not less than \( - (\lambda - \varrho)^{n+1} y(\tau, \lambda_0)/2(n+1)! > 0 \). This contradicts (4.7) and (4.8n) for \( k = 0, \ldots, n-1 \), and so (4.9) cannot hold. Thus \( y = (-1)^n X^{(\alpha)} \geq 0 \) on \( T \times A^0 \), i.e., (4.4a) holds. Furthermore, (4.7), (4.8a) for \( k = 0, \ldots, n-1 \), and (4.11) imply (4.8a). This completes the induction and the proof of Theorem 1.2 in the case (4.7) of (4.2).

We now consider the case \( \int t^{\beta} p(t) = \infty \) of (4.2). By induction, we prove (4.4a) and

\[
X^{(\alpha)}(t, \cdot) \text{ is bounded } t \to \beta \text{ in } L_{1,\infty}(A^0).
\]
This proof is similar to that above if it is noted that (4.9) implies not only $y'(t, \lambda_0) \leq 0$ for $t \leq \tau$, but also that $y(t, \lambda_0) \rightarrow -\infty$ as $t \rightarrow \beta$. This follows from a convexity argument since (4.5) shows that $d^2y(t, \lambda_0)/ds^2 \leq 0$ for $0 \leq s < \infty$, where $ds/dt = 1/p(t)$, $s(\tau) = 0$, and $t = t(s)$ is the inverse of $s = s(t)$.

5. A regular singular point.

The next result concerns the case of the family of differential equations

\begin{equation}
-t^2x'' + r_0tx' - \left[q_0(\lambda) + \sum_{n=1}^{\infty} q_n t^n\right] x = 0.
\end{equation}

**THEOREM 5.1.** Let $r_0$ be arbitrary and $q_1, q_2, \ldots$ non-negative constants such that the power series in (5.1) is convergent for $|t| < \beta(\leq \infty)$. Let $q_0(\lambda) \in C^\infty(\Lambda)$ satisfy

\begin{equation}
q_0(\lambda) > 0 \quad \text{and} \quad \partial q_0/\partial \lambda \text{ is completely monotone on } \Lambda.
\end{equation}

The indicial polynomial $P(\nu) = P(\nu, \lambda)$,

\begin{equation}
P(\nu, \lambda) = \nu(\nu - 1) + r_0\nu - q_0(\lambda)
\end{equation}

has the (unique) positive zero $\nu = \nu(\lambda) > 0$,

\begin{equation}
\nu(\lambda) = (1 - r_0)/2 + [(1 - r_0)^2/4 + q_0(\lambda)]^{1/2},
\end{equation}

and $\partial \nu/\partial \lambda$ is completely monotone on $\Lambda$. Let $c(\lambda) \neq 0$ be arbitrary. Then, for fixed $\lambda$, (5.1) has a solution

\begin{equation}
X(t, \lambda) = c(\lambda)t^{\nu(\lambda)} \sum_{n=0}^{\infty} x_n(\lambda) t^n
\end{equation}

on $0 < t < \beta$ such that $x_n(\lambda) = 1$ and $x_n(\lambda)$ is completely monotone on $\Lambda$ for $n = 1, 2, \ldots$. Hence, if $c(\lambda)t^{\nu(\lambda)}$ is completely monotone on $\Lambda$ for $0 < t < \beta(\leq \beta)$, then the same is true of (5.5). (This is the case if $c(\lambda) \equiv 1$ and $\beta_0 \leq 1$.)

**PROOF.** We shall make use of the following simple fact in this proof and in the next section.

**PROPOSITION 5.1.** Let $g(\mu)$ be completely monotone for $\mu \geq 0$. Let $\mu = \varphi(\lambda)$ be continuous for $\lambda \geq 0$, $\varphi(0) = 0$, $\varphi \in C^\infty(0, \infty)$, and $d\varphi/d\lambda$ is completely monotone for $\lambda > 0$. Then $G(\lambda) = g(\varphi(\lambda))$ is completely monotone for $\lambda \geq 0$. 
Substituting (5.5) into (5.1) gives the recursion formula

\[ x_n(\lambda) = \sum_{i=1}^{n} x_{n-i}(\lambda) g_i P(n + \nu(\lambda), \lambda) \quad \text{and} \quad x_n(\lambda) = 1. \]

Here \( P(\nu, \lambda) \) is the indicial polynomial (5.3) for fixed \( \lambda \), so that, by (5.4),

\[ P(n + \nu(\lambda), \lambda) = 2n[(1 - r_0)^2/4 + g_0(\lambda)]^{1/4} + n^2. \]

An analogue of Proposition 5.1 with \( g(\mu) = (2n\mu + n^2)^{-1} \) and \( \varphi(\lambda) = (1 - r_0)^2/4 + g_0(\lambda) \) shows that \( \lambda \) is completely monotone. Another application gives that \( q_1/2P(n + \nu(\lambda), \lambda) \) is completely monotone on \( \lambda \). An induction and (5.6) imply that \( x_1(\lambda), x_2(\lambda), \ldots \) are completely monotone.

6. – A property of the \( I \)-function.

The modified Bessel differential equation (1.1) is of the type (5.1) with \( g_0(\lambda) = \lambda = \mu^2 > 0 \). The solution \( I_\mu(t) \) has an expansion,

\[ I_\mu(t) = \left[(t/2)^\mu/\Gamma(1 + \mu)\right] \left[1 + \sum_{n=1}^{\infty} \frac{(t/2)^n}{n!(\mu + 1) \cdots (\mu + n)}\right], \]

analogous to (5.6); cf. [15], p. 77. From Theorem 5.1, we can only deduce that \( I_\mu(t) \) is a completely monotone function of \( \lambda = \mu^2 > 0 \) for \( 0 < t \leq 2 e^{-\gamma} \) (rather than for all \( t > 0 \)). In fact, we have the following:

**Proposition 6.1.** Let \( t > 0 \) be fixed and \( f(\mu) = t^\mu/\Gamma(1 + \mu) \). Then \( F(\lambda) = f(\lambda^2) \) is completely monotone for \( \lambda \geq 0 \) if and only if

\[ 0 < t \leq e^{-\gamma} = 0.56 \ldots, \]

where \( \gamma \) is the Euler-Mascheroni constant. In this case, \( -\partial \log F(\lambda)/\partial \lambda \) is also completely monotone for \( \lambda > 0 \).

**Remark 1.** By the Hausdorff-Bernstein theorem (cf. [16], p. 60), it follows that if (6.2) holds, then there exists a distribution function \( V(\mu) = V(r, t) \) for \( r \geq 0 \) satisfying

\[ t^\mu/\Gamma(1 + \mu) = \int_0^\infty \exp(-r\mu^2) V(dr, t) \quad \text{for} \quad \mu > 0; \]

\[ V(r, t) = 0 \quad \text{or} \quad V(r, t) = V(r + \log(te^\gamma), e^{-\gamma}) \quad \text{according as} \quad r + \log(te^\gamma) \leq 0 \quad \text{or} \quad > 0. \]
Remark 2. The exponent $\frac{1}{2}$ in $F(\lambda) = f(\lambda^\frac{1}{2})$ is the best possible in the sense that $f(\lambda^\delta)$ is not completely monotone on $\lambda > 0$, for any $\delta > \frac{1}{2}$ and any $t > 0$. In fact, it is readily verified, by using the formulas in the proof to follow, that $\frac{d^2f(\lambda^\delta)}{d\lambda^2} \to -\infty$ as $\lambda \to 0$ if $\frac{1}{2} < \delta < 1$, $t > 0$.

Proof. Let $\psi(\mu)$ be the logarithmic derivative of $\Gamma(\mu)$, $\psi(\mu) = \Gamma''(\mu)/\Gamma(\mu)$. By a standard formula,

$$\psi(1 + \mu) + \gamma = \sum_{n=0}^{\infty} \mu/(n + 1)(n + 1 + \mu) \geq 0 \quad \text{for} \quad \mu \geq 0;$$

cf. [4], p. 15. Put

$$\varphi(\mu) = \int_0^\mu [\psi(1 + s) + \gamma] \, ds \quad \text{and} \quad \varphi(\lambda) = \varphi(\lambda^\frac{1}{2}).$$

Then Proposition 5.1 is applicable to $\varphi(\lambda)$, for, in this case,

$$\frac{d\varphi}{d\lambda} = [\psi(1 + \lambda^\frac{1}{2}) + \gamma]/2\lambda^\frac{1}{2} = \sum_{n=0}^{\infty} 1/2(n + 1)(n + 1 + \lambda^\frac{1}{2})$$

is completely monotone for $\lambda \geq 0$. By (6.4) and $f(\mu) = \psi(\mu)/\Gamma(1 + \mu)$,

$$F(\lambda) = f(\mu) = \exp (-\varphi(\mu)(te^\mu)^\mu \quad \text{with} \quad \mu = \lambda^\frac{1}{2} \geq 0.$$

When (6.2) holds, $(te^\mu)^\mu$ is a completely monotone function of $\mu \geq 0$, hence of $\lambda \geq 0$ (by Proposition 5.1 with $\varphi(\lambda) = \lambda^\frac{1}{2}$). Also, $e^\varphi$ is a completely monotone function of $\varphi \geq 0$, so that $\exp (-\varphi(\mu))$ is a completely monotone function of $\lambda$ (by the case (6.4) of Proposition 5.1). Consequently, (6.6) is completely monotone for $\lambda \geq 0$ when (6.2) holds. Also, in this case, $-F(\lambda)/F(\lambda) = d\varphi/d\lambda + [\log(1/[te^\varphi])]/2\lambda^\frac{1}{2}$ is completely monotone for $\lambda > 0$, where $d\varphi/d\lambda$ is given by (6.5).

Let $te^\varphi > 1$. Then, by (6.6), $df(\mu)/d\mu = [-d\varphi(\mu)/d\mu + \log(te^\varphi)]f(\mu)$. Since $d\varphi/d\mu = 0$ at $\mu = 0$, by (6.3)-(6.4), and $\log(te^\varphi) > 0$, we have $df/d\mu > 0$ at $\mu = 0$. Hence $F(\lambda) = f(\lambda^\frac{1}{2})$ is increasing for small $\lambda > 0$ and cannot be completely monotone.

Appendix 1: Monotone families of solutions.

We use the notation of Sections 1-3.

Definition. The class $M_\mu(A)$. A function $h(\lambda)$, $\lambda \in A$, is said to be of class $M_1 = M_\mu(A)$ if $h$ is nonnegative and nonincreasing. A function $h \in M_\mu(A)$
is said to be of class \( M_2 = M_2(A) \) if it is continuous and convex on \( A \). If \( \mu = 2, 3, \ldots \), \( h \) is said to be of class \( M_\mu = M_\mu(A) \) if \( h \in O^\mu(A) \cap O^{\mu-2}(A^\theta) \) and \( (-1)^m h^{(m)} \in M_q(A^\theta) \) for \( m = 0, \ldots, \mu - 2 \). \( M_\mu(A) \) is the class of completely monotone functions on \( A \).

Note that \( h \in M_\mu \) is continuous if \( \mu > 1 \). The class \( M_\mu \) is closed under point-wise convergence if \( A = A^\theta \) is open. This is false if if \( A = [0, \infty) \) as is seen from the example \( h_n(\lambda) = \exp(-n\lambda) \).

**Lemma.** Let \( A = A^\theta \) be an open interval. Let \( h_n \in M_\mu \) for \( n = 1, 2, \ldots \) such that \( h(\lambda) = \lim h_n(\lambda) \) exists as \( n \to \infty \) for \( \lambda \in A \). Then \( h \in M_\mu \). Furthermore, if \( \mu \geq 2 \), then

\[
(1) \quad h_n(\lambda) \to h(\lambda) \quad \text{as} \quad n \to \infty \quad \text{for} \quad j = 0, \ldots, \mu - 2
\]

uniformly on compacts of \( A \).

**Proof.** It is clear that \( h \in M_\mu \) if \( \mu = 1 \) or \( \mu = 2 \). Let \( \mu = 2 \), \( a < c < \beta \), and \( (a, \beta) \subset A \). Then \( c \leq \sigma < \tau < \beta \) implies that

\[
0 \leq [h_n(\sigma) - h_n(\tau)](\tau - \sigma) \leq [h_n(a) - h_n(c)](c - a),
\]

so that \( h_n \) is uniformly Lipschitz continuous on \([c, \beta]\) with a Lipschitz constant independent of \( n \). Thus (1) with \( j = 0 \) holds uniformly on \( \lambda \)-compacts.

Let \( \mu = 3 \). The argument just completed shows that the sequence of first order derivatives \( h'_1, h'_2, \ldots \) are uniformly bounded on \( \lambda \)-compacts. Thus, since they are convex, they are uniformly Lipschitz continuous with a Lipschitz constant independent of \( n \) on \( \lambda \)-compacts. Hence the Arzela selection theorem implies that there exist subsequences of \( h'_1, h'_2, \ldots \) uniformly convergent on \( \lambda \)-compacts. But the limit of such a subsequence is necessarily \( h'_1 \), independent of the subsequence. Consequently, (2) with \( j = 1 \) holds uniformly on \( \lambda \)-compacts. This proves the case \( \mu = 3 \). The proof of the Lemma can be completed by a simple induction.

\((\Lambda 2_{kn})\) Let \( \mu \geq 1 \) and \( k > 0 \) be fixed integers. Let \( \partial^n q/\partial \lambda^m \) exist, be continuous, and satisfy \( (-1)^{m+k} \partial^n q/\partial \lambda^m \geq 0 \) on \( T \times A^\theta \) for \( m = 1, 2, \ldots, \mu \) (in particular, \( (-1)^{k+1} \partial q(t, \cdot)/\partial \lambda \in M_{\mu-1}(A^\theta) \)).

The Lemma just proved and the arguments in Section 2 and 3 have the following consequences which are analogues of statements in Sections 1-3.

**Theorem 1.1.** Assume (A1) and (\(\Lambda 2_{kn}\)) for fixed integers \((k, \mu)\), \(0 < k < n\) and \(\mu > 1\). Let \( \tau \in T^\theta \). Then \( D^i\eta_{n-k}(t, \tau, \lambda) \in O^\mu(T \times T^\theta \times A^\theta) \) for \( 0 \leq i \leq n \) and \( \eta_{n-k}(t, \tau, \cdot) \in M_\mu(A^\theta) \) for fixed \((t, \tau)\), \(\alpha < \tau \leq t < \beta\).
THEOREM 1.2. Assume (A1), (A2) with $k = n - 1$ and $\mu > 1$, and (A3) with $\sigma = \alpha$ [or $\sigma = \beta$]. Then $D^i \xi(t, \lambda) \in C^0(T \times A)$ for $0 \leq i \leq n$, and $\xi(t, \cdot)$ [or $1/\xi(t, \cdot)$] is of class $M_\mu(A)$ for fixed $t \in T$.

LEMMA 2.1. Assume (A1), (A2) with fixed $(k, \mu)$, $0 < k < n$ and $\mu > 0$. Let $[a, b] \subset T$. Then $(-1)^n G_{\omega}(t, s, \cdot) \in M_\mu(A)$.

PROPOSITION 3.1. Assume (A1), (A2) with fixed $(k, \mu)$, $0 < k < n$ and $\mu > 0$. Then $(-1)^{k+1} X^s(t, \alpha, s, \cdot) \in M_\mu(A)$ for fixed $(t, \alpha, s)$, $\alpha < \alpha < t < s < \beta$, and $X^{n-k}(t, b, s, \cdot) \in M_\mu(A)$ for fixed $(t, b, s)$, $\alpha < s < t < b$.

Note that $A$ (rather than $A^0$) and $\mu > 0$ (rather than $\mu > 1$) occur in the last two assertions, since no limit process is involved in the proof of Lemma 2.1 and the limit process in the proof of Proposition 3.1 is uniform on compacts of $A$.

Appendix 2: Cauchy functions of n-th order equations.

In the $N$-th order linear differential equation

$$Lx = \begin{cases} p_{m+1}^{-1} D p_m^{-1} D \ldots p_1^{-1} D - \sum_{j=0}^{k-1} q_j D^j \end{cases} x = 0,$$

let $D = d/dt$; $k \geq 1$; $m \geq 0$; $n = k + m$; $p_j = p_j(t, \lambda) > 0$, $q_j(t, \lambda) \geq 0$ continuous on $T \times A$ such that $p_j(t, \cdot)$, $q_j(t, \cdot)$ are completely monotone on $A$ (for fixed $t \in T$). For suitable functions $x(t)$ on $T$, define the vector $y(t; x) = (y_1, \ldots, y_n)$ by $y_j = D^{j-1} x$ for $1 \leq j \leq k$ and $y_{k+j} = p_1^{-1} D \ldots D p_1^{-1} D^k x$ for $1 \leq j \leq m$, so that (1) is equivalent to the first order system

$$\begin{align*}
y_1' &= y_{i+1} & \text{for } 1 \leq j < k; \\
y_{k+i}' &= p_{i+1}y_{k+i} & \text{for } 0 \leq j < m; \\
y_n' &= p_{m+1} \sum_{j=0}^{k-1} q_j y_{j+1};
\end{align*}$$

ef., e.g., [5], pp. 309-310.

For fixed $\lambda$, let $X(t, s, \lambda)$ be the Cauchy function for (1), i.e., if $y(t; x)$ is the vector belonging to the solution $x(t) = X(t, s, \lambda)$ of (1), then $y$ satisfies the initial conditions

$$y_j = 0 \text{ for } 1 \leq j < n \quad \text{and} \quad y_n = 1 \text{ at } t = s.$$
We denote this vector $y(t; x)$, with $x = X(t, s, \lambda)$, by $Y(t, s, \lambda) = (Y_1(t, s, \lambda), \ldots, Y_n(t, s, \lambda))$. For example, $Y_j(t, s, \lambda) = D^{i-1}X(t, s, \lambda)$ for $1 \leq j \leq k$.

**Theorem.** Under the conditions above, $Y_j(t, s, \cdot)$ is completely monotone on $\Lambda$ for fixed $t, s \in T$, $t > s$, and $j = 1, \ldots, n$.

Note that, in this theorem, $q_j(t, \cdot)$ is completely monotone, so that assumption (A2a) holds, but the disconjugacy of (1) is not assumed. It is clear that the theorem is contained in the following

**Lemma.** Let $A(t, \lambda)$ be a continuous $n \times n$ matrix function on $T \times \Lambda$ such that every entry is completely monotone on $\Lambda$, for fixed $t \in T$. Let $U(t) = U(t, s, \lambda)$ be the fundamental matrix of

$$U' = A(t, \lambda)U$$

reducing to the identity at $t = s$. Then each entry of $U(t, s, \cdot)$ is completely monotone on $\Lambda$ for fixed $s, t \in T$, $t > s$.

**Remark.** It follows that if $h(t, \lambda)$ is a continuous vector on $T \times \Lambda$ such that each component of $h(t, \cdot)$ is completely monotone on $\Lambda$ and $c(\lambda)$ is a vector with completely monotone components, then

$$u' = A(t, \lambda)u + h(t, \lambda), \quad u(s) = c(\lambda),$$

has the solution

$$u(t, s, \lambda) = U(t, s, \lambda)c(\lambda) + \int_s^t U(t, r, \lambda)h(r, \lambda)\,dr,$$

with completely monotone components on $\Lambda$, for fixed $t, s \in T$, $t \geq s$.

**Proof of Lemma.** Since each entry of $A(t, \lambda)$ is non-negative, it is clear that each entry of $U(t, s, \lambda)$ is non-negative for $t \geq s$. Let $U = U(t, s, \lambda)$ in (4) and differentiate with respect to $\lambda \in A^0$ to get

$$U'_\lambda(\cdot, s, \lambda) = A(\cdot, \lambda)U_\lambda(\cdot, s, \lambda) + A_\lambda(\cdot, \lambda)U(\cdot, s, \lambda).$$

Since $U_\lambda(t, s, \lambda) = 0$ when $t = s$, the variations of constants formula gives

$$U_\lambda(t, s, \lambda) = \int_s^t U(t, r, \lambda)A_\lambda(r, \lambda)U(r, s, \lambda)\,dr.$$

This shows that each entry of $U_\lambda(t, s, \lambda)$ is non-positive for $t \geq s$. Repeated differentiations with respect to $\lambda \in A^0$ and an induction give the lemma.
Appendix 3: Special principal solutions.

Let the differential equation

\[ D^n x + q_{n-1}(t)D^{n-1}x + \ldots + q_1(t)Dx + q_0(t)x = 0 \]

have continuous coefficients on a \( t \)-interval \( T \) with endpoints \( \alpha, \beta \), where \(-\infty \leq \alpha < \beta \leq \infty\). Let \( T^0 = \text{int} T \).

**Theorem.** Let (1) be disconjugate on \( T \) and \([s, b]\subset T^0\). Let \( x = X^j(t, b, s) \) be the solution satisfying the boundary conditions

\[ \text{exists in } C^n \text{ on arbitrary } (t, s)-\text{compacts of } T \times T^0 \text{ and is the } j\text{-th special principal solution of (1) at } t = b, \text{ determined by } s; \text{ cf. (1.10) above.} \]

**Proof.** The case \( j = 1 \) follows from Theorem 7.1(ii), [7], p. 330. In particular, the theorem is correct if \( n = 2 \). Assume its validity for disconjugate differential equations of order \( n - 1 \geq 1 \). Let \( 1 \leq j \leq n \). Let \( x(t, s) \). There exists a disconjugate differential equation of order \( n - 1 \), say \( L_{n-1}v = 0 \), such that \( x \) is a solution of (1) if and only if \( v = W(\eta_1, x) = \eta_1'v' - \eta_1''v \) is a solution of \( L_{n-1}v = 0 \); furthermore, for \( 1 \leq j < n \), \( x \) is a \( j \)-th principal solution of (1) at \( t = \beta \) if and only if \( v = W(\eta_1, x) \) is a \((j - 1)\)-st principal solution of \( L_{n-1}v = 0 \) at \( \text{Theorem 7.2}\text{iv}, [7], p. 332. \)

Put \( V^{j-1}(t, b, s) = W(\eta_1, X^j(t, b, s)) \) for \( j = 2, \ldots, n - 1 \). Then \( v = V^{j-1} \) is a solution of \( L_{n-1}v = 0 \) and satisfies the same boundary conditions as \( X^j(t, b, s) \) as does \( X^{j-1}(t, b, s) \), i.e., (2) with \( j \) replaced by \( j - 1 \). By the induction hypothesis, \( \psi_{j-1}(t, s) = \lim W(\eta_1, X^j(t, b, s)) \), as \( b \to \beta \), exists in \( C^{n-1} \) on arbitrary \((t, s)\)-compacts of \( T \times T^0 \), and is the \((j - 1)\)-st special principal solution of \( L_{n-1}v = 0 \) at \( t = \beta \), determined by \( s \).

This implies that \( X^j/\eta_1 \to \psi_{j-1}/\eta_1^n \) as \( b \to \beta \). Hence, also \( q(t) = \lim X^j(t, b, s) \) exists, as \( b \to \beta \), in \( C^n \) on arbitrary \((t, s)\)-compacts of \( T \times T^0 \), and

\[ q(t) = \eta_1(t, s) \int_0^t [\psi_{j-1}(r, s)/\eta_1^n(r, s)] dr + c\eta_1(t, s), \]
where $c$ is a constant. Since $\psi(s) = 0$ and $\eta_1 > 0$, we have $c = 0$, so that $\psi$ satisfies the condition (1.10) with $r = s$. Also, from $\psi_{r-1} = W(\eta_1, g)$, it follows that $x = \psi(t)$ is a $j$-th principal solution of (1) at $t = \beta$. Hence $\psi(t) = \eta_j(t, s)$, and the proof is complete.

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