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An Asymptotic Formula Relating the Siegel Zero and the Class Number of Quadratic Fields.

DORIAN GOLDFELD (*)

§ 1. - For a fundamental discriminant d , let $h(d) = \sum_{a,b,c} 1$ denote the number of reduced, primitive, inequivalent binary quadratic forms $ax^2 + bxy + cy^2$ with $d = b^2 - 4ac$. In view of the correspondence

$$ax^2 + bxy + cy^2 \leftrightarrow \left[a, \frac{b + \sqrt{d}}{2} \right] \quad (\mathbb{Z}\text{-module}),$$

$h(d)$ is also the narrow class number of the quadratic field $Q(\sqrt{d})$.

If, for a real primitive character $\chi(\text{mod } d)$

$$(1) \quad L(1, \chi) \ll (\log|d|)^{-1},$$

then $L(s, \chi)$ will have a real zero β in the interval $0 < 1 - \beta \ll (\log|d|)^{-1}$. This can be seen, by simply considering the following integral (see [1])

$$(2) \quad 1 \ll \int_{2-i\infty}^{1+i\infty} \zeta(s + \beta') L(s + \beta', \chi) \frac{x^s}{s(s+1)(s+2)} ds = \\ = L(1, \chi) \frac{x^{1-\beta'}}{(1-\beta')(2-\beta')(3-\beta')} + \frac{1}{2} \zeta(\beta') L(\beta', \chi) + O(|d|x^{-\frac{1}{2}})$$

after shifting the line of integration to $\text{Re}(s) = -\frac{1}{2}$. Now, if $1 - \beta' = c_1(\log|d|)^{-1}$ for suitable c_1 , and $L(s, \chi) \neq 0$ for $\beta' \leq s < 1$ then $\zeta(\beta') L(\beta', \chi) < 0$. Choosing $x = |d|^3$, say, it follows from (2) that $L(1, \chi) \gg (\log|d|)^{-1}$, which contradicts (1) unless $L(s, \chi)$ has a zero in the interval. The precise location

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of this zero can be given as follows

THEOREM 1. *Let $d < 0$. If the Siegel zero β exists, then*

$$1 - \beta = \frac{6}{\pi^2} \{L(1, \chi) + O(1 - \beta)^2 (\log |d|)^3\} \left\{ \sum_{a,b,c} \frac{1}{a} + O(L(1, \chi) \log |d|) \right\}^{-1},$$

$$1 - \beta = (L(1, \chi) + O(1 - \beta)^2 (\log |d|)^3) L'(1, \chi)^{-1},$$

$$L'(1, \chi) = \frac{-\pi}{\sqrt{|d|}} \sum_{m=1}^{|d|} \chi(m) \log \Gamma \left(\frac{m}{|d|} \right) + \frac{h(d) \pi (\gamma + \log 2\pi)}{\sqrt{|d|}},$$

where γ is Euler's constant, and all other constants occurring in the O -symbols are effectively computable.

When $d > 0$, let $\omega = (-b + \sqrt{d})/2a$, $\bar{\omega} = (-b - \sqrt{d})/2a$, p_n/q_n (with $q_1 = 1$) be the principal convergents to ω , and $\|q_n \omega\| = |q_n \omega - p_n|$. If ε is the fundamental unit of $Q(\sqrt{d})$, define M to be 0 if $a\varepsilon/\sqrt{d} < 1$, and otherwise the unique integer satisfying $q_M < a\varepsilon/\sqrt{d} < q_{M+1}$.

THEOREM 2. *Let $d > 0$. If the Siegel zero β exists, then*

$$1 - \beta = \frac{6}{\pi^2} \{L(1, \chi) + O(1 - \beta)^2 (\log |d|)^3\} \left\{ \sum_{a,b,c} \left(\frac{1}{a} + \frac{2}{\pi \sqrt{d}} Q \right) + O(L(1, \chi) \log d) \right\}^{-1},$$

$$Q = \sum_{m=1}^{M-1} \frac{1}{q_m \|q_m \omega\|} \left[\text{Arctan} \left(\frac{q_{m+1} \|q_m \omega\|}{q_m} \right) - \text{Arctan} (q_m \|q_m \omega\|) \right] +$$

$$+ \frac{1}{q_M \|q_M \omega\|} \left[\text{Arctan} \left(\frac{a^2 \varepsilon^2 \|q_M \omega\|}{d q_M} \right) - \text{Arctan} (q_M \|q_M \omega\|) \right]$$

and all constants occurring in O -symbols are effectively computable.

§ 2. – Following a suggestion of Gallagher, a simple proof of Theorem (1) can be given by use of Kronecker's limit formula. Let, for $z = x + iy$

$$(3) \quad f(z, s) = y^s \sum_{\substack{m,n=-\infty \\ (m,n) \neq (0,0)}}^{\infty} |m + nz|^{-2s} =$$

$$= 2y^s \zeta(2s) + 2\sqrt{\pi} y^{1-s} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \zeta(2s - 1) +$$

$$+ 4 \frac{\pi \sqrt{y}}{\Gamma(s)} \sum_{n=1}^{\infty} n^{s-\frac{1}{2}} \sigma_{1-2s}(n) \cos(2\pi xn) \int_0^{\infty} \exp\left(-n\pi y \left(t + \frac{1}{t}\right)\right) t^{s-\frac{1}{2}} \frac{dt}{t},$$

a result which easily follows from [2].

Now, by a classical theorem of Dirichlet (for $d < 0$)

$$(4) \quad \zeta(s)L(s, \chi) = \frac{1}{2} \sum_{a,b,c} 2^s |d|^{-s/2} f(z, s), \quad \left(z = \frac{b + \sqrt{d}}{2a} \right),$$

and by equations (3) and (4)

$$(5) \quad \lim_{s \rightarrow 1} \left(\zeta(s)L(s, \chi) - \frac{L(1, \chi)}{s-1} \right) = \gamma L(1, \chi) + L'(1, \chi) = \\ = \frac{\pi^2}{6} \sum_{a,b,c} \frac{1}{a} + O(L(1, \chi) \log |d|).$$

On the other hand, the Taylor series expansion for L about β gives

$$(6) \quad 0 = L(1, \chi) + (\beta - 1)L'(1, \chi) + O(1 - \beta)^2(\log |d|)^3.$$

The first part of Theorem 1 now follows from (5) and (6). The second part follows from (6) and the formula for $L'(1, \chi)$ (see [2], p. 110).

§ 3. - In the case $d > 0$, let

$$x = \frac{\bar{\omega} + \omega u^2}{u^2 + 1}, \quad y = \frac{\sqrt{d}}{a} \frac{u}{u^2 + 1}.$$

Following Hecke, Siegel [3]

$$(7) \quad \zeta(s)L(s, \chi) = \sum_{a,b,c} \frac{\Gamma(s)}{\Gamma^2(s/2)} d^{-s/2} \int_{\eta}^{\eta e^s} f(z, s) \frac{du}{u}$$

where $\eta > 0$ is arbitrary. Henceforth, we take $\eta = a/\sqrt{d}$.

Now, $f(z, s)$ is invariant under a unimodular transformation

$$(8) \quad z \rightarrow z^* = \frac{\alpha z + \lambda}{\theta z + \delta}, \\ y^* = \frac{y}{(\theta x + \delta)^2 + (\theta y)^2}.$$

It follows that for

$$(9) \quad q_n < y^{-\frac{1}{2}} < q_{n+1}$$

and the choice $\theta = q_n$, $\delta = -p_n$ that

$$(10) \quad 1 \ll y^* \ll \sqrt{d}.$$

The condition (9) can be expressed

$$Q_n \leq u \leq Q_{n+1}$$

$$Q_n = \frac{1}{2} \left| \frac{\sqrt{d}}{a} q_n^2 + \left(\frac{d}{a^2} q_n^4 - 4 \right)^{\frac{1}{2}} \right|, \quad (1 \leq n \leq M),$$

$$Q_{M+1} = a\varepsilon^2/\sqrt{d}.$$

Consequently, the integral in (7) can be written as a sum of integrals

$$(11) \quad \zeta(s)L(s, \chi) = \frac{\Gamma(s)}{\Gamma^2(s/2)} d^{-s/2} \sum_{a,b,c} \left[\int_{a/\sqrt{d}}^{Q_1} f(z, s) \frac{du}{u} + \sum_{m=1}^M \int_{Q_m}^{Q_{m+1}} f(z^*, s) \frac{du}{u} \right].$$

We now get from (3), (10), and (11) that

$$(12) \quad \lim_{s \rightarrow 1} \left(\zeta(s)L(s, \chi) - \frac{L(1, \chi)}{s-1} \right) = \gamma L(1, \chi) + L'(1, \chi) =$$

$$= \frac{\pi}{3\sqrt{d}} \sum_{m=0}^M I_m + R_1 + R_2$$

where

$$I_0 = \int_{a/\sqrt{d}}^{Q_1} y \frac{du}{u}, \quad I_m = \int_{Q_m}^{Q_{m+1}} y^* \frac{du}{u}, \quad (1 \leq m \leq M)$$

and

$$(13) \quad \begin{cases} R_1 \ll d^{-\frac{1}{2}} \sum_{a,b,c} \log d \int_{a/\sqrt{d}}^{\varepsilon^2 a/\sqrt{d}} \frac{du}{u} \ll L(1, \chi) \log d, \\ R_2 \ll d^{-\frac{1}{2}} \sum_{a,b,c} \int_{a/\sqrt{d}}^{\varepsilon^2 a/\sqrt{d}} \frac{du}{u} \ll L(1, \chi). \end{cases}$$

Since $x = \omega - \sqrt{d}/a(u^2 + 1)$, it follows from (8) that

$$y^* = \frac{\sqrt{d}}{a} \|q_m \omega\|^{-2} \frac{u}{u^2 + B_m^2},$$

$$B_m = \frac{\sqrt{d}}{a} \frac{q_m}{\|q_m \omega\|} - 1.$$

Therefore (for $1 \leq m \leq M$)

$$I_0 = \frac{\sqrt{d}}{a} [\operatorname{Arctan}(Q_1) - \operatorname{Arctan}(a/d)],$$

$$I_m = \frac{\sqrt{d}}{a} \|q_m \omega\|^{-2} B_m^{-1} \left[\operatorname{Arctan} \left(\frac{Q_{m+1}}{B_m} \right) - \operatorname{Arctan} \left(\frac{Q_m}{B_m} \right) \right].$$

We get

$$(14) \quad \left\{ \begin{aligned} \sum_{a,b,c} I_0 &= \frac{\pi \sqrt{d}}{2} \sum_{a,b,c} \frac{1}{a} + O(h(d)), \\ \sum_{a,b,c} \sum_{m=1}^{M-1} I_m &= \sum_{a,b,c} \sum_{m=1}^{M-1} \frac{1}{q_m \|q_m \omega\|} \left[\operatorname{Arctan} \left(\frac{q_{m+1} \|q_m \omega\|}{q_m} \right) - \right. \\ &\quad \left. - \operatorname{Arctan}(q_m \|q_m \omega\|) \right] + O(h(d)), \\ \sum_{a,b,c} I_M &= \sum_{a,b,c} \frac{1}{q_M \|q_M \omega\|} \left[\operatorname{Arctan} \left(\frac{a^2 \varepsilon^2 \|q_M \omega\|}{dq_M} \right) - \operatorname{Arctan}(q_M \|q_M \omega\|) \right] + \\ &\quad + O(1). \end{aligned} \right.$$

Theorem (2) now follows from (6), (12), (13) and (14).

* * *

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