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On Siegel's Zero.

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1. - Let d be fundamental discriminant, and let

$$\chi(n) = \left(\frac{d}{n}\right) \quad (\text{Kronecker's symbol}).$$

It is well known (see [1]) that $L(s, \chi)$ has at most one zero β in the interval $(1 - c_1/\log |d|, 1)$ where c_1 is an absolute positive constant. The main aim of this paper is to prove:

THEOREM 1. *Let d , χ and β have the meaning defined above. Then the following asymptotic relation holds*

$$(1) \quad 1 - \beta = \frac{6}{\pi^2} \frac{L(1, \chi)}{\sum' 1/a} \left[1 + O\left(\frac{(\log \log |d|)^2}{\log |d|}\right) + O((1 - \beta) \log |d|) \right]$$

where \sum' is taken over all quadratic forms (a, b, c) of discriminant d such that

$$(2) \quad -a < b \leq a < \frac{1}{4} \sqrt{|d|},$$

and the constants in the O -symbols are effectively computable.

In order to apply the above theorem we need some information about the size of the sum $\sum' 1/a$. This is supplied by the following.

THEOREM 2. *If (a, b, c) runs through a class C of properly equivalent primitive forms of discriminant d , supposed fundamental, then*

$$\sum_{\substack{\frac{1}{4}\sqrt{|d|} \geq |a| \geq b > -a \\ (a,b,c) \in C}} \frac{1}{|a|} < \begin{cases} 1/m_0 & \text{if } d < 0, \\ \frac{\log \varepsilon_0}{\log(\frac{1}{2}\sqrt{d}-1)} + \frac{4}{\sqrt{d}} & \text{if } d > 676, \end{cases}$$

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where m_0 is the least positive integer represented by C and ε_0 is the least totally positive unit of the field $Q(\sqrt{d})$.

Theorems 1 and 2 together imply

COROLLARY. For any $\eta > 0$ and $|d| > c(\eta)$ (d fundamental) we have

$$1 - \beta \gg \begin{cases} \left(\frac{6}{\pi} - \eta\right) \frac{1}{\sqrt{|d|}} & \text{if } d < 0, \\ \left(\frac{6}{\pi^2} - \eta\right) \frac{\log d}{\sqrt{d}} & \text{if } d > 0, \end{cases}$$

where $c(\eta)$ is an effectively computable constant.

REMARK. In the case $d < 0$, the constant $6/\pi$ could be improved by using the knowledge of all fields with class number ≤ 2 .

Similar inequalities with $6/\pi$ and $6/\pi^2$ replaced by unspecified positive constants have been claimed by Hanecke [3], however, as pointed out by Pintz [8], Hanecke's proof is defective and when corrected gives inequalities weaker by a factor $\log \log |d|$. Pintz himself has proved the first inequality of the corollary with the constant $6/\pi$ replaced by $12/\pi$ (see [8]).

For $d < 0$, the first named author [2] has obtained (1) with a better error term by an entirely different method. M. Huxley has also found a proof in the case $d < 0$ by a more elementary method different, however, from the method of the present paper.

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2. - The proofs of Theorems 1 and 2 are based on several lemmata.

LEMMA 1. Let $f(d) = (\log |d| / \log \log |d|)^2$. Then

$$\sum_{N\mathfrak{a} \leq \frac{1}{4}\sqrt{|d|}f(d)} \frac{1}{N\mathfrak{a}} = \frac{\pi^2}{6} \sum' \frac{1}{a} \left(1 + O\left(\frac{(\log \log |d|)^2}{\log |d|}\right) \right),$$

where the left hand sum goes over all ideals $\mathfrak{a} \in Q(\sqrt{d})$ with norm $\leq \frac{1}{4}\sqrt{|d|}f(d)$ and the constant in the O -symbol is effectively computable.

PROOF. Every ideal \mathfrak{a} of $\Omega(\sqrt{d})$ can be represented in the form

$$\mathfrak{a} = u \left[a, \frac{b + \sqrt{d}}{2} \right]$$

where u, a are positive integers and $b^2 \equiv d \pmod{4a}$ (see [5], Theorem 59). If we impose the condition that

$$-a < b \leq a$$

then the representation becomes unique. Since $Na = u^2 a$, it follows that

$$\begin{aligned} (3) \quad \sum_{Na \leq \frac{1}{4}\sqrt{|d|}f(d)} \frac{1}{Na} &= \sum' \frac{1}{a} \sum_{1 \leq u^2 \leq \frac{\sqrt{|d|}f(d)}{4a}} \frac{1}{u^2} + O\left(\sum_{\frac{1}{4}\sqrt{|d|} < a < \frac{1}{4}\sqrt{|d|}f(d)} \frac{1}{a}\right) = \\ &= \sum' \frac{1}{a} \left(\frac{\pi^2}{6} + O((f(d))^{-\frac{1}{2}})\right) + O(S). \end{aligned}$$

To estimate the sum S , we divide it into two sums S_1 and S_2 . In the sum S_1 , we gather all the terms $1/a$ such that a has at least one prime power factor

$$\begin{aligned} p^\alpha > l(d) &= d^{1/21 \log \log |d|}, \\ p^\alpha | a, \end{aligned}$$

and in S_2 all the other terms.

Let $\nu(a)$ be the number of representations of a as Na where a has no rational integer divisor > 1 . Then $\nu(a)$ is a multiplicative function satisfying

$$\nu(p^\alpha) = \begin{cases} 1 + \left(\frac{d}{p^\alpha}\right) & \text{if } p \nmid d \text{ or } \alpha = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly

$$\begin{aligned} S_1 &\leq \sum' \frac{1}{a} \sum'' \nu(p^\alpha) p^{-\alpha} \\ &\leq \sum' \frac{1}{a} \sum'' 2p^{-\alpha} \end{aligned}$$

where \sum'' goes over all prime powers p^α with

$$\max(l(d), \sqrt{|d|}/4a) < p^\alpha \leq \sqrt{|d|}f(d)/4a.$$

Now, by a well known result of Mertens

$$\sum_{p^\alpha < x} p^{-\alpha} = \log \log x + c + O((\log x)^{-1})$$

where c is a constant.

Hence

$$\begin{aligned} \sum_{x < p^\alpha < y} p^{-\alpha} &= \log \left(\frac{\log y}{\log x} \right) + O((\log x)^{-1}) \ll \\ &\ll \frac{\log y}{\log x} - 1 + O((\log x)^{-1}) = \\ &= \frac{\log y/x + O(1)}{\log x}. \end{aligned}$$

This gives

$$\sum^n p^{-\alpha} \ll \frac{\log f(d) + O(1)}{\log l(d)} \ll \frac{(\log \log |d|)^2}{\log d}$$

and we get

$$(4) \quad S_1 = O\left(\frac{(\log \log |d|)^2}{\log |d|}\right) \sum \frac{1}{a}.$$

To estimate S_2 , we notice that each a occurring in it must have at least

$$k_0 = \frac{\log \left(\frac{1}{4} \sqrt{|d|}\right)}{\log l(d)} \geq 10 \log \log |d|$$

distinct prime factors. Therefore

$$\begin{aligned} S_2 &\leq \sum_{k \geq k_0} (1/k!) \left(\sum_{p^\alpha < l(d)} \nu(p^\alpha) p^{-\alpha} \right)^k \\ &< (1/k_0!) \sigma^{k_0} e^\sigma \end{aligned}$$

where

$$\begin{aligned} \sigma &= \sum_{p^\alpha < l(d)} \nu(p^\alpha) p^{-\alpha} < 2 \log \log l(d) + O(1) \\ &= 2 \log \log |d| + O(1). \end{aligned}$$

Now, Stirling's formula gives $k! > k_0^{k_0} \exp[-k_0]$. Hence

$$\begin{aligned} \log S_2 &\leq -k_0 \log k_0 + k_0(\log \sigma + 1) + \sigma \\ &\leq -k_0[\log 10 + \log \log \log |d| - \log 2 - \log \log \log |d| - 1] + \sigma \\ &\leq -3 \log \log |d| + O(1) \end{aligned}$$

and

$$(5) \quad S_2 = O((\log |d|)^{-3}).$$

The lemma now follows from equations (3), (4) and (5). The next lemma gives the growth conditions for the Riemann zeta-function and Dirichlet L -functions on the imaginary axis.

LEMMA 2. For all real t

$$(6) \quad |\zeta(it)| \ll (|t|^{\frac{1}{2}} + 1) \log(|t| + 2)$$

$$(7) \quad |L(it, \chi)| \ll \sqrt{|d|} (|t|^{\frac{1}{2}} + 1) \log(|d|(|t| + 2)).$$

PROOF. If $|t| > t_0$, the estimate

$$|\zeta(it)| \ll |t|^{\frac{1}{2}} \log|t|$$

holds (see [10], p. 19). Since $\zeta(s)$ has no pole on the imaginary axis, we have

$$|\zeta(it)| \ll 1 \quad \text{for } |t| \leq t_0$$

and the inequality (6) now follows.

To prove (7), we note that

$$|L(1 - it, \chi)| \ll \log(|d|(|t| + 2))$$

(see [1], p. 17, lemma 2 with $q = |d|$, $x = 2|d|(|t| + 2)$).

Now, by the fundamental equation for L -functions

$$|L(it, \chi)| = |L(1 - it, \chi)| |d|^{\frac{1}{2}} |\Gamma(\frac{1}{2}it + A) \Gamma(\frac{1}{2}it + A) \Gamma^{-1}(\frac{1}{2} - \frac{1}{2}it + A)|$$

where

$$A = \frac{1}{4}(1 - \chi(-1)).$$

Using the formula

$$|\Gamma(s)| = \sqrt{2\pi} |t|^{\sigma - \frac{1}{2}} \exp[-\frac{1}{2}\pi t] (1 + O(|t|^{-1}))$$

valid for $s = \sigma + it$, $0 \leq \sigma \leq \frac{1}{2}$, $|t| > 1$ (see [9], p. 395), equation (7) follows, upon noting that

$$|\Gamma(\frac{1}{2}t + A) \Gamma^{-1}(\frac{1}{2} - \frac{1}{2}t + A)| \ll 1 \quad \text{for } |t| < 1.$$

PROOF OF THEOREM 1. By the standard argument ([4], p. 31)

$$\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{y^s}{s(s+2)(s+3)} ds = \begin{cases} \frac{1}{6} - \frac{y^{-2}}{2} + \frac{y^{-3}}{3} & \text{if } y \geq 1, \\ 0 & \text{if } 0 < y < 1. \end{cases}$$

Since for $\text{Re}(s) > 1$

$$\zeta(s)L(s, \chi) = \sum (N\alpha)^{-s},$$

it follows that for any $x > 0$

$$\begin{aligned} I &= \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \zeta(s+\beta)L(s+\beta, \chi) \frac{x^s}{s(s+2)(s+3)} ds \\ &= \sum_{N\alpha \leq x} (N\alpha)^{-\beta} \left| \frac{1}{6} - \frac{(N\alpha)^2}{2x^2} + \frac{(N\alpha)^3}{3x^3} \right|. \end{aligned}$$

Choose $x = \frac{1}{4} \sqrt{|d|} f(d)$ with $f(d) = (\log |d| / \log \log |d|)^2$.

If $N\alpha \leq x$, we have

$$(N\alpha)^{-\beta} = (N\alpha)^{-1} (1 + O((1-\beta) \log |d|)).$$

Hence

$$\begin{aligned} I &= \frac{1}{6} \sum_{N\alpha \leq x} (N\alpha)^{-1} (1 + O((1-\beta) \log |d|)) \\ &\quad + O\left(\sum_{N\alpha \leq x/f(d)} (N\alpha)^{-1} f(d)^{-2} \right) + O\left(\sum_{x/f(d) \leq N\alpha \leq x} (N\alpha)^{-1} \right), \end{aligned}$$

and by lemma (1) (cf. formula (3))

$$(8) \quad I = \frac{1}{6} \sum' \frac{1}{\alpha} \left| 1 + O\left(\frac{(\log \log |d|)^2}{\log |d|} \right) + O((1-\beta) \log |d|) \right|.$$

On the other hand, after shifting the line of integration to $\text{Re}(s) = -\beta$

$$(9) \quad I = \frac{L(1, \chi) x^{1-\beta}}{(1-\beta)(3-\beta)(4-\beta)} + \frac{1}{2\pi i} \int_{-\beta-i\infty}^{-\beta+i\infty} \zeta(s+\beta)L(s+\beta, \chi) \frac{x^s}{s(s+2)(s+3)} ds.$$

By lemma (2), the integral on the right does not exceed

$$O(x^{-\beta} \sqrt{|d|} \log |d|)$$

and since

$$\begin{aligned}x^{1-\beta} &= 1 + O((1-\beta) \log |d|) \\(1-\beta)(3-\beta)(4-\beta) &= 6 + O(1-\beta)\end{aligned}$$

we get from (8) and (9)

$$1-\beta = \frac{6}{\pi^2} \frac{L(1, \chi)}{\sum' 1/a} \left| 1 + O\left(\frac{(\log \log |d|)^2}{\log |d|}\right) + O((1-\beta) \log |d|) \right|.$$

3. - PROOF OF THEOREM 2. For $d < 0$ it is enough to prove that every class contains at most one form satisfying

$$(10) \quad -|a| < b \leq |a| < \frac{1}{4} \sqrt{|d|}.$$

Now, since

$$|d| = 4ac - b^2$$

we infer from (10) that

$$a < \sqrt{|d|} < |d|/4a \leq c,$$

thus every form satisfying (10) is reduced, and it is well known that every class contains at most one such form.

For $d > 0$, let us choose in the class C a form (*) (α, β, γ) reduced in the sense of Gauss, i.e. such that

$$(11) \quad \beta + \sqrt{d} > 2|\alpha| > -\beta + \sqrt{d} > 0.$$

We can assume without loss of generality that $\alpha > 0$. Now, for any form $f \in C$, there exists a properly unimodular transformation

$$T = \begin{pmatrix} p & r \\ q & s \end{pmatrix}$$

taking (α, β, γ) into f . The first column of this transformation can be made to consist of positive rational integers by Theorem 79 of [5]. If f satisfies (10), we infer from

$$(12) \quad \alpha p^2 + \beta pq + \gamma q^2 = a$$

(*) β is not to be confused with Siegel's zero.

that

$$\left| p + \frac{\beta - \sqrt{d}}{2\alpha} q \right| = a \left| \alpha p + \frac{\beta + \sqrt{d}}{2} q \right|^{-1} \leq \frac{1}{4} \sqrt{d} \cdot 2(\sqrt{d}q)^{-1} = \frac{1}{2} q^{-1}$$

and by lemma (16), p. 175 from [5], p/q is a convergent of the continued fraction expansion for

$$\omega = \frac{-\beta + \sqrt{d}}{2\alpha}.$$

From this point onwards, we shall use the notation of Perron's monograph [7]. Since by (11)

$$\omega^{-1} > 1 \quad \text{and} \quad O > (\omega')^{-1} > -1,$$

ω^{-1} is a reduced quadratic surd and it has a pure periodic expansion into a continued fraction. Hence

$$\omega = [0, \overline{b_1, b_2 \dots b_k}]$$

where the bar denotes the primitive period. The corresponding complete quotients form again a periodic sequence

$$\omega_v = \frac{P_v + \sqrt{d}}{Q_v}, \quad \omega_0 = \omega$$

where for all $v \geq 1$, ω_v is reduced,

$$(13) \quad \omega_v = \omega_{v+k},$$

and k is the least number with the said property.

LEMMA 3. *Let $[0, \overline{b_1, b_2, \dots, b_k}]$ be the continued fraction for ω defined above. Then*

$$\sum_{(a,b,c) \in C} \frac{1}{|a|} \leq \frac{2}{\sqrt{d}} \sum_{\substack{v=2 \\ \sqrt{d} \geq b_v \geq 2}}^{[k,2]} \min \left(\frac{\sqrt{d}}{2}, b_v + 1 \right)$$

where the sum on the left is taken over all (a, b, c) in the class C satisfying (10).

PROOF. If A_j/B_j is the j -th convergent of ω , we have by formula (18), § 20 of [7]

$$(A_{v-1}Q_0 - B_{v-1}P_0)^2 - d(B_{v-1})^2 = (-1)^v Q_0 Q_v$$

which gives on simplification

$$(14) \quad \alpha A_{v-1}^2 + \beta A_{v-1} B_{v-1} + \gamma B_{v-1}^2 = (-1)^v Q_v / 2.$$

Similarly, eliminating Q_v from formulae (16) and (17) in § 20 of [7], we get

$$(15) \quad 2\alpha A_{v-1} A_{v-2} + \beta(A_{v-1} B_{v-2} + B_{v-2} A_{v-2}) + 2\alpha B_{v-1} B_{v-2} = (-1)^{v-1} P_v.$$

Let $p = A_{v-1}$, $q = B_{v-1}$ ($v \geq 1$). By (12)

$$a = (-1)^v Q_v / 2.$$

Hence, by formula (1) of § 6 of [7]

$$\begin{vmatrix} A_{v-1} & A_{v-2} \\ B_{v-1} & B_{v-2} \end{vmatrix} = (-1)^v.$$

and since

$$\begin{vmatrix} A_{v-1} & r \\ B_{v-1} & s \end{vmatrix} = 1$$

it follows that

$$T = \begin{pmatrix} A_{v-1} & A_{v-2} \\ B_{v-1} & B_{v-2} \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & (-1)^v \end{pmatrix}, \quad t \in Z.$$

Thus we find using (14) and (15)

$$\begin{aligned} f &= (\alpha, \beta, \gamma) \begin{pmatrix} A_{v-1} & A_{v-2} \\ B_{v-1} & B_{v-2} \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & (-1)^v \end{pmatrix} = \\ &= \left((-1)^v \frac{Q_v}{2}, (-1)^{v-1} P_v, (-1)^v \frac{Q_{v-1}}{2} \right) \begin{pmatrix} 1 & t \\ 0 & (-1)^v \end{pmatrix}. \end{aligned}$$

In order to make f satisfy (10) we must choose

$$t = (-1)^v \left[\frac{P_v}{Q_v} + \frac{1}{2} \right].$$

Thus f is uniquely determined by ω_v and in view of (13), we have

$$(16) \quad \sum_{(a,b,c) \in C} \frac{1}{|a|} < \sum_{\substack{v=1 \\ q_v < \frac{1}{2}\sqrt{d}}}^{[k,2]} 2(Q_v)^{-1}.$$

Since ω_v is reduced, we have further for v in question

$$\sqrt{d} \geq \frac{2\sqrt{d}}{Q_v} > \frac{P_v + \sqrt{d}}{Q_v} > \frac{\sqrt{d}}{Q_v} \geq 2.$$

Hence for

$$b_v = [\omega_v],$$

we get the inequalities

$$\sqrt{d} > b_v \geq 2, \quad b_v + 1 > \sqrt{d}/Q_v,$$

and by (16), lemma (3) follows.

Now, let ε_0 be the least totally positive unit $\varepsilon_0 > 1$ of the ring $Z(\sigma)$ where

$$\sigma = \begin{cases} \frac{1}{2}\sqrt{d} & \text{if } d \equiv 0 \pmod{4}, \\ \frac{1 + \sqrt{d}}{2} & \text{if } d \equiv 1 \pmod{4}. \end{cases}$$

By Theorem (7) of Chapter IV of [6]

$$\varepsilon_0 = \frac{u + v\sqrt{d}}{2},$$

where for $l = [k, 2]$,

$$v = (q_{l-1}, p_{l-1} - q_{l-2}, p_{l-2}), \quad u = p_{l-1} + q_{l-2}$$

and p_j, q_j are the numerator and denominator, respectively, of the j -th convergent for ω^{-1} . Moreover, since ω^{-1} satisfies the equation

$$-\gamma\omega^2 - \beta\omega^{-1} - \alpha = 0, \quad (-\gamma > 0)$$

we find from formula (1) of § 2 of Chapter IV of [6] that

$$q_{l-2} - p_{l-1} = -\beta v, \quad -p_{l-2} = -\alpha v.$$

Hence

$$\varepsilon_0 = \frac{p_{l-1} + q_{l-2}}{2} + \frac{p_{l-2}\sqrt{d}}{2\alpha} = q_{l-2} + \frac{\beta + \sqrt{d}}{2\alpha} p_{l-2}.$$

Since $p_i = B_{i+1}$, $q_i = A_{i+1}$, we get

$$(17) \quad \varepsilon_0 = B_{l-1} \left(\frac{A_{l-1}}{B_{l-1}} + \frac{\beta + \sqrt{d}}{2\alpha} \right) > B_{l-1} \left(\omega + \frac{\beta + \sqrt{d}}{2\alpha} \right) = \frac{\sqrt{d}}{\alpha} B_{l-1}.$$

Now,

$$\omega_i = b_i + \omega_{i+1}^{-1} = b_i + \omega_1^{-1} = b_i + \omega, \quad \omega'_i = b_i + \omega'$$

and since ω_i is reduced $0 > b_i + \omega' > -1$

$$b_i = [-\omega'] = \left\lfloor \frac{\beta + \sqrt{d}}{2\alpha} \right\rfloor < \frac{\sqrt{d}}{\alpha}.$$

Thus (17) gives

$$\varepsilon_0 > b_l B_{l-1} > \prod_{v=1}^l b_v,$$

and by (16)

$$(18) \quad \sum_{(a,b,c) \in \mathcal{C}} \frac{1}{|a|} \leq \frac{2}{\sqrt{d}} \max \sum (x_i + 1) = \frac{2}{\sqrt{d}} M$$

where maximum is taken over all non-decreasing sequence of at most l numbers satisfying

$$2 < x_i \leq \frac{1}{2}\sqrt{d} - 1 = D, \quad \prod x_i < \varepsilon_0.$$

Let (x_1, x_2, \dots, x_m) be a point in which the maximum is taken with the least number m . We assert that the sequence contains at most one term x with $2 < x < D$. Indeed, if we had $2 < x_i < x_{i+1} < D$, we could replace the numbers x_i, x_{i+1} by

$$\frac{x_i}{\min(x_i/2, D/x_{i+1})}, \quad x_{i+1} \min\left(\frac{x_i}{2}, \frac{D}{x_{i+1}}\right)$$

and the sum $\sum (x_i + 1)$ would increase. Also, if we had $x_1 = x_2 = x_3 = 2$, we could replace them by $x_1 = 8$, and the sum $\sum (x_i + 1)$ would remain the same while m would decrease.

Let

$$\frac{\varepsilon_0}{4} = D^e \theta, \quad \text{where } e = \frac{\log(\varepsilon_0/4)}{\log D}.$$

Using $d > 676$, we get

$$M = \begin{cases} \frac{1}{2} e \sqrt{d} + \max(4\theta + 1, 2\theta + 4) & \text{if } 4\theta < D, \\ \frac{1}{2} e \sqrt{d} + 2\theta + 4 & \text{if } 2\theta < D \leq 4\theta, \\ \frac{1}{2} e \sqrt{d} + \theta + 7 & \text{if } D \leq 2\theta. \end{cases}$$

Now,

$$e = \frac{\log \varepsilon_0}{\log D} - \frac{\log 4\theta}{\log D}.$$

Since for $1 \leq x < y$, $y(\log x / \log y) \geq x - 1$, and for $d > 676$, $D/\log D > 12/\log 12 > 4.8$, we obtain if $4\theta < D$.

$$\begin{aligned} M - \frac{1}{2} \sqrt{d} \frac{\log \varepsilon_0}{\log D} &= \max(4\theta + 1, 2\theta + 4) - D \frac{\log 4\theta}{\log D} = \frac{\log 4\theta}{\log D} < \\ &< \max(4\theta + 1, 2\theta + 4) - \max(4\theta - 1, 6) \leq 2, \end{aligned}$$

if $2\theta < D \leq 4\theta$

$$\begin{aligned} M - \frac{1}{2} \sqrt{d} \frac{\log \varepsilon_0}{\log D} &= 2\theta + 4 - D \frac{\log 2\theta}{\log D} - D \frac{\log 2}{\log D} - \frac{\log 4\theta}{\log D} < \\ &< 2\theta + 4 - 2\theta + 1 - 3 - 1 = 1, \end{aligned}$$

if $D \leq 2\theta$

$$\begin{aligned} M - \frac{1}{2} \sqrt{d} \frac{\log \varepsilon_0}{\log D} &= \theta + 7 - D \frac{\log \theta}{\log D} - D \frac{\log 4}{\log D} - \frac{\log 4\theta}{\log D} < \\ &< \theta + 7 - \theta + 1 - 6 - 1 = 1. \end{aligned}$$

This together with (18) gives the theorem.

4. - PROOF OF COROLLARY. We can assume $1 - \beta < (\log |d|)^{-2}$. It then by Theorem (1) that for every $\eta > 0$, there exists $c(\eta)$ such that if $d > c(\eta)$

$$(19) \quad 1 - \beta \geq \frac{6}{\pi^2} \frac{L(1, \chi)}{\sum' 1/a} \left(1 - \frac{\eta}{2}\right).$$

Let h_0 be the number of classes of forms in question. For $d < -4$, we have

$$L(1, \chi) = \frac{\pi h_0}{\sqrt{|d|}},$$

and by Theorem (2)

$$\sum' \frac{1}{a} \leq h_0.$$

Hence by (19)

$$1 - \beta \geq \frac{6}{\pi^2} \frac{h_0 \pi}{h_0 \sqrt{|d|}} \left(1 - \frac{\eta}{2}\right) > \left(\frac{6}{\pi} - \eta\right) \frac{1}{\sqrt{|d|}}.$$

For $d > 0$, we have

$$L(1, \chi) = \frac{h_0 \log \varepsilon_0}{\sqrt{d}}.$$

Now, for any class C of forms

$$\sum'_{(a,b,c) \in C} \frac{1}{|a|} = \sum_{\substack{(a,b,c) \in C \\ \frac{1}{2}\sqrt{d} \geq a \geq b > -a}} \frac{1}{a} + \sum_{\substack{(-a,b,-c) \in C \\ \sqrt{d} \geq -a \geq b > a}} \frac{1}{|a|}.$$

If (a, b, c) runs through C , $(-a, b, -c)$ runs through another class which we denote by $-C$ (It may happen that $-C = C$). If $C_1 \neq C_2$, then $-C_1 \neq -C_2$. Hence

$$\sum_C \sum_{\substack{(a,b,c) \in C \\ \frac{1}{2}\sqrt{d} \geq |a| \geq b > -|a|}} \frac{1}{|a|} = 2 \sum' \frac{1}{a}$$

and by Theorem (2)

$$\sum' \frac{1}{a} \leq \frac{h_0}{2} \left(\frac{\log \varepsilon_0}{\log(\frac{1}{2}\sqrt{d}-1)} + \frac{4}{\sqrt{d}} \right) < \frac{h_0 \log \varepsilon_0}{\log d} \left(1 + O\left(\frac{1}{\sqrt{d}}\right) \right),$$

where the constant in the O -symbol is effective. (Note that $\varepsilon_0 > \frac{1}{2}\sqrt{d}$). This together with (19) gives the corollary.

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