ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

D. M. GOLDFELD

A. SCHINZEL

On Siegel's zero

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 2, nº 4 (1975), p. 571-583

<http://www.numdam.org/item?id=ASNSP_1975_4_2_4_571_0>

On Siegel's Zero.

D. M. GOLDFELD (*) - A. SCHINZEL (**)

1. – Let d be fundamental discriminant, and let

$$\chi(n) = \left(\frac{d}{n}\right) \qquad \text{(Kronecker's symbol)} .$$

It is well known (see [1]) that $L(s, \chi)$ has at most one zero β in the interval $(1 - c_1/\log|d|, 1)$ where c_1 is an absolute positive constant. The main aim of this paper is to prove:

THEOREM 1. *Let d, χ and β have the meaning defined above. Then the following asymptotic relation holds*

$$1 - \beta = \frac{6}{\pi^2}\frac{L(1, \chi)}{\sum' 1/a}\left[1 + O\left(\frac{(\log\log|d|)^2}{\log|d|}\right) + O((1-\beta)\log|d|)\right] \tag{1}$$

where $\sum'$ is taken over all quadratic forms (a, b, c) of discriminant d such that

$$-a < b \leqslant a < \tfrac{1}{4}\sqrt{|d|}\,, \tag{2}$$

and the constants in the O-symbols are effectively computable.

In order to apply the above theorem we need some information about the size of the sum $\sum' 1/a$. This is supplied by the following.

THEOREM 2. *If (a, b, c) runs through a class C of properly equivalent primitive forms of discriminant d, supposed fundamental, then*

$$\sum_{\substack{\frac{1}{4}\sqrt{|d|} \geqslant |a| \geqslant b > -|a| \\ (a,b,c)\in C}} \frac{1}{|a|} \leqslant \begin{cases} 1/m_0 & \text{if } d < 0\,, \\ \dfrac{\log\varepsilon_0}{\log\left(\frac{1}{2}\sqrt{d} - 1\right)} + \dfrac{4}{\sqrt{d}} & \text{if } d > 676\,, \end{cases}$$

(*) Scuola Normale Superiore, Pisa.
(**) Institut of Mathematics, Warsaw.
Pervenuto alla Redazione il 30 Maggio 1975.

where m_0 is the least positive integer represented by C and ε_0 is the least totally positive unit of the field $Q(\sqrt{d})$.

Theorems 1 and 2 together imply

COROLLARY. *For any $\eta > 0$ and $|d| > c(\eta)$ (d fundamental) we have*

$$1-\beta \geqslant \begin{cases} \left(\frac{6}{\pi}-\eta\right)\frac{1}{\sqrt{|d|}} & \text{if } d<0\,, \\ \left(\frac{6}{\pi^2}-\eta\right)\frac{\log d}{\sqrt{d}} & \text{if } d>0\,, \end{cases}$$

where $c(\eta)$ is an effectively computable constant.

REMARK. In the case $d<0$, the constant $6/\pi$ could be improved by using the knowledge of all fields with class number $\leqslant 2$.

Similar inequalities with $6/\pi$ and $6/\pi^2$ replaced by unspecified positive constants have been claimed by Hanecke [3], however, as pointed out by Pintz [8], Hanecke's proof is defective and when corrected gives inequalities weaker by a factor $\log\log|d|$. Pintz himself has proved the first inequality of the corollary with the constant $6/\pi$ replaced by $12/\pi$ (see [8]).

For $d<0$, the first named author [2] has obtained (1) with a better error term by an entirely different method. M. Huxley has also found a proof in the case $d<0$ by a more elementary method different, however, from the method of the present paper.

The authors wish to thank Scuola Normale Superiore which gave them the opportunity for this joint work.

2. – The proofs of Theorems 1 and 2 are based on several lemmata.

LEMMA 1. *Let $f(d) = (\log|d|/\log\log|d|)^2$. Then*

$$\sum_{N\mathfrak{a}\leqslant\frac{1}{4}\sqrt{|d|}f(d)}\frac{1}{N\mathfrak{a}} = \frac{\pi^2}{6}\sum{}'\frac{1}{a}\left(1+O\left(\frac{(\log\log|d|)^2}{\log|d|}\right)\right),$$

where the left hand sum goes over all ideals $\mathfrak{a}\in Q(\sqrt{d})$ with norm $\leqslant\frac{1}{4}\sqrt{|d|}f(d)$ and the constant in the O-symbol is effectively computable.

PROOF. Every ideal $\mathfrak{a}$ of $\Omega(\sqrt{d})$ can be represented in the form

$$\mathfrak{a} = u\left[a, \frac{b+\sqrt{d}}{2}\right]$$

where u, a are positive integers and $b^2 \equiv d \pmod{4a}$ (see [5], Theorem 59). If we impose the condition that

$$-a < b \leqslant a$$

then the representation becomes unique. Since $N\mathfrak{a} = u^2 a$, it follows that

$$\text{(3)} \qquad \sum_{N\mathfrak{a} \leqslant \frac{1}{4}\sqrt{|d|}f(d)} \frac{1}{N\mathfrak{a}} = \sum{}' \frac{1}{a} \sum_{1 \leqslant u^2 \leqslant \frac{\sqrt{|d|}f(d)}{4a}} \frac{1}{u^2} + O\Bigg(\sum_{\frac{1}{4}\sqrt{|d|} < a < \frac{1}{4}\sqrt{|d|}f(d)} \frac{1}{a}\Bigg) =$$

$$= \sum{}' \frac{1}{a}\left(\frac{\pi^2}{6} + O\left(\left(f(d)\right)^{-\frac{1}{2}}\right)\right) + O(S).$$

To estimate the sum S, we divide it into two sums S_1 and S_2. In the sum S_1, we gather all the terms $1/a$ such that a has at least one prime power factor

$$p^\alpha > l(d) = d^{1/21 \log\log|d|},$$
$$p^\alpha | a,$$

and in S_2 all the other terms.

Let $\nu(a)$ be the number of representations of a as $N\mathfrak{a}$ where $\mathfrak{a}$ has no rational integer divisor > 1. Then $\nu(a)$ is a multiplicative function satisfying

$$\nu(p^\alpha) = \begin{cases} 1 + \left(\dfrac{d}{p^\alpha}\right) & \text{if } p \nmid d \text{ or } \alpha = 1, \\ 0 & \text{otherwise}. \end{cases}$$

Clearly

$$S_1 \leqslant \sum{}' \frac{1}{a} \sum{}'' \nu(p^\alpha) p^{-\alpha}$$
$$\leqslant \sum{}' \frac{1}{a} \sum{}'' 2p^{-\alpha}$$

where $\sum''$ goes over all prime powers p^α with

$$\max\left(l(d), \sqrt{|d|}/4a\right) < p^\alpha \leqslant \sqrt{|d|}\, f(d)/4a.$$

Now, by a well known result of Mertens

$$\sum_{p^\alpha < x} p^{-\alpha} = \log\log x + c + 0\left((\log x)^{-1}\right)$$

where c is a constant.

Hence

$$\sum_{x<p^\alpha<y} p^{-\alpha} = \log\left(\frac{\log y}{\log x}\right) + O\big((\log x)^{-1}\big) \leqslant$$

$$\leqslant \frac{\log y}{\log x} - 1 + O\big((\log x)^{-1}\big) =$$

$$= \frac{\log y/x + O(1)}{\log x}\,.$$

This gives

$$\sum{}'' p^{-\alpha} \leqslant \frac{\log f(d) + O(1)}{\log l(d)} \ll \frac{(\log\log|d|)^2}{\log d}$$

and we get

$$S_1 = O\left(\frac{(\log\log|d|)^2}{\log|d|}\right)\sum{}' \frac{1}{a}\,. \tag{4}$$

To estimate S_2, we notice that each a occuring in it must have at least

$$k_0 = \frac{\log\left(\frac{1}{4}\sqrt{|d|}\right)}{\log l(d)} \geqslant 10\log\log|d|$$

distinct prime factors. Therefore

$$S_2 \leqslant \sum_{k\geqslant k_0} (1/k!)\Big(\sum_{p^\alpha<l(d)} \nu(p^\alpha)p^{-\alpha}\Big)^k$$

$$< (1/k_0!)\sigma^{k_0}e^\sigma$$

where

$$\sigma = \sum_{p^\alpha<l(d)} \nu(p^\alpha)p^{-\alpha} < 2\log\log l(d) + O(1)$$

$$= 2\log\log|d| + 0(1)\,.$$

Now, Stirling's formula gives $k! > k_0^{k_0}\exp[-k_0]$. Hence

$$\log S_2 \leqslant -k_0\log k_0 + k_0\big((\log\sigma) + 1\big) + \sigma$$

$$\leqslant -k_0[\log 10 + \log\log\log|d| - \log 2 - \log\log\log|d| - 1] + \sigma$$

$$< -3\log\log|d| + 0(1)$$

and

$$S_2 = O\big((\log|d|)^{-3}\big)\,. \tag{5}$$

The lemma now follows from equations (3), (4) and (5). The next lemma gives the growth conditions for the Riemann zeta-function and Dirichlet L-functions on the imaginary axis.

LEMMA 2. *For all real t*

$$|\zeta(it)| \ll (|t|^{\frac{1}{2}}+1)\log(|t|+2) \tag{6}$$

$$|L(it,\chi)| \ll \sqrt{|d|}(|t|^{\frac{1}{2}}+1)\log\big(|d|(|t|+2)\big)\,. \tag{7}$$

PROOF. If $|t|>t_0$, the estimate

$$|\zeta(it)| \ll |t|^{\frac{1}{2}}\log|t|$$

holds (see [10], p. 19). Since $\zeta(s)$ has no pole on the imaginary axis, we have

$$|\zeta(it)| \ll 1 \qquad \text{for } |t|\leqslant t_0$$

and the inequality (6) now follows.

To prove (7), we note that

$$|L(1-it,\chi)| \ll \log(|d|(|t|+2))$$

(see [1], p. 17, lemma 2 with $q=|d|$, $x=2|d|(|t|+2)$).

Now, by the fundamental equation for L-functions

$$|L(it,\chi)| = |L(1-it,\chi)|\,|d|^{\frac{1}{2}}|\Gamma(\tfrac{1}{2}it+A)\Gamma(\tfrac{1}{2}it+A)\Gamma^{-1}(\tfrac{1}{2}-\tfrac{1}{2}it+A)|$$

where

$$A=\tfrac{1}{4}(1-\chi(-1))\,.$$

Using the formula

$$|\Gamma(s)| = \sqrt{2\pi}|t|^{\sigma-\frac{1}{2}}\exp[-\tfrac{1}{2}\pi t](1+O(|t|^{-1}))$$

valid for $s=\sigma+it$, $0\leqslant\sigma\leqslant\frac{1}{2}$, $|t|>1$ (see [9], p. 395), equation (7) follows, upon noting that

$$|\Gamma(\tfrac{1}{2}t+A)\Gamma^{-1}(\tfrac{1}{2}-\tfrac{1}{2}t+A)| \ll 1 \qquad \text{for } |t|<1\,.$$

PROOF OF THEOREM 1. By the standard argument ([4], p. 31)

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{y^s}{s(s+2)(s+3)}\,ds=\begin{cases}\dfrac{1}{6}-\dfrac{y^{-2}}{2}+\dfrac{y^{-3}}{3} & \text{if } y\geqslant 1\,,\\ 0 & \text{if } 0<y<1\,.\end{cases}$$

Since for $\operatorname{Re}(s)>1$

$$\zeta(s)L(s,\chi)=\sum(N\mathfrak{a})^{-s},$$

it follows that for any $x>0$

$$\begin{aligned}I&=\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\zeta(s+\beta)L(s+\beta,\chi)\frac{x^s}{s(s+2)(s+3)}\,ds\\&=\sum_{N\mathfrak{a}\leqslant x}(N\mathfrak{a})^{-\beta}\left|\frac{1}{6}-\frac{(N\mathfrak{a})^2}{2x^2}+\frac{(N\mathfrak{a})^3}{3x^3}\right|.\end{aligned}$$

Choose $x=\frac{1}{4}\sqrt{|d|}f(d)$ with $f(d)=(\log|d|/\log\log|d|)^2$.

If $N\mathfrak{a}\leqslant x$, we have

$$(N\mathfrak{a})^{-\beta}=(N\mathfrak{a})^{-1}\big(1+O((1-\beta)\log|d|)\big).$$

Hence

$$\begin{aligned}I=\tfrac{1}{6}&\sum_{N\mathfrak{a}\leqslant x}(N\mathfrak{a})^{-1}\big(1+O((1-\beta)\log|d|)\big)\\&+O\Big(\sum_{N\mathfrak{a}\leqslant x/f(d)}(N\mathfrak{a})^{-1}f(d)^{-2}\Big)+O\Big(\sum_{x/f(d)\leqslant N\mathfrak{a}\leqslant x}(N\mathfrak{a})^{-1}\Big),\end{aligned}$$

and by lemma (1) (cf. formula (3))

$$I=\frac{1}{6}\sum{}'\frac{1}{a}\left|1+O\left(\frac{(\log\log|d|)^2}{\log|d|}\right)+O((1-\beta)\log|d|)\right|. \tag{8}$$

On the other hand, after shifting the line of integration to $\operatorname{Re}(s)=-\beta$

$$I=\frac{L(1,\chi)x^{1-\beta}}{(1-\beta)(3-\beta)(4-\beta)}+\frac{1}{2\pi i}\int_{-\beta-i\infty}^{-\beta+i\infty}\zeta(s+\beta)L(s+\beta,\chi)\frac{x^s}{s(s+2)(s+3)}\,ds\,. \tag{9}$$

By lemma (2), the integral on the right does not exceed

$$O\big(x^{-\beta}\sqrt{|d|}\log|d|\big)$$

and since

$$x^{1-\beta} = 1 + O\big((1-\beta)\log|d|\big)$$

$$(1-\beta)(3-\beta)(4-\beta) = 6 + O(1-\beta)$$

we get from (8) and (9)

$$1-\beta = \frac{6}{\pi^2}\frac{L(1,\chi)}{\sum' 1/a}\left|1 + O\left(\frac{(\log\log|d|)^2}{\log|d|}\right) + O\big((1-\beta)\log|d|\big)\right|.$$

3. – Proof of Theorem 2. For $d<0$ it is enough to prove that every class contains at most one form satisfying

$$-|a| < b \leqslant |a| < \tfrac{1}{4}\sqrt{|d|}\,. \tag{10}$$

Now, since

$$|d| = 4ac - b^2$$

we infer from (10) that

$$a < \sqrt{|d|} < |d|/4a \leqslant c\,,$$

thus every form satisfying (10) is reduced, and it is well known that every class contains at most one such form.

For $d>0$, let us choose in the class C a form (*) (α, β, γ) reduced in the sense of Gauss, i.e. such that

$$\beta + \sqrt{d} > 2|\alpha| > -\beta + \sqrt{d} > 0\,. \tag{11}$$

We can assume without loss of generality that $\alpha > 0$. Now, for any form $f\in C$, there exists a properly unimodular transformation

$$T = \begin{pmatrix} p & r \\ q & s\end{pmatrix}$$

taking (α, β, γ) into f. The first column of this transformation can be made to consist of positive rational integers by Theorem 79 of [5]. If f satisfies (10), we infer from

$$\alpha p^2 + \beta pq + \gamma q^2 = a \tag{12}$$

(*) β is not to be confused with Siegel's zero.

that

$$\left|p+\frac{\beta-\sqrt{d}}{2\alpha}q\right|=a\left|\alpha p+\frac{\beta+\sqrt{d}}{2}q\right|^{-1}\leqslant$$

$$\leqslant\frac{1}{4}\sqrt{d}\cdot 2(\sqrt{d}\,q)^{-1}=\frac{1}{2}q^{-1}$$

and by lemma (16), p. 175 from [5], p/q is a convergent of the continued fraction expansion for

$$\omega=\frac{-\beta+\sqrt{d}}{2\alpha}.$$

From this point onwards, we shall use the notation of Perron's monograph [7]. Since by (11)

$$\omega^{-1}>1 \quad\text{and}\quad O>(\omega')^{-1}>-1,$$

ω^{-1} is a reduced quadratic surd and it has a pure periodic expansion into a continued fraction. Hence

$$\omega=[0,\overline{b_1,b_2\ldots b_k}]$$

where the bar denotes the primitive period. The corresponding complete quotients form again a periodic sequence

$$\omega_v=\frac{P_v+\sqrt{d}}{Q_v}, \qquad \omega_0=\omega$$

where for all $v\geqslant 1$, ω_v is reduced,

$$\omega_v=\omega_{v+k}, \tag{13}$$

and k is the least number with the said property.

LEMMA 3. *Let* $[0,\overline{b_1,b_2,\ldots,b_k}]$ *be the continued fraction for* ω *defined above. Then*

$$\sum_{(a,b,c)\in C}'' \frac{1}{|a|}\leqslant\frac{2}{\sqrt{d}}\sum_{\substack{v=2\\ \sqrt{d}\geqslant b_v\geqslant 2}}^{[k,2]}\min\left(\frac{\sqrt{d}}{2},\,b_v+1\right)$$

where the sum on the left is taken over all (a,b,c) *in the class* C *satisfying* (10).

PROOF. If A_j/B_j is the j-th convergent of ω, we have by formula (18), § 20 of [7]

$$(A_{v-1}Q_0 - B_{v-1}P_0)^2 - d(B_{v-1})^2 = (-1)^v Q_0 Q_v$$

which gives on simplification

$$\alpha A_{v-1}^2 + \beta A_{v-1}B_{v-1} + \gamma B_{v-1}^2 = (-1)^v Q_v/2 . \tag{14}$$

Similarly, eliminating Q_v from formulae (16) and (17) in § 20 of [7], we get

$$2\alpha A_{v-1}A_{v-2} + \beta(A_{v-1}B_{v-2} + B_{v-2}A_{v-2}) + 2\alpha B_{v-1}B_{v-2} = (-1)^{v-1}P_v . \tag{15}$$

Let $p = A_{v-1}$, $q = B_{v-1}$ $(v \geqslant 1)$. By (12)

$$a = (-1)^v Q_v/2 .$$

Hence, by formula (1) of § 6 of [7]

$$\begin{vmatrix} A_{v-1} & A_{v-2} \\ B_{v-1} & B_{v-2} \end{vmatrix} = (-1)^v.$$

and since

$$\begin{vmatrix} A_{v-1} & r \\ B_{v-1} & s \end{vmatrix} = 1$$

it follows that

$$T = \begin{pmatrix} A_{v-1} & A_{v-2} \\ B_{v-1} & B_{v-2} \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & (-1)^v \end{pmatrix}, \qquad t \in Z .$$

Thus we find using (14) and (15)

$$f = (\alpha, \beta, \gamma) \begin{pmatrix} A_{v-1} & A_{v-2} \\ B_{v-1} & B_{v-2} \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & (-1)^v \end{pmatrix} =$$
$$= \left((-1)^v \frac{Q_v}{2}, (-1)^{v-1} P_v, (-1)^v \frac{Q_{v-1}}{2} \right) \begin{pmatrix} 1 & t \\ 0 & (-1)^v \end{pmatrix} .$$

In order to make f satisfy (10) we must choose

$$t = (-1)^v \left[\frac{P_v}{Q_v} + \frac{1}{2} \right] .$$

Thus f is uniquely determined by ω_v and in view of (13), we have

$$\sum_{(a,b,c)\in C}'' \frac{1}{|a|} \leqslant \sum_{\substack{v=1 \\ Q_v < \frac{1}{2}\sqrt{d}}}^{[k,2]} 2(Q_v)^{-1} . \tag{16}$$

Since ω_v is reduced, we have further for v in question

$$\sqrt{d} \geqslant \frac{2\sqrt{d}}{Q_v} > \frac{P_v+\sqrt{d}}{Q_v} > \frac{\sqrt{d}}{Q_v} \geqslant 2 .$$

Hence for

$$b_v = [\omega_v] ,$$

we get the inequalities

$$\sqrt{d} > b_v \geqslant 2 , \qquad b_v + 1 > \sqrt{d}/Q_v ,$$

and by (16), lemma (3) follows.

Now, let ε_0 be the least totally positive unit $\varepsilon_0 > 1$ of the ring $Z(\sigma)$ where

$$\sigma = \begin{cases} \frac{1}{2}\sqrt{d} & \text{if } d \equiv 0 \pmod 4 , \\ \frac{1+\sqrt{d}}{2} & \text{if } d \equiv 1 \pmod 4 . \end{cases}$$

By Theorem (7) of Chapter IV of [6]

$$\varepsilon_0 = \frac{u+v\sqrt{d}}{2} ,$$

where for $l = [k, 2]$,

$$v = (q_{l-1}, p_{l-1} - q_{l-2}, p_{l-2}) , \qquad u = p_{l-1} + q_{l-2}$$

and p_j, q_j are the numerator and denominator, respectively, of the j-th convergent for ω^{-1}. Moreover, since ω^{-1} satisfies the equation

$$-\gamma\omega^2 - \beta\omega^{-1} - \alpha = 0 , \qquad (-\gamma > 0)$$

we find from formula (1) of § 2 of Chapter IV of [6] that

$$q_{l-2} - p_{l-1} = -\beta v , \qquad -p_{l-2} = -\alpha v .$$

Hence

$$\varepsilon_0 = \frac{p_{l-1}+q_{l-2}}{2} + \frac{p_{l-2}\sqrt{d}}{2\alpha} = q_{l-2} + \frac{\beta+\sqrt{d}}{2\alpha} p_{l-2} .$$

Since $p_j = B_{j+1}$, $q_j = A_{j+1}$, we get

$$(17)\qquad \varepsilon_0 = B_{l-1}\left(\frac{A_{l-1}}{B_{l-1}} + \frac{\beta+\sqrt{d}}{2\alpha}\right) \geqslant B_{l-1}\left(\omega + \frac{\beta+\sqrt{d}}{2\alpha}\right) = \frac{\sqrt{d}}{\alpha} B_{l-1}\,.$$

Now,

$$\omega_l = b_l + \omega_{l+1}^{-1} = b_l + \omega_1^{-1} = b_l + \omega\,, \qquad \omega_l' = b_l + \omega'$$

and since ω_l is reduced $0 > b_l + \omega' > -1$

$$b_l = [-\omega'] = \left[\frac{\beta+\sqrt{d}}{2a\alpha}\right] < \frac{\sqrt{d}}{\alpha}\,.$$

Thus (17) gives

$$\varepsilon_0 > b_l B_{l-1} > \prod_{v=1}^{l} b_v\,,$$

and by (16)

$$(18)\qquad \sum_{(a,b,c)\in C}{}'' \frac{1}{|a|} \leqslant \frac{2}{\sqrt{d}} \max \sum (x_i+1) = \frac{2}{\sqrt{d}} M$$

where maximum is taken over all non-decreasing sequence of at most l numbers satisfying

$$2 \leqslant x_i \leqslant \tfrac{1}{2}\sqrt{d} - 1 = D\,, \qquad \prod x_i < \varepsilon_0\,.$$

Let $(x_1, x_2, \ldots, x_m)$ be a point in which the maximum is taken with the least number m. We assert that the sequence contains at most one term x with $2 < x < D$. Indeed, if we had $2 < x_i < x_{i+1} < D$, we could replace the numbers x_i, x_{i+1} by

$$\frac{x_i}{\min(x_i/2,\, D/x_{i+1})}\,, \qquad x_{i+1}\min\left(\frac{x_i}{2}, \frac{D}{x_{i+1}}\right)$$

and the sum $\sum(x_i+1)$ would increase. Also, if we had $x_1 = x_2 = x_3 = 2$, we could replace them by $x_1 = 8$, and the sum $\sum(x_i+1)$ would remain the same while m would decrease.

Let

$$\frac{\varepsilon_0}{4} = D^e\theta\,, \qquad \text{where } e = \frac{\log(\varepsilon_0/4)}{\log D}\,.$$

Using $d > 676$, we get

$$M = \begin{cases} \tfrac{1}{2}e\sqrt{d} + \max(4\theta+1, 2\theta+4) & \text{if } 4\theta < D\,, \\ \tfrac{1}{2}e\sqrt{d} + 2\theta + 4 & \text{if } 2\theta < D \leqslant 4\theta\,, \\ \tfrac{1}{2}e\sqrt{d} + \theta + 7 & \text{if } D \leqslant 2\theta\,. \end{cases}$$

Now,

$$e = \frac{\log \varepsilon_0}{\log D} - \frac{\log 4\theta}{\log D}.$$

Since for $1 \leqslant x \leqslant y$, $y(\log x/\log y) \geqslant x - 1$, and for $d > 676$, $D/\log D \geqslant 12/\log 12 > 4.8$, we obtain if $4\theta < D$.

$$M - \frac{1}{2}\sqrt{d}\frac{\log \varepsilon_0}{\log D} = \max(4\theta + 1, 2\theta + 4) - D\frac{\log 4\theta}{\log D} = \frac{\log 4\theta}{\log D} <$$
$$< \max(4\theta + 1, 2\theta + 4) - \max(4\theta - 1, 6) \leqslant 2\,,$$

if $2\theta < D \leqslant 4\theta$

$$M - \frac{1}{2}\sqrt{d}\frac{\log \varepsilon_0}{\log D} = 2\theta + 4 - D\frac{\log 2\theta}{\log D} - D\frac{\log 2}{\log D} - \frac{\log 4\theta}{\log D} <$$
$$< 2\theta + 4 - 2\theta + 1 - 3 - 1 = 1\,,$$

if $D \leqslant 2\theta$

$$M - \frac{1}{2}\sqrt{d}\frac{\log \varepsilon_0}{\log D} = \theta + 7 - D\frac{\log \theta}{\log D} - D\frac{\log 4}{\log D} - \frac{\log 4\theta}{\log D} <$$
$$< \theta + 7 - \theta + 1 - 6 - 1 = 1\,.$$

This together with (18) gives the theorem.

4. – Proof of Corollary. We can assume $1 - \beta < (\log|d|)^{-2}$. It then by Theorem (1) that for every $\eta > 0$, there exists $c(\eta)$ such that if $d > c(\eta)$

$$1 - \beta \geqslant \frac{6}{\pi^2}\frac{L(1, \chi)}{\sum' 1/a}\left(1 - \frac{\eta}{2}\right). \tag{19}$$

Let h_0 be the number of classes of forms in question. For $d < -4$, we have

$$L(1, \chi) = \frac{\pi h_0}{\sqrt{|d|}}\,,$$

and by Theorem (2)

$$\sum{}' \frac{1}{a} \leqslant h_0\,.$$

Hence by (19)

$$1 - \beta \geqslant \frac{6}{\pi^2}\frac{h_0 \pi}{h_0\sqrt{|d|}}\left(1 - \frac{\eta}{2}\right) > \left(\frac{6}{\pi} - \eta\right)\frac{1}{\sqrt{|d|}}\,.$$

For $d > 0$, we have

$$L(1, \chi) = \frac{h_0 \log \varepsilon_0}{\sqrt{d}}.$$

Now, for any class C of forms

$$\sum_{(a,b,c)\in C}'' \frac{1}{|a|} = \sum_{\substack{(a,b,c)\in C \\ \frac{1}{4}\sqrt{d}\geqslant a\geqslant b>-a}} \frac{1}{a} + \sum_{\substack{(-a,b,-c)\in C \\ \sqrt{d}\geqslant -a\geqslant b>a}} \frac{1}{|a|}.$$

If (a, b, c) runs through C, $(-a, b, -c)$ runs through another class which we denote by $-C$ (It may happen that $-C = C$). If $C_1 \neq C_2$, then $-C_1 \neq -C_2$. Hence

$$\sum_C \sum_{\substack{(a,b,c)\in C \\ \frac{1}{4}\sqrt{d}\geqslant |a|\geqslant b>-|a}} \frac{1}{|a|} = 2 \sum' \frac{1}{a}$$

and by Theorem (2)

$$\sum' \frac{1}{a} \leqslant \frac{h_0}{2}\left(\frac{\log \varepsilon_0}{\log\left(\frac{1}{2}\sqrt{d}-1\right)} + \frac{4}{\sqrt{d}}\right) < \frac{h_0 \log \varepsilon_0}{\log d}\left(1 + O\left(\frac{1}{\sqrt{d}}\right)\right),$$

where the constant in the O-symbol is effective. (Note that $\varepsilon_0 > \frac{1}{2}\sqrt{d}$). This together with (19) gives the corollary.

REFERENCES

[1] H. Davenport, *Multiplicative Number Theory*, Chicago, 1967.
[2] D. Goldfeld, *An asymptotic formula relating the Siegel zero and the class number of quadratic fields*, Ann. Scuola Normale Sup., this volume.
[3] W. Hanecke, *Über die reellen Nullstellen der Dirichletschen L-Reihen*, Acta Arith., **22** (1973), pp. 391-421.
[4] A. E. Ingham, *The Distribution of Prime Numbers*, Cambridge, 1932.
[5] B. W. Jones, *The Arithmetic Theory of Quadratic Forms*, Baltimore, 1950.
[6] S. Lang, *Introduction to Diophantine Approximation*, Reading, 1966.
[7] O. Perron, *Die Lehre von den Kettenbrugchen*, Band I, Basel-Stuttgart, 1954.
[8] J. Pintz, *Elementary methods in the theory of L-functions II*, to appear in Acta Arithmetica.
[9] K. Prachar, *Primzahlverteilung*, Berlin-Göttingen-Heidelberg, 1957.
[10] E. Titchmarsh, *The zeta function, of Riemann*, Cambridge, 1930.