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A Relative Embedding Theorem for Stein Spaces (*).

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Introduction.

In this work we prove the following theorems:

THEOREM 1. Let X be a reduced Stein space, $n = \dim X < \infty$, Y a closed subspace of X and $\varphi: Y \to \boldsymbol{C}^l$ an embedding with $l \geqslant 2n + 1$.

Then the set of all maps $f: X \to \boldsymbol{C}^l$ which extend φ to X and are proper, one—one and regular at each regular point of X is dense in the space of all maps extending φ to X.

THEOREM 2. Let $(X, \mathcal{O}_X)$ be a Stein space of dimension n and locally of type N, possibly non reduced. Let $(Y, \mathcal{O}_Y)$ be a closed subspace of X and $\varphi: Y \to \boldsymbol{C}^l$ an embedding with $l \geqslant n + N$. Then the set of all maps $f: X \to \boldsymbol{C}^l$ which extend φ to X and are embeddings of X is dense in the space of all maps extending φ to X.

A similar result is obtained in the real case.

Theorems 1 and 2 were proved by R. Narasimhan in [2] and K. Wiegmann in [5] in the case $Y = \emptyset$.

1. – Convex open sets and admissible systems.

Let $(X, \mathcal{O}_X)$ be a Stein analytic complex space; if $\dim_{\boldsymbol{C}} X = n < \infty$ and X is locally of finite type N, by [2], [5] we can always embed $(X, \mathcal{O}_X)$ as a closed subspace of $\boldsymbol{C}^m$ for suitable m. Let us consider a fixed embedding

$$\vartheta: (X, \mathcal{O}_X) \hookrightarrow (\boldsymbol{C}^m, \mathcal{O}_m).$$

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In this way the Fréchet topology of $\Gamma(X, \mathcal{O}_X)$ can be considered as the quotient of the compact-open topology of $\Gamma(\boldsymbol{C}^m, \mathcal{O}_m)$.

DEFINITION 1. An open set $U \subset X$ is called X-convex if for each compact $K \subset U$, the set

$$\hat{K} = \{x \in U : |f(x)| \leqslant \sup_K |f| \text{ for all } f \in \Gamma(X, \mathcal{O}_X)\}$$

is compact.

REMARK 1. Let X_{red} be the reduced space associated with X; an obvious consequence of definition 1 is that an open set $U \subset X$ is X-convex if and only if U is X_{red}-convex.

REMARK 2. Let $\Omega \subset \boldsymbol{C}^m$ be a $\boldsymbol{C}^m$-convex open set (usually called a Runge-open set). Then $\vartheta^{-1}(\Omega) = \Omega \cap X$ is X-convex. In fact since $\Gamma(\boldsymbol{C}^m, \mathcal{O}_m) \to \Gamma(X, \mathcal{O}_X)$ is surjective, if $K \subset \Omega \cap X$ is a compact set, then

$$\hat{K} = \left(\{x \in \Omega : |f(x)| \leqslant \sup_K |f| \forall f \in \Gamma(\boldsymbol{C}^m, \mathcal{O}_m)\}\right) \cap X \quad \text{is compact}.$$

If X is a reduced space and $U \subset X$ is a X-convex open set it is known that each section $s \in \Gamma(U, \mathcal{O}_X)$ can be approximated by global sections (see for instance [1]). We give now a generalization of this:

THEOREM 1. *Let $\Omega \subset \boldsymbol{C}^m$ be a $\boldsymbol{C}^m$-convex open set and $U = \Omega \cap X$. Then the restriction map*

$$r : \Gamma(X, \mathcal{O}_X) \to \Gamma(U, \mathcal{O}_X)$$

has dense image.

PROOF. Let $g \in \Gamma(U, \mathcal{O}_X)$. Since U is a closed Stein subspace of the Stein manifold Ω, there exists $G \in \Gamma(\Omega, \mathcal{O}_m)$ such that $G|_U = g$.

Since Ω is $\boldsymbol{C}^m$-convex, G can be approximated by a sequence $\{F_n\}$ of holomorphic functions defined on $\boldsymbol{C}^m$. Let $\{f_n\} \subset \Gamma(X, \mathcal{O}_X)$ be the sequence induced by $\{F_n\}$. Since $\Gamma(U, \mathcal{O}_X)$ has the quotient topology of $\Gamma(\Omega, \mathcal{O}_m)$, the sequence $\{f_n|_U\}$ converges to g.

DEFINITION 2. A locally-finite family $\{U_i\}_{i \in I}$ of relatively compact open sets in X is called an admissible system, if it verifies the following properties:

(1) $U = \bigcup_{i \in I} U_i$ is X-convex;

(2) $U_i \cap U_j = \emptyset$ if $i \neq j$;

(3) is given a sequence $\{B_n\}$ of open sets such that:

$$B_n \subset\subset B_{n+1}, \quad \bigcup_n B_n = X, \quad B_n \cup U \text{ is } X\text{-convex for each } n.$$

Moreover we can suppose $U_i \subset B_n$ if $B_n \cap U_i \neq \emptyset$ (see [2]). In this case the sequence $\{B_n\}$ is called associated to $\{U_i\}$.

REMARK. If $\{\Omega_i\}$ is an admissible system for $\boldsymbol{C}^m$ and $\{A_n\}$ is an associated sequence then $\{\Omega_i \cap X\}$ and $\{A_n \cap X\}$ are respectively an admissible system for X and an associated sequence.

Lemmas 1 and 2, theorems 1 and 2 of [2] still hold for admissible systems obtained as in the remark.

Only theorem 1 of [2] needs some care. We have to show that it is possible to find $2n+1$ admissible systems $\{\Omega_i^\lambda\}$ $\lambda = 1, \ldots, 2n+1$ such that

$$X = \bigcup_{\lambda=1}^{2n+1} \Big(\bigcup_{i \in I} \Omega_i^\lambda \cap X \Big).$$

Following the proof of Theorem 1 of [2] it is enough to prove that there is an admissible system $\{\Omega_i\}$ in $\boldsymbol{C}^m$ such that $A = X - \bigcup_i (\Omega_i \cap X)$ is a real analytic set which doesn't contain any point in a chosen countable subset of X.

Consider a family of open cubes (1) $\{Q_h\}$ in $\boldsymbol{C}^m$, with sides parallel to the real axes, such that if $\bar{Q}_h \cap \bar{Q}_k \neq \emptyset$ then $\bar{Q}_h \cap \bar{Q}_k$ is exactly a common face of some dimension. Then $\boldsymbol{C}^m - \bigcup_i \Omega_i$ is the union of a countable family of real hyperplanes. We can clearly choose them so that their equations are satisfied by none of the points of the chosen set.

Now let $(Y, \mathcal{O}_Y)$ be a closed subspace of $(X, \mathcal{O}_X)$ and let $\varphi\colon Y \to \boldsymbol{C}^l$ be a fixed embedding. We have $\varphi = (\varphi_1, \ldots, \varphi_l) \in \Gamma(Y, \mathcal{O}_Y)^l$ and we suppose $l \geqslant n + N$.

We define $\Gamma(X)_\varphi \subset \Gamma(X, \mathcal{O}_X)^l$ to be the set of the maps: $X \to \boldsymbol{C}^l$ extending φ. Clearly, since Y is a Stein subspace, $\Gamma(X)_\varphi$ is a closed non void subset of $\Gamma(X, \mathcal{O}_X)^l$.

(1) For a cube in $\boldsymbol{C}^m$ we mean a subset of $\boldsymbol{C}^m$ defined by

$$\{x \in \boldsymbol{C}^m \colon a_i < |\operatorname{Re} z_i| < b_i,\ c_i < |\operatorname{Im} z_i| < d_i\},$$

where $z_1, \ldots, z_m$ are coordinates in $\boldsymbol{C}^m$.

THEOREM 2. *Let $\Omega \subset \boldsymbol{C}^m$ be a $\boldsymbol{C}^m$-convex open set and $U = \Omega \cap X$. Let $F \in \Gamma(U, \mathcal{O}_X)$ be such that*

$$F|_{U \cap Y} = g|_{U \cap Y}$$

where $g \in \Gamma(Y, \mathcal{O}_Y)$.

For each neighborhood A of $F|_{U \cap Y}$ in $\Gamma(U, \mathcal{O}_X)$, there exists a section $h \in \Gamma(X, \mathcal{O}_X)$ such that $h|_Y = g$ and $h|_U \in A$.

PROOF. Since Y is a Stein subspace of X, there is a section $G \in \Gamma(X, \mathcal{O}_X)$ such that $G|_Y = g$: therefore we need only to prove that it is possible to approximate the section $f = F - G \in \Gamma(U, \mathcal{O}_X)$, which vanishes on $U \cap Y$, with a section in $\Gamma(X, \mathcal{J}_Y)$, where $\mathcal{J}_Y$ denotes the ideal sheaf defining Y.

Let us firstly suppose U to be relatively compact in X.

Because of theorem A, for each $y \in Y$ the stalk $\mathcal{J}_{Y,y}$ is generated by a finite number of sections in $\Gamma(X, \mathcal{J}_Y)$. Since U is relatively compact we can choose $t_1, \ldots, t_q \in \Gamma(X, \mathcal{J}_Y)$ generating $\mathcal{J}_{Y,y}$ for each $y \in U$. These sections define a surjective homomorphism of sheaves:

$$\mathcal{O}_X^q|_U \xrightarrow{t_1, \ldots, t_q} \mathcal{J}_Y|_U .$$

Because of theorem B the restriction homomorphism between $\Gamma(U, \mathcal{O}_X)^q$ and $\Gamma(U, \mathcal{J}_Y)^q$ is again surjective. In particular we get:

$$f = \sum_{i=1}^{q} \alpha_i t_i$$

where $\alpha_i \in \Gamma(U, \mathcal{O}_X|_U)$. For each open neighborhood A of $f|_U$, we can obviously find neighborhoods A_i of α_i in such a way that if $\beta_i \in A_i$ then $\sum \beta_i t_i \in A$. Because of theorem 1 for each i there is $\gamma_i \in \Gamma(X, \mathcal{O}_X)$ such that $\gamma_i|_U \in A_i$; therefore $\tilde{f} = \sum_{i=1}^{q} \gamma_i t_i$ is such that $\tilde{f}|_U \in A$.

If U is not relatively compact, consider any neighborhood A of $f|_U$. Since $\Gamma(U, \mathcal{O}_X)$ has the quotient topology of the compact-open topology on $\Gamma(U, \mathcal{O}_m)$ for each extension $h \in \Gamma(\Omega, \mathcal{O}_m)$ of $f|_U$ there is a compact $K \subset \Omega$ and a constant $\varepsilon > 0$ such that if $s \in \Gamma(\boldsymbol{C}^m, \mathcal{O}_m)$ and $\|s - h\|_k < \varepsilon$ then $s|_U \in A$.

Let $\Omega' \subset\subset \Omega$ be a $\boldsymbol{C}^m$-convex open neighborhood of K and let $t_1, \ldots, t_q \in \Gamma(\boldsymbol{C}^m, \mathcal{J}_Y)$ be global generators of $\mathcal{J}_{Y,y}$ for each $y \in \Omega'$. Then $h|_{\Omega'} = \sum_{i=1}^{q} \alpha_i t_i$. Since Ω' is convex we can approximate α_i on K with $\beta_i \in \Gamma(\boldsymbol{C}^m, \mathcal{O}_m)$, in such a way that $\|\sum \beta_i t_i - \sum \alpha_i t_i\|_k < \varepsilon$. Define $\tilde{f} = \left(\sum_{i=n}^{q} \beta_i t_i\right)\Big|_X$ and, by construction, $\tilde{f}|_U \in A$.

REMARK. If $(X, \mathcal{O}_X)$ is a reduced space, theorem 2 holds for any X-convex open set U. In fact we can apply the proof of theorem 2 without considering Ω and Ω'. Since in the reduced case the topology of $\Gamma(X, \mathcal{O}_X)$ is again the compact-open topology, it is sufficient to approximate f in the topology of $\Gamma(V, \mathcal{O}_X)$ where V is any relatively compact X-convex open set containing the compact set K which defines the neighborhood A.

THEOREM 3. *Let $\{\Omega_i\}$ be an admissible system for $\boldsymbol{C}^m$ and $U_i = \Omega_i \cap X$. Let g be in $\Gamma(X, \mathcal{O}_X)$ and $f_i \in \Gamma(U_i, \mathcal{O}_X)$ be such that*

$$f_i|_{U \cap Y} = g|_{U \cap Y}.$$

For any choice of open neighborhoods A_i of f_i in $\Gamma(U_i, \mathcal{O}_X)$ for each i, there is a section $f \in \Gamma(X, \mathcal{O}_X)$ such that:

1) $f|_Y = g|_Y$
2) $f|_{U_i} \in A_i$ *for each* i.

PROOF. Suppose firstly that $(X, \mathcal{O}_X)$ is reduced (2) and the neighborhoods A_i are given by compact sets K_i and constants ε_i. Let $\{B_n\}_{n \in N}$ be associated to the admissible system $\{U_i\}$. Let $U_i \subset B_1$ for $i \leqslant i_1$, $U_i \subset B_n - B_{n-1}$ for $i_{n-1} < i \leqslant i_n$.

Let $K_n' \subset B_n$ be a compact set such that $B_{n-1} \subset K_n'$ and $K_i \subset K_n'$ for $i \leqslant i_n$. Choose δ_n such that $\sum_{\mu=n+1}^{\infty} \delta_\mu < \frac{1}{2} \min_{i \leqslant i_n} \varepsilon_i$. Because of theorem 2, since $U = \bigcup_{i \in I} U_i$ is X-convex, there is $F_1 \in \Gamma(X, \mathcal{O}_X)$ such that $F_1|_Y = g|_Y$ and $\|F_1 - f_i\|_{K_i} < \frac{1}{2}\varepsilon_i$ for $i \leqslant i_1$. Since $B_1 \cup U$ is X-convex there is $F_2 \in \Gamma(X, \mathcal{O}_X)$ such that $F_2|_Y = g_Y$, $\|F_2 - F_1\|_{K_1} < \delta_1$ and $\|F_2 - f_i\|_{K_i} < \frac{1}{2}\varepsilon_i$ for $i_1 < i \leqslant i_2$ and so on.

Since the constructed sequence converges uniformely on compact sets, its limit $f = \lim_{n \to \infty} F_n$ is in $\Gamma(X, \mathcal{O}_X)$. It is clear that $\|f - f_i\|_{K_i} < \varepsilon_i$ and by construction $f|_Y = g|_Y$.

If X is not reduced let $\tilde{g} \in \Gamma(\boldsymbol{C}^m, \mathcal{O}_m)$ be an extension of g and $\tilde{f}_i \in \Gamma(\Omega_i, \mathcal{O}_m)$ be extensions of f_i. Fix compact sets $K_i \subset \Omega_i$ and constants ε_i in such a way that by the open map

$$r_i \colon \Gamma(\Omega_i, \mathcal{O}_m) \to \Gamma(U_i, \mathcal{O}_X)$$

we have $r_i(\{h \in \Gamma(\Omega_i, \mathcal{O}_m) \mid \|h - \tilde{f}_i\|_{K_i} < \varepsilon_i\}) \subset A_i$.

(2) In the reduced case our proof is the proof of theorem 2 of [2].

Applying the previous proof to $(\boldsymbol{C}^m, \mathcal{O}_m)$, $\{\Omega_i\}$, $\tilde{g}$ and $\tilde{f}_i$, we can find $\tilde{f} \in \Gamma(\boldsymbol{C}^m, \mathcal{O}_m)$ such that $\tilde{f}|_Y = g|_Y$ and $\|\tilde{f} - \tilde{f}_i\|_{K_i} < \varepsilon_i$ for each i. Defining $f = \tilde{f}|_X$ we complete the proof.

LEMMA 1. *Let N be a fixed integer and $f \in \Gamma(Y, \mathcal{O}_Y)^N$. For each compact set $K \subset X$ let $A(K) \subset \Gamma(X, \mathcal{O}_X)^N$ be a family with the following properties:*

(1) *$A(K)$ is dense in $\Gamma(X, \mathcal{O}_X)^N$.*

(2) *$A(K) \cap \Gamma(X)_f$ is dense in $\Gamma(X)_f$* (3).

(3) *If $K \subset \mathring{K}'$ then $A(K) \supset A(K')$.*

(4) *If K' is a compact neighborhood of K in $\boldsymbol{C}^m$ and $\tilde{f} \in \Gamma(\boldsymbol{C}^m, \mathcal{O}_m)$ is such that $f = \tilde{f}|_X \in A(K)$, there is an $\varepsilon > 0$ such that if $\|\tilde{g} - \tilde{f}\|_{K'} < \varepsilon$ then $g = \tilde{g}|_X \in A(K)$. This implies that $A(K)$ is open in $\Gamma(X, \mathcal{O}_X)^N$.*

Let $\{\Omega_i^\lambda\}$, $\lambda = 1, \ldots, N$, be admissible systems in $\boldsymbol{C}^m$, $U_i^\lambda = \Omega_i^\lambda \cap X$; fix $g \in \Gamma(X)_f$; for any choice of an open neighborhood A of g in $\Gamma(X, \mathcal{O}_X)^N$ and open neighborhoods A_i^λ of $g_\lambda|_{U_i^\lambda}$ in $\Gamma(U_i^\lambda, \mathcal{O}_X)$, there is $\varphi \in \Gamma(X)_f$ such that $\varphi \in A$, $\varphi_\lambda|_{U_i^\lambda} \in A_i^\lambda$ for $\lambda = 1, \ldots, N$ and $\varphi \in A(K)$ for each compact set $K \subset X$.

PROOF. Suppose firstly $(X, \mathcal{O}_X)$ is reduced (4). Let the open neighborhoods A, A_i^λ be given by compact sets $C \subset X$, $K_i^\lambda \subset U_i^\lambda$ and constants ε, $\varepsilon_i > 0$.

Consider a sequence $\{B_n^\lambda\}_{n \in N}$ associated to the admissible system $\{U_i^\lambda\}$, $\lambda = 1, \ldots, N$. Let $'K_n^\lambda$ be a compact neighborhood of B_{n-1}^λ in B_n^λ such that if $K_i^\lambda \subset B_n^\lambda$ then $K_i^\lambda \subset {}'K_n^\lambda$. Let C_n^λ be a compact neighborhood of $'K_n^\lambda$ in B_n^λ. Define $K_n' = \bigcap_{\lambda=1}^N {}'K_n^\lambda$, $C_n = \bigcap_{\lambda=1}^N C_n^\lambda$. Suppose $U_i^\lambda \subset B_n^\lambda - B_{n-1}^\lambda$ for $i_{n-1}^\lambda < i \leqslant i_n^\lambda$.

Clearly $K_n' \subset (\mathring{K}_{n+1}')$, $\bigcup_n K_n' = X$. If $f \in A(K_n')$ for all n then, by property (3), $f \in A(K)$ for all compact subsets K of X.

We may suppose $C \subset K_1'$ and $\varepsilon < \varepsilon_i$ for $i \leqslant \max_\lambda i_1^\lambda$.

$A(K_1')$ is dense; therefore we can find $f^1 = (f_1^1, \ldots, f_N^1) \in A(K_1') \cap \Gamma(X)_f$ such that $\|f^1 - g\|_{K_i'} < \frac{1}{2}\varepsilon$ and $\|f_\lambda^1 - g_\lambda\|_{K_i^\lambda} < \frac{1}{2}\varepsilon < \frac{1}{2}\varepsilon_i$ for $i \leqslant i_1^\lambda$. By property (4) there is a δ_1 such that if $\|F - f^1\|_{C_1} < \delta_1$ then $F \in A(K_1')$. Since $B_1^\lambda \cup U^\lambda$ is X-convex (here $U^\lambda = \bigcup U_i^\lambda$) there is $'f^2 \in \Gamma(X)_f$ such that $\|'f_\lambda^2 - f_\lambda^1\|_{C_1^\lambda} < \frac{1}{4}\delta_1$ (therefore $'f^2 \in A(K_1')$) and $\|'f_\lambda^2 - g_\lambda\|_{K_i^\lambda} < \frac{1}{4}\varepsilon_i$ for $i_1^\lambda < i \leqslant i_2^\lambda$. Since $A(K_2')$ is dense there is $f^2 = (f_1^2, \ldots, f_N^2) \in \Gamma(X)_f \cap A(K_2')$ such that $\|f^2 - f^1\|_{C_1} < \frac{1}{2}\delta_1$, $\|f_\lambda^2 - g_\lambda\|_{K_i^\lambda} < \frac{1}{2}\varepsilon_i$ for $i \leqslant i_2^\lambda$. There is δ_2, with $0 < \delta_2 < \frac{1}{2}\delta_1$, such that if

(3) We recall that $\Gamma(X)_f$ is the set of all extensions of f to X.

(4) The proof of the reduced case is, with slight differences, the proof of lemma 2 of [2].

$\|F-f^2\|_{C_2}<\delta_2$ then $F\in A(K_2')$. Iterating this process we get a sequence $\{f^n\}$ with the following properties: $f^n\in A(K_n')\cap\Gamma(X)_f$ for all n; $\|f^n-f^{n-1}\|_{C_{n-1}}<\frac{1}{2}\delta_{n-1}$; $\|f_\lambda^n-g_\lambda\|_{K_i^\lambda}<\frac{1}{2}\varepsilon_i$ for $i\leqslant i_n^\lambda$. Let us define $\varphi=\lim_{n\to\infty}f^n$; then $\varphi\in\Gamma(X)_f$. Since $\delta_n<\frac{1}{2}\delta_{n-1}$ we have $\sum_{\mu=n+1}^{\infty}\delta_\mu<\delta_n$; so we obtain $\|\varphi-f^n\|_{C_n}<\delta_n$ and therefore $\varphi\in A(K_n')$ for all n, i.e. $\varphi\in A(K)$ for every K. Moreover if δ_n is chosen sufficiently small, we have $\|\varphi-g\|_C<\varepsilon$ and $\|g_\lambda-\varphi_\lambda\|_{K_i^\lambda}<\varepsilon_i$.

If X is not reduced, consider the restriction map $r\colon\Gamma(\boldsymbol{C}^m,\mathcal{O}_m)^N\to\Gamma(X,\mathcal{O}_X)^N$. Since r is open $r^{-1}(A(K))$ is dense in $\Gamma(\boldsymbol{C}^m,\mathcal{O}_m)$ and $r^{-1}(A(K))\cap\Gamma(\boldsymbol{C}^m)_f$ is dense in $\Gamma(\boldsymbol{C}^m)_f$. If $K\not\subset X$, K compact subset of $\boldsymbol{C}^m$, let us define $A'(K)=r^{-1}(A(K\cap X))$. It is easy to verify properties (1), ..., (4) for the family $\{A'(K)\}$. If $g\in\Gamma(X)_f$, fix any extension $\tilde{g}\in\Gamma(\boldsymbol{C}^m,\mathcal{O}_m)^N$ of g. Applying lemma 1 for $\tilde{g}$ to the reduced space $\boldsymbol{C}^m$ we prove lemma 1 in the general case.

LEMMA 2. *Let $\varphi\in\Gamma(Y,\mathcal{O}_Y)^l$ be an embedding and $\phi\in\Gamma(X)_\varphi$. There is an open neighborhood V of Y in X such that $\phi|_V$ is a proper map.*

PROOF. Let $\{P_n\}$ be a sequence of concentric polidiscs invading $\boldsymbol{C}^m$. Define $K_n=P_n\cap X$ and $H_n=K_n\cap Y$. Choose a subsequence $\{H_{n_i}\}$ such that $|\varphi_\lambda(x)|\geqslant i$ for each $x\in\bar{H}_{n_i}-H_{n_{i-1}}$. Each compact $\bar{H}_{n_i}-H_{n_{i-1}}$ has an open neighborhood U_i in X such that $\inf_{U_i}|\phi_\lambda(x)|\geqslant i-\frac{1}{2}$. Take $V_i=U_i\cap K_{n_{i+1}}$; V_i is a relatively compact open neighborhood of $\bar{H}_{n_i}-H_{n_{i-1}}$ and clearly $\inf_{V_i}|\phi_\lambda(x)|\geqslant i-\frac{1}{2}$. Therefore $V=\bigcup V_i$ has the required property.

For an extension ϕ of an embedding $\varphi\in\Gamma(Y,\mathcal{O}_Y)^l$ we give the following

DEFINITION 3. An admissible system $\{U_i\}$ is called relative to ϕ if there is an open neighborhood V of Y in X such that:

(1) $\phi|_V$ is a proper map;

(2) If $U_h\cap Y=\emptyset$ then $U_h\subset V$.

LEMMA 3. *Let $\phi\in\Gamma(X)_\varphi$; there is an admissible system $\{\Omega_i\}$ in $\boldsymbol{C}^m$ such that $\{U_i=\Omega_i\cap X\}$ is relative to ϕ.*

PROOF. Let V be an open neighborhood of Y in X such that $\varphi|_V$ is a proper map. We want to construct a countable family of open cubes $\{Q_h\}$ in $\boldsymbol{C}^m$, as done after the remark to definition 2, with the following properties:

(1) $\boldsymbol{C}^m=\bigcup_{h=0}^{\infty}\bar{Q}_h$.

(2) $Q_h\cap Q_{h'}=\emptyset$ if $h\neq h'$.

(3) If $\bar{Q}_h \cap \bar{Q}_{h'} = F \neq \emptyset$ then F is exactly a face of one of them.

(4) There is a sequence $\{P_n\}$ of open cubes such that: $C^m = \bigcup_{n=0}^{\infty} P_n$; $P_n \subset\subset P_{n+1}$; $\bar{P}_0 = \bigcup_{h=0}^{q_0} \bar{Q}_h$; ...; $\bar{P}_{k+1} = \bigcup_{h=q_k+1}^{q_{k+1}} Q_h$; ....

For this purpose let us consider a sequence of open cubes $\{P_n\}$ invading C^m. If $K_n = P_n \cap X$ and $H_n = K_n \cap Y$, take a subsequence $\{H_{n_i}\}$ as in lemma 2. Divide P_{n_0} into cubes Q_h, $h = 0, \ldots, q_0$ so small that if $Q_h \cap H_{n_0} \neq \phi$ then $Q_h \subset V$. Divide $P_{n_1} - \bar{P}_{n_0}$ in small cubes Q_h, $h = q_0 + 1, \ldots, q_1$, in such a way that property (3) holds and if $Q_h \cap H_{n_i} \neq \phi$ then $Q_h \subset V$; and so on.

Since $\{Q_h\}$ is an admissible system, $\{Q_h \cap X\}$ is an admissible system and by construction it is relative to ϕ. Moreover it is easy to show, with the same argument used before, that X can be covered by $2n+1$ admissible systems obtained in this way and relative to ϕ.

2. – The first relative embedding theorem.

We want to prove the relative case of theorem 5 of [2]; in other words we try to find in $\Gamma(X)_\varphi$ all the maps of X into C^l which are one-one, proper and regular in each regular point of X.

Since a non reduced space might have no regular points, from now on X will be supposed a reduced space.

We want to prove the following theorem:

THEOREM 4. *Let X be a reduced Stein space, Y a closed subspace of X and φ: $Y \to \boldsymbol{C}^l$ an embedding. Suppose $l \geqslant 2n+1$, where $n = \dim_{\boldsymbol{C}} X$.*

Then the set of all maps which are proper, one-one and regular in each regular point of X is dense in $\Gamma(X)_\varphi$.

If $l < n+1$ the same result is obtained by adding to $(\varphi_1, \ldots, \varphi_l)$ $n+1-l$ arbitrary components.

In order to prove this theorem let us give some definitions, following [2]. Let S be the set of singular points of X. Then, if f is any section in $\Gamma(X, \mathcal{O}_X)^k$, we define:

$$X(f, m) = \{x \in X - S \mid \text{rank of } f \text{ in } x \leqslant m\} \text{ for } m \leqslant n.$$

$M(f) =$ union of all the irreducible components different from the diagonal of $\{(x, y) \in X \times X \mid f(x) = f(y)\}$.

Here, if x is a regular point of X, for rank of f in x we mean the rank of the jacobian matrix of f.

If $f \in \Gamma(X)_\varphi$ then, for each $x \in Y - S$, rank of f in x is greater or equal to the dimension of the Zarisky tangent space to Y in x; in particular the equality holds if this dimension is equal to $\dim_x = X$. Moreover, since $f|_Y = \varphi$ is one-one, $M(f) \cap Y \times Y = \emptyset$.

We now need the following lemmas.

LEMMA 4. *Let $x_0, \ldots, x_p \in X$, $x_1, \ldots, x_p \notin Y$. Then the set of sections $f \in \Gamma(X, \mathfrak{J}_Y)$ separating $x_0, \ldots, x_p$ is dense in $\Gamma(X, \mathfrak{J}_Y)$.*

(Here $\mathfrak{J}_Y$ is the ideal sheaf of $\mathcal{O}_X$ which defines Y).

PROOF. Consider the exact sequence of coherent sheaves:

$$0 \to \mathfrak{J} \to \mathcal{O}_X \to \mathcal{O}_X/\mathfrak{J} \to 0$$

where $\mathfrak{J}$ is the sheaf of germs of holomorphic functions vanishing on Y and in $x_0, \ldots, x_p$. Since $\mathfrak{J}$ is coherent, because of theorem B, the map:

$$\Gamma(X, \mathcal{O}_X) \to \Gamma(X, \mathcal{O}_X/\mathfrak{J})$$

is surjective. But $\Gamma(X, \mathcal{O}_X/\mathfrak{J}) = \Gamma(Y, \mathcal{O}_Y) \underbrace{\oplus \boldsymbol{C} \oplus \ldots \oplus \boldsymbol{C}}_{k \text{ times}}$, where $k = p$ if $x_0 \in Y$, $k = p+1$ if $x_0 \notin Y$. Then there is a function f such that $f(x_i) = i$, $i = 0, \ldots, p$. If h is any section in $\Gamma(X, \mathfrak{J}_Y)$, then $h + \varepsilon f$ does not separate $x_0, \ldots, x_p$ only if ε takes a finite number of values. If ε is less than all these values then $h + \varepsilon f$ separates $x_0, \ldots, x_p$ and approximates h.

LEMMA 5. *Let $x_1, \ldots, x_p \in X - S$ be such that if $x_i \in Y$ then the dimension of the Zarisky tangent space $\mathcal{T}(Y)_{x_i}$ to Y in x_i is less than $n = \dim_{\boldsymbol{C}} X$. Let $f_1, \ldots, f_k$ be holomorphic functions on X such that $f_i|_Y = \varphi_i$ for each $i \leqslant l$. Then the set of $f \in \Gamma(X, \mathcal{O}_X)$ such that:*

(1) $f|_Y = \varphi_{k+1}$ (if $k+1 \leqslant l$).

(2) *rank of $(f_1, \ldots, f_k, f)$ in $x_i >$ rank of $(f_1, \ldots, f_k)$ in x_i,*

is dense in $\Gamma(X)_{\varphi_{k+1}}$.

PROOF. For each $x_i \notin Y$ we choose n sections $z_1^i, \ldots, z_n^i \in \Gamma(X, \mathfrak{J}_Y)$ giving a system of local coordinates in a neighborhood of x_i. If $x_i \in Y$ we choose $z_1^i, \ldots, z_q^i \in \Gamma(X, \mathfrak{J}_Y)$ giving a minimal system of equations for $\mathcal{T}(Y)_{x_i}$ in a neighborhood of x_i. Then if ξ is a suitable linear combination of all

these sections

$$\operatorname{rank}(f_1, \ldots, f_k, \xi) \text{ in } x_i > \operatorname{rank}(f_1, \ldots, f_k) \text{ in } x_i .$$

(This is not true only for a finite number of values of the coefficients of ξ). If h is any section in $\Gamma(X)_{\varphi_{k+1}}$, consider $h + \varepsilon\xi$. Then

$$\operatorname{rank}(f_1, \ldots, f_k, h + \varepsilon\xi) \text{ in } x_i = \operatorname{rank}(f_1, \ldots, f_k) \text{ in } x_i$$

only for a finite number of values. Taking ε less than their minimum we get the thesis.

LEMMA 6. *Let $(f_1, \ldots, f_k) \in \Gamma(X, \mathcal{O}_X)^k$, $k \geqslant 0$, be a holomorphic map verifying the following conditions:*

1) $f_i|_Y = \varphi_i$ *for each* $i \leqslant l$.

a_k) $\dim_C M(f_1, \ldots, f_k) \leqslant 2n - k$.

b_k) $\dim_C X(f_1, \ldots, f_k, m) \leqslant n - k + m$ *for each* $0 \leqslant m < n$.

Let $h \in \Gamma(X)_{\varphi_{k+1}}$ be a holomorphic function. For any choice of an admissible system $\{U_i\}$ relative to h, compact sets $K_i \subset U_i$, a compact set $C \subset X$ and constants ε_i, $\varepsilon > 0$ there is a function $f \in \Gamma(X, \mathcal{O}_X)$ such that $f|_Y = h|_Y$, $\|f - h\|_{K_i} < \varepsilon_i$, $\|f - h\|_C < \varepsilon$ and moreover the map $(f_1, \ldots, f_k, f)$ verifies a_{k+1} and b_{k+1}.

PROOF. We can suppose that $\varphi_1, \ldots, \varphi_l$ are ordered in such a way that $(\varphi_1, \ldots, \varphi_k)$ verifies a_k and b_k relatively to Y for each $k \leqslant l$. Obviously in this case, if $m =$ rank of $(\varphi_1, \ldots, \varphi_k)$ in $y \in Y$, then $0 \leqslant m \leqslant \dim \mathcal{T}(Y)_y$. If $y \in Y$ is a regular point of X, then $\dim \mathcal{T}(Y)_y \leqslant n$ and so $0 \leqslant m \leqslant n$.

Let M_q be one of the irreducible components of $M(f_1, \ldots, f_k)$ whose dimension is $2n - k$. It cannot be $M_q \subset Y \times Y$, since in this case we should have

$$\dim M_q \leqslant 2 \dim Y - k < 2n - k .$$

Therefore we can choose a point $(x_q, y_q) \in M - \Delta - Y \times Y$.

Let X_m be the union of those irreducible components X_m^p of $X(f_1, \ldots, f_k, m)$ whose dimension is $n - k + m$. We choose a point $x_m^p \in X_m^p$ in such a way that $x_m^p \notin S \cup Y$. This is possible since if $X_m^p - S \subset Y$ then

$$\dim X_m^p \leqslant \dim Y + m - k < n + m - k .$$

Both the set $\{(x_q, y_q)\} \subset X \times X$ and the set $\{x_m^p\} \subset X$ are discrete.

For each compact set $K \subset X$ we define $A(K)$ as follows:

$$g \in A(K) \text{ if and only if } \begin{cases} 1)\ g(x_q) \neq g(y_q) & \text{for each } (x_q, y_q) \in K \times K \\ 2)\ \text{rank of } (f_1, \ldots, f_k, g) \text{ in } x_m^p = m+1 & \\ & \text{for each } x_m^p \in K. \end{cases}$$

By the results of [2], $A(K)$ is dense in $\Gamma(X, \mathcal{O}_X)$; it is dense in $\Gamma(X)_{\varphi_{k+1}}$ by lemmas 1 and 2; all the other hypothesis of lemma 1 are satisfied.

Therefore we can find $g \in \Gamma(X)_{\varphi_{k+1}}$ such that $\|g-h\|_{K_i} < \varepsilon_i$, $\|g-h\|_C < \varepsilon$ and $g \in A(K)$ for each compact set $K \subset X$. Therefore $g(x_q) \neq g(y_q)$ for any q and rank$(f_1, \ldots, f_k, g)$ in x_m^p is equal to $m+1$ for any p and m. Then we get:

1) $\dim M(f_1, \ldots, f_k, g) < \dim M(f_1, \ldots, f_k)$ since every component of $M(f_1, \ldots, f_k, g)$ different from Δ must be contained in some component of $M(f_1, \ldots, f_k)$ and cannot be one of those of greatest dimension.

2) $\dim X(f_1, \ldots, f_k, g, m) < \dim X(f_1, \ldots, f_k, m)$ since every component of the first one is contained in some component of the second one and cannot be one X_m^p because $(f_1, \ldots, f_k, g)$ has rank $m+1$ in $x_m^p \in X_m^p$.

Then $(f_1, \ldots, f_k, g)$ satisfies a_{k+1} and b_{k+1} and the lemma is proved.

Proof of Theorem 4. Let us fix an element $\phi \in \Gamma(X)_\varphi$, $\phi = (\phi_1, \ldots, \phi_l)$. Suppose $l \geqslant 2n+1$ (if not it is sufficient to add arbitrarily to $(\phi_1, \ldots, \phi_l)$ $2n+1-l$ components). Choose l admissible systems $\{U_i^\lambda\}$ $\lambda = 1, \ldots, l$ relative to ϕ. Choose compact sets $K_i^\lambda \subset U_i^\lambda$ in such a way that $X = \bigcup_{i,\lambda} K_i^\lambda$. Because of theorem 3 there is $h = (h_1, \ldots, h_l) \in \Gamma(X)_\varphi$ such that:

1) $\|h_\lambda - \varphi_\lambda\|_C < \frac{1}{2}\varepsilon$.

2) If $K_i^\lambda \cap Y \neq \emptyset$ then $\|h_\lambda - \phi_\lambda\|_{K_i^\lambda} < \frac{1}{2}$.

3) If $K_i^\lambda \cap C = \emptyset$, $K_i^\lambda \cap Y = \emptyset$ then $\|h_\lambda\|_{K_i^\lambda} \geqslant i+1$.

Since conditions a_0 and b_0 are void, a function $f_1 \in \Gamma(X, \mathcal{O}_X)_{\varphi_1}$ exists such that $\dim M(f_1) \leqslant 2n-1$, $\dim X(f_1, m) \leqslant n-1+m$ for $0 \leqslant m < n$. We can successively choose functions $f_2 \in \Gamma(X)_{\varphi_2}, \ldots, f_l \in \Gamma(X)_{\varphi_l}$ in such a way that: $\dim M(f_1, \ldots, f_k) \leqslant 2n-k$; $\dim X(f_1, \ldots, f_k, m) \leqslant n-k+m$ for $0 \leqslant m < n$; $\|f_\lambda - h_\lambda\|_{K_i^\lambda} < \frac{1}{2}$; $\|f_\lambda - h_\lambda\|_C < \frac{1}{2}\varepsilon$.

Therefore the map $F = (f_1, \ldots, f_l)$ verifies the following properties:

1) $\|\phi - F\|_C < \varepsilon$;

2) Since $\dim M(F) \leqslant 2n-1 \leqslant -1$ and $\dim X(F, m) \leqslant n-1+m \leqslant -1$, the map F is one-one and regular on the regular points of X;

3) It is a proper map: in fact if $\{x_n\} \subset X$ is a divergent sequence, then either $\{x_n\}$ is definitively in $\bigcup_{K_i^\lambda \cap Y \neq \phi} K_i^\lambda$ or it is definitively outside; in the first case $\{F(x_n)\}$ diverges, since $\{\phi(x_n)\}$ does so and $\|F_\lambda - \phi_\lambda\|_{K_i^\lambda} < 1$, in the second case $\{F(x_n)\}$ diverges because $\|F_\lambda\|_{K_i^\lambda} > i$.

3. – The second relative embedding theorem.

From now on the Stein space $(X, \mathcal{O}_X)$ will not be supposed reduced. We want to show the following theorem:

THEOREM 5. *Let $(X, \mathcal{O}_X)$ be a Stein space of dimension n and locally of type N. Let $(Y, \mathcal{O}_Y)$ be a closed subspace and $\varphi = (\varphi_1, \ldots, \varphi_l) \in \Gamma(Y, \mathcal{O}_Y)^l$ a fixed embedding; we may suppose $l \geqslant n + N$* (5). *Then the set of all maps of X into $\boldsymbol{C}^l$ which are embeddings is dense in the space $\Gamma(X)_\varphi = \{f \in \Gamma(X, \mathcal{O}_X)^l \mid f|_Y = \varphi\}$.*

Let us recall the following

DEFINITION 4. A map $f: X \to \boldsymbol{C}^l$ is called regular at the point $x \in X$ if f_x can be extended to a map $\tilde{f}_x$ defined in an open neighborhood of x in the Zarisky tangent space $\mathcal{T}(X)_x$ such that the jacobian matrix of $\tilde{f}_x$, which is a morphism between two complex manifolds, has rank at x equal to $\dim \mathcal{T}(X)_x$.

In this case we shall say that f has maximal rank at x.

Now for each compact set $K \subset X$ let us define $A(K) \subset \Gamma(X, \mathcal{O}_X)^l$ to be the set of all maps which are one-one and regular on K. The following lemmas show that the family $A(K)$ verifies the hypothesis of lemma 1.

LEMMA 7. *Let K be a compact subset of X and suppose $f \in A(K)$. Let $\tilde{f} \in \Gamma(\boldsymbol{C}^m, \mathcal{O}_m)$ be an extension of f. For each compact neighborhood K' of K in $\boldsymbol{C}^m$ there is an $\varepsilon > 0$ such that if $\tilde{g} \in \Gamma(\boldsymbol{C}^m, \mathcal{O}_m)$ and $\|\tilde{g} - \tilde{f}\|_{K'} < \varepsilon$ then $g = \tilde{g}|_X \subset A(K)$.*

PROOF. If $(X, \mathcal{O}_X)$ is a reduced space this is lemma 5 of [2]. Since the property of being one-one is a topological property we need only to prove that the set

$$\{\tilde{g} \in \Gamma(\boldsymbol{C}^m, \mathcal{O}_m) \mid \tilde{g}|_X \text{ is regular at each point of } K\}$$

(5) Otherwise it is sufficient to add to $(\varphi_1, \ldots, \varphi_l)$ exactly $n + N - l$ arbitrary components.

is open in $\Gamma(\boldsymbol{C}^m, \mathcal{O}_m)^l$; but this is clear from the definition of regularity, since by small changes the rank of a map does not decrease.

LEMMA 8. *$A(K)$ is dense in $\Gamma(X, \mathcal{O}_X)^l$ and in $\Gamma(X)_\varphi$.*

PROOF. Choose $g \in \Gamma(X, \mathcal{O}_X)^p$. Define $M(g)$ to be the union of all irreducible components of the analytic set $\{(x, y) \in X \times X \; g(x) = g(y)\}$ which are not contained in the diagonal. Define $V(g, m)$ as follows: a point x is an element of $V(g, m)$ if and only if any extension $\tilde{g}: \boldsymbol{C}^m \to \boldsymbol{C}^p$ of g has rank $\leqslant m$ at x. So $V(g, m)$ is defined in X to be the set of common zeros to a family of holomorphic functions over $\boldsymbol{C}^m$; hence $V(g, m)$ is an analytic subset of X.

We begin by proving the following:

Let x be a point of X: if $x \in Y$ suppose $\dim \mathcal{T}(Y)_x < N$. Suppose $g \in \Gamma(X, \mathcal{O}_X)^p$ has an extension $\tilde{g} \in \Gamma(\boldsymbol{C}^m, \mathcal{O}_m)^p$ of rank $r < N$ at x. Then the set of all $h \in \Gamma(X, \mathcal{O}_X)$ such that the following properties hold:

(1) $h|_Y = \varphi_{p+1}$ (if $p + 1 \leqslant l$)

(2) (g, h) has an extension in $\Gamma(\boldsymbol{C}^m, \mathcal{O}_m)^{p+1}$ of rank $r + 1$ at x

is open and dense in $\Gamma(X)_{\varphi_{p+1}}$.

In order to prove this let $\xi \in \Gamma(\boldsymbol{C}^m, \mathcal{J}_Y)$ (6) be fixed as follows: if $x \notin Y$ it is a linear combination of elements in $\Gamma(\boldsymbol{C}^m, \mathcal{J}_Y)$ giving a system of local coordinates at x: if $x \in Y$ it is a linear combination of elements in $\Gamma(\boldsymbol{C}^m, \mathcal{J}_Y)$ giving a minimal system of equations for the Zarisky tangent space $\mathcal{T}(Y)_x$, as a submanifold of $\boldsymbol{C}^m$.

In both cases $(\tilde{g}, \xi)$ has rank $r + 1$ if the coefficients are suitably chosen (there is only a finite number of possibilities that this does not happen). Now if $\tilde{\varphi}_{p+1}$ is any extension to $\boldsymbol{C}^m$ of φ_{p+1} for small ε, $\tilde{\varphi}_{p+1} + \varepsilon\xi$ approximates $\tilde{\varphi}_{p+1}$ and is such that $(\tilde{g}, \tilde{\varphi}_{p+1} + \varepsilon\xi)$ has rank $r + 1$ at x.

This argument shows that the set we speak about is non void and dense since its inverse image in $\Gamma(\boldsymbol{C}^m, \mathcal{O}_m)$ and in $\Gamma(\boldsymbol{C}^m, \mathcal{O}_m)_{\varphi_{p+1}}$ is non void and dense; but this set is clearly open (same argument as lemma 7), so our statement is proved.

Now let $g = (g_1, \ldots, g_p) \in \Gamma(X, \mathcal{O}_X)^p$ have the following properties:

(1) $g_i|_Y = \varphi_i$ for each $i \leqslant l$;

(α_p) all the irreducible components of $M(g)$ meeting $K \times K$ have dimension $\leqslant 2n - p$;

(β_p) all the irreducible components of $V(g, m)$ meeting K have dimension $\leqslant n - p + m$ for $0 \leqslant m < n$;

(6) Here $\mathcal{J}_Y$ is the ideal sheaf defining $(Y, \mathcal{O}_Y)$ as a closed subspace in $\boldsymbol{C}^m$.

then the set of those $h \in \Gamma(X, \mathcal{O}_X)$ such that

(1) $h|_Y = \varphi_{p+1}$ if $p+1 \leqslant l$;

(2) $(g, h) \in \Gamma(X, \mathcal{O})^{p+1}$ satisfies (α_{p+1}) and (β_{p+1});

is dense in $\Gamma(X)_{\varphi_{p+1}}$.

In order to prove this statement in each component of $M(g)$ of dimension $2n-p$ meeting $K \times K$ choose a point $(x_q, y_q) \notin Y \times Y$; in each component of $V(g, m)$ of dimension $n-p+m$ meeting K choose a point x_i^m in such a way that if $x_i^m \in Y$ then $\dim \mathcal{T}(Y)_{x_i^m} < \dim \mathcal{T}(X)_{x_i^m}$.

This is possible because φ is one-one and regular; therefore it separates the points of Y and if $x \in Y$ any extension of φ has rank at x at least equal to the dimension of $\mathcal{T}(Y)_x$. In this way we have two finite sets $\{(x_q, y_q)\} \subset \subset K \times K$ and $\{x_i^m\} \subset K$. Moreover by hypothesis g admits an extension of rank m at x_i^m.

The same argument used in lemma 6 shows that condition (1) and (2) are equivalent to the following:

(a) $h(x_q) \neq h(y_q)$ for each q;

(b) (g, h) admits an extension to $\boldsymbol{C}^m$ of rank $m+1$ at x_i^m for each r and each m.

To prove (a) it is sufficient to show that the set of extensions of φ_{p+1} separating a given pair (x, y), where $x \notin Y$, is open and dense in $\Gamma(X)_{\varphi_{p+1}}$. To prove this consider the exact sequence of coherent sheaves

$$0 \to \mathcal{J} \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{J} \to 0$$

where $\mathcal{J}$ is the coherent ideal sheaf defining the subspace $Y \cup \{x\}$. By theorem B the map

$$\Gamma(X, \mathcal{O}_X) \to \Gamma(X, \mathcal{O}_X/\mathcal{J}) = \Gamma(Y, \mathcal{O}_Y) \oplus \boldsymbol{C}$$

is surjective. Hence there is a global section $s \in \Gamma(X, \mathcal{O}_X/\mathcal{J})$ such that $s(x) \neq 0$, $s(y) = 0$. If $h \in \Gamma(X)_{\varphi_{p+1}}$ we extend h to an $\tilde{h} \in \Gamma(\boldsymbol{C}^m, \mathcal{O}_m)$ and s to $\tilde{s} \in \Gamma(\boldsymbol{C}^m, \mathcal{O}_m)$; if ε is sufficiently small $\tilde{h} + \varepsilon\tilde{s}$ approximates $\tilde{h}$, therefore $h + \varepsilon s$ approximates h. On the other hand it is clear that our set is open.

We have already proved that the set of h satisfying (b) is open and dense, as intersection of a finite number of dense open sets.

So we obtain the thesis by intersecting the dense open set relative to condition (a) with the dense open set relative to condition (b).

Now, since conditions α_0, β_0 are void, what we have shown proves that

the set of all $h \in \Gamma(X, \mathcal{O}_X)^l$ such that

1) $h_i|_Y = \varphi_i$ for $i = 1, \ldots, l \geqslant n + N$;
2) $\dim\big(M(h) \cap K \times K\big) \leqslant 2n - l \leqslant -1$ (i.e. h is one-one on K);
3) $\dim\big(V(h, m) \cap K\big) \leqslant n - l + m \leqslant -1$ (i.e. h is regular on K);

is dense in $\Gamma(X)_\varphi$.

The same argument, except for the condition of extending φ, proves that $A(K)$ is dense in $\Gamma(X, \mathcal{O}_X)^l$.

Proof of Theorem 5. For each compact set $K \subset X$ let us define as before

$$A(K) = \{f \in \Gamma(X, \mathcal{O}_X)^l | f \text{ is one-one and regular on } K\}.$$

Because of lemmas 7 and 8 all the hypothesis of lemma 1 hold. Given $g = (g_1, \ldots, g_l) \in \Gamma(X)_\varphi$, choose l admissible systems $\{\Omega_i^\lambda\}$, $\lambda = 1, \ldots, l$, such that $\{U_i^\lambda = \Omega_i^\lambda \cap X\}$ is relative to g for each λ and compact sets $K_i^\lambda \subset \Omega_i^\lambda$ such that $X \subset \bigcup_{i,\lambda} K_i^\lambda$.

Choose any open neighborhood A of g in $\Gamma(X, \mathcal{O}_X)^l$ and any extension $\tilde{g}$ of g to $\boldsymbol{C}^m$; let C be a compact in $\boldsymbol{C}^m$ and ε be a constant such that if $\tilde{f} \in \Gamma(\boldsymbol{C}^m, \mathcal{O}_m)$ and $\|\tilde{f} - \tilde{g}\|_C < \varepsilon$ then $\tilde{f}|_X \in A$.

By theorem 3 there are holomorphic functions $\tilde{h}_1, \ldots, \tilde{h}_l$ in $\Gamma(\boldsymbol{C}^m, \mathcal{O}_m)$ such that:

(1) $\tilde{h}_i|_Y = \varphi_i$, $i = 1, \ldots, l$;
(2) $\|\tilde{h}_i - \tilde{g}_i\|_C < \varepsilon/2$, $i = 1, \ldots, l$;
(3) if $i_0(C) = \sup\{i: K_i^\lambda \cap C \neq \emptyset\}$, then $\|h_\lambda\|_{K_i^\lambda} > i + 1$ for $i > i_0(C)$ and $K_i^\lambda \cap Y = \emptyset$;
(4) $\|h_\lambda - g_\lambda\|_{K_i^\lambda} < \frac{1}{2}$ for $K_i^\lambda \cap Y \neq \emptyset$.

By lemma 1 there is $\tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_l) \in \Gamma(\boldsymbol{C}^m, \mathcal{O}_m)$ with the following properties:

(1) $\tilde{f}_\lambda|_Y = \varphi_\lambda$, $\lambda = 1, \ldots, l$;
(2) $\|\tilde{f} - \tilde{h}\|_C < \varepsilon/2$;
(3) $\|\tilde{f}_\lambda - \tilde{h}_\lambda\|_{K_i^\lambda} < \frac{1}{2}$ for $K_i^\lambda \cap Y \neq \emptyset$;
(4) $\|\tilde{f}_\lambda - \tilde{h}_\lambda\|_{K_i^\lambda} < \frac{1}{2}$ for $i > i_0(C)$ and $K_i^\lambda \cap Y = \emptyset$;
(5) $\tilde{f} \in r^{-1}\big(A(K \cap X)\big)$ for each compact set $K \subset \boldsymbol{C}^m$.

Let f be $\tilde{f}|_X$. Conditions (3) and (4) show that f is a proper map, since the set of points x such that $|f_\lambda(x)| < i$, $\lambda = 1, \ldots, l$, is contained in the compact set

$$\Big\{C \cup \Big(\bigcup_{j \leqslant \max(i_0(C), i)} \Big(\bigcup_{\lambda=1}^{l} K_i^\lambda\Big)\Big) \cup H\Big\} \cap X$$

where H is the compact set where the proper map $g' = g|_V$ is less than $i+1$ (V is the neighborhood of Y defined in lemma 2).

Condition 5 says that f is one-one and regular on all of X and therefore theorem 5 is proved.

4. – The real case.

We can deduce from theorems 4 and 5 the statements for real analytic spaces (7). More precisely we obtain the following theorems.

THEOREM 6. *Let $(X, \mathcal{O}_X)$ be a real analytic space of dimension n and $(Y, \mathcal{O}_Y)$ a closed subspace. Let $\varphi: Y \to \boldsymbol{R}^l$ be an embedding, $l \geqslant 2n+1$. The set of all analytic maps of X into $\boldsymbol{R}^l$ which are proper, one-one, of maximal rank in each regular point of X and extend φ is dense in*

$$\Gamma(X)_\varphi = \{f \in \Gamma(X, \mathcal{O}_X)^l : f|_Y = \varphi\} .$$

THEOREM 7. *Let $(X, \mathcal{O}_X)$ be a real analytic space of dimension n and locally of type N. Let $(Y, \mathcal{O}_Y)$ be a closed subspace and $\varphi: Y \to \boldsymbol{R}^l$ an embedding, $l \geqslant n+N$. Then the set of all maps which are embeddings of X into $\boldsymbol{R}^l$ and extend φ is dense in $\Gamma(X)_\varphi$.*

PROOF. The proof is reduced (as in [3]) to the existence of suitable admissible systems. Such systems can be constructed as in [4].

(7) Any real analytic space $(X, \mathcal{O}_X)$ is coherent, but it may be non reduced.

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