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**On the capillarity problem with constant volume**

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# On the Capillarity Problem with Constant Volume (*).

CLAUS GERHARDT (**)

## 0. – Introduction.

In this paper we shall discuss capillary problems which arise physically when the equilibrium surface of a liquid of fixed volume in a cylinder is analysed. The surface $u$ is determined by the principle of virtual work which leads to the variational problem

$$(0.1)\qquad I(v)=\int_\Omega (1+|Dv|^2)^{\frac12}\,dx+\frac{\varkappa}{2}\cdot\int_\Omega |v|^2\,dx-\int_{\partial\Omega}\beta v\,d\mathcal{H}_{n-1}\to\min$$

$$\text{in } K_1=\left\{v\in BV(\Omega)\colon v\geqslant\psi,\ \int_\Omega(v-\psi)\,dx=V\right\}.$$

where $\Omega$ is the cross-section of the cylinder, $\varkappa$ is the (nonnegative) capillarity constant, the obstacle $\psi$ represents the bottom of the cylinder, and $V$ is the prescribed volume. The function $\beta\in L^\infty(\partial\Omega)$ is the cosine of the angle between the free surface and the bounding cylinder walls, *i.e.* $\beta$ is absolutely bounded by 1.

The solvability of the variational problem depends on estimating the boundary integral $\int_{\partial\Omega}|\beta v|\,d\mathcal{H}_{n-1}$ by

$$(0.2)\qquad \int_{\partial\Omega}|\beta v|\,d\mathcal{H}_{n-1}\leqslant\int_{\Omega_\varepsilon}|Dv|\,dx+c_\varepsilon\cdot\int_\Omega|v|\,dx\qquad\forall v\in BV(\Omega),$$

where $\Omega_\varepsilon$ is the boundary strip $\{x\in\Omega\colon \operatorname{dist}(x,\partial\Omega)<\varepsilon\}$ and $c_\varepsilon$ a constant depending on $\varepsilon$ and $\partial\Omega$.

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The simplest and most far-reaching condition which we shall impose on $\partial\Omega$, in order that an estimate of this kind holds, is an interior sphere condition (ISC):

DEFINITION 0.1. $\Omega$ is said to satisfy an ISC of radius $R$, if for any boundary point $x_0 \in \partial\Omega$ there exists a ball $B$ of radius $R$ such that $B \subset \Omega$ and $x_0 \in \bar{B}$.

REMARK 0.1. An equivalent statement is to say that every interior point $x \in \Omega$ is contained in ball $B$ of radius $R$ which lies entirely in $\Omega$.

The main theorem which we shall prove is the following

THEOREM 0.1. *Let $\Omega$ be a bounded domain of $\mathbb{R}^n$, $n \geq 2$, with Lipschitz boundary $\partial\Omega$ satisfying an* ISC *of radius $R$, and let $\psi \in C^{0,1}(\bar{\Omega})$ and $\beta \in L^\infty(\partial\Omega)$, $|\beta| \leq 1$, be given functions. Then the following results are valid*

(i) *The variational problem* (0.1) *has a solution $u \in C^{0,1}(\Omega) \cap L^\infty(\Omega)$ provided that $\beta$ is bounded by*

$$|\beta| \leq 1 - a, \quad a > 0. \tag{0.3}$$

*Moreover, $u$ also solves the variational problem*

$$I_\lambda(v) = I(v) + \lambda \cdot \int_\Omega v\,dx \to \min \quad in \quad K_2 = \{v \in BV(\Omega) : v \geq \psi\} \tag{0.4}$$

*where $\lambda$ is a suitable Lagrange multiplier.*

(ii) *If $\psi$ is supposed to be of class $H^{2,p}(\Omega)$, $n < p \leq \infty$, then $u$ has the same degree of smoothness locally, i.e.*

$$u \in H^{2,p}_{\text{loc}}(\Omega). \tag{0.5}$$

(iii) *In the case that $\varkappa$ is strictly positive the solution is uniquely determined in $BV(\Omega)$ and the preceding results are valid for any $\beta \in L^\infty(\partial\Omega)$ with $|\beta| \leq 1$, and there exists a positive number $V^*$ such that*

$$u > \psi \tag{0.6}$$

*provided that* $\int_\Omega (u - \psi)\,dx \geq V^*$.

## 1. – A priori bounds for $|u|$.

In the following, we shall consider a slightly more general variational problem than the preceding one: Let $H \in C^{0,1}(\mathbb{R}^n \times \mathbb{R})$ and $j\colon \partial\Omega \times \mathbb{R} \to \mathbb{R}$ be given functions such that

$$\frac{\partial H}{\partial t} \geqslant 0\,, \tag{1.1}$$

$$\sup_{\Omega} H(x, t) \leqslant \alpha\cdot(1+t) \qquad \forall t > 0\,, \tag{1.1a}$$

where $\alpha$ is some positive constant, and $j$ satisfies a Carathéodory condition, *i.e.* it is measurable in $x$ (with respect to the $(n-1)$-dimensional Hausdorff measure on $\partial\Omega$) and continuous in the second variable. Furthermore, we assume that for $\mathcal{H}_{n-1}$ — *a.e.* $j(x,\cdot)$ is a strict contraction, *i.e.*

$$|j(x, r) - j(x, s)| \leqslant (1-a)\cdot|r-s|\,, \qquad a > 0\,, \tag{1.2}$$

where $a$ is independent of $x$, that

$$j(x,\cdot) \quad \textit{is convex}\,, \tag{1.3}$$

and that

$$j(\cdot\,, 0) \in L^1(\partial\Omega)\,. \tag{1.4}$$

Then, we consider the functional

$$J(v) = \int_{\Omega} (1 + |Dv|^2)^{\frac{1}{2}}\, dx + \int_{\Omega}\int_0^v H(x, t)\, dt\, dx + \int_{\partial\Omega} j(x, v)\, d\mathcal{H}_{n-1}\,. \tag{1.5}$$

The functional $I$ is contained in this general setting taking $H(x, t) = \varkappa\cdot t$ and $j(x, t) = -\beta(x)\cdot t$. Furthermore, let us remark that

$$j(x, t) = (1-a)\cdot|t - \varphi(x)|\,,$$

$\varphi \in L^1(\partial\Omega)$, also satisfies the conditions imposed on $j$.

Under the preceding assumptions on $\Omega$, $H$, and $j$ we can prove

LEMMA 1.1. *Let $u$ be a solution of the variational problem*

$$J(v) \to \min \quad \textit{in } K_2\,. \tag{1.6}$$

*Then $u$ can be estimated by*

$$\max\left\{\inf_\Omega \psi,\ -c_1\right\} \leqslant u \leqslant \max\left\{\sup_\Omega \psi,\ c_1\right\}, \tag{1.7}$$

*where the constant $c_1$ depends on $|\Omega|$, $\|u\|_1$, $\|H(\cdot, 0)\|_p$ $(p > n)$, $a$, $n$, and on a constant $c_0$ which will be defined in the following.*

Here, we denote by $\|\cdot\|_q$, $q \geqslant 1$, the norm in $L^q(\Omega)$.

Before proving the lemma, let us mention a result which has been derived in [7; Remark 2].

LEMMA 1.2. *Let $\Omega$ be a bounded domain of $\mathbf{R}^n$, $n \geqslant 2$, with Lipschitz boundary satisfying an* ISC *of radius $R$. Then the following estimate is valid*

$$\int_{\partial\Omega} |v|\, d\mathcal{H}_{n-1} \leqslant \int_{\Omega_\varepsilon} |\mathrm{D}v|\, dx + c_\varepsilon \cdot \int_\Omega |v|\, dx \qquad \forall v \in BV(\Omega), \tag{1.8}$$

*where $\Omega_\varepsilon = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \varepsilon\}$, $c_\varepsilon$ depends on $\varepsilon$, $R$, and $\partial\Omega$, and $\varepsilon$ is any positive number less than or equal to $R/2$.*

PROOF OF LEMMA 1.1. Let $k$ be a positive number greater than $\sup_\Omega \psi$, and set $u_k = \min(u, k)$. Then $u_k$ belongs to $K_2$ and from the minimum property of $u$ we get

$$J(u) \leqslant J(u_k). \tag{1.9}$$

Hence, using the notation $A(k) = \{x \in \Omega : u(x) \geqslant k\}$ and supposing for a moment $u$ to be smooth, we obtain

$$\int_{A(k)} (1 + |Du|^2)^{\frac{1}{2}}\, dx + \int_\Omega \int_{u_k}^{u} H(x, t)\, dt\, dx + \int_{\partial\Omega} \{j(x, \mathrm{J}) - j(x, u_k)\}\, d\mathcal{H}_{n-1} \leqslant |A(k)|, \tag{1.10}$$

where $|A(k)|$ denotes the Lebesgue measure in $\mathbf{R}^n$ of $A(k)$.

In view of the condition (1.2) the boundary integral can be estimated by

$$(1 - a) \cdot \int_{\partial\Omega} |u - u_k|\, d\mathcal{H}_{n-1}, \tag{1.11}$$

or, taking Lemma 1.2 into account, by

$$(1 - a) \cdot \int_\Omega |Dw|\, dx + c_0 \cdot (1 - a) \cdot \int_\Omega |w|\, dx, \tag{1.12}$$

where we have set $w = \max(u - k, 0) = u - u_k$ and $c_0 = c_{R/2}$.

Furthermore, observing that

$$(1.13)\qquad \int_{u_k}^{u} H(x,t)\,dt \geqslant H(x,0)\cdot(u-u_k) = H_0\cdot w \qquad (H_0 = H(\cdot,0))$$

in view of the monotonicity of $H(x,\cdot)$, we deduce from (1.10) and (1.12) the inequality

$$(1.14)\qquad a\cdot\int_\Omega |Dw|\,dx + \int_\Omega H_0 w\,dx - (1-a)\cdot c_0\cdot\int_\Omega w\,dx \leqslant |A(k)|$$

which will also be valid for $u\in BV(\Omega)$ using an approximation argument (cf. [7; Lemma A4]).

To estimate the integral $\int_\Omega H_0 w\,dx$, we use the Hölder inequality

$$(1.15)\qquad \left|\int_\Omega H_0 w\,dx\right| \leqslant \|w\|_{n^*}\cdot\left\{\int_{A(k)} |H_0|^n\,dx\right\}^{1/n} \leqslant \|w\|_{n^*}\cdot\|H_0\|_p\cdot|A(k)|^{(p-n)/n\cdot p},$$

where we denote by $n^*$ the conjugate exponent, $1/n^* = 1 - 1/n$.

Thus, using the Hölder inequality again, we obtain from (1.14)

$$(1.16)\qquad a\cdot\int_\Omega |Dw|\,dx + a\cdot c_0\cdot\int_\Omega w\,dx - \{\|H_0\|_p\cdot|A(k)|^{(p-n)/n\cdot p} + c_0\cdot|A(k)|^{1/n}\}\cdot \cdot\|w\|_{n^*} \leqslant |A(k)|\,.$$

Now, applying the Sobolev imbedding theorem and using the fact that

$$(1.17)\qquad |A(k)| \leqslant \frac{1}{k}\cdot\int_\Omega |u|\,dx\,,$$

we derive from (1.16)

$$(1.18)\qquad \|w\|_{n^*} \leqslant c_2\cdot|A(k)|$$

for $k \geqslant k_0$, where $k_0$ and $c_2$ depend on $\|u\|_1$, $\|H_0\|_p$, $a$, $c_0$, and known quantities. Hence, we conclude

$$(1.19)\qquad \int_{A(k)} (u-k)\,dx = \int_\Omega w\,dx \leqslant c_2\cdot|A(k)|^{1+1/n}$$

or finally

$$(1.20)\qquad (h-k)\cdot|A(h)| \leqslant c_2\cdot|A(k)|^{1+1/n} \qquad \text{for } h > k \geqslant k_0\,.$$

From a lemma due to Stampacchia [13; Lemma 4.1] we now deduce

$$u \leqslant k_0 + c_2 \cdot |\Omega|^{1/n} \cdot 2^{(n+1)} . \tag{1.21}$$

Though $u$ is obviously bounded from below by $\psi$, it would be worth to get the sharper estimate (1.7), for by this we had also derived a bound for solutions to the free problem

$$J(v) \to \min \quad \text{in } BV(\Omega) \tag{1.22}$$

setting formally $\psi = -\infty$.

In order to obtain the lower bound we insert $u_k = \max(u, -k)$ in (1.9). The proof of Lemma 1.1 can then be completed by similar considerations as above.

## 2. – Existence and regularity of solutions to the variational problem.

Generally, the functional $J$ does not have a minimum in the convex set $K_2$, unless we impose some growth condition on $H$. However, we can prove a rather abstract existence theorem which will be very useful in the following.

THEOREM 2.1. *Let $\Omega$ and $J$ be as in Lemma* 1.1, *where we may now assume that $j$ is only a contraction, i.e.*

$$|j(x, r) - j(x, s)| \leqslant |r - s| \quad \mathcal{H}_{n-1} - a.e. \text{ in } \partial\Omega . \tag{2.1}$$

*Let $K \subset BV(\Omega)$ be convex and closed with respect to convergence in $L^1(\Omega)$. Furthermore, let $v_\varepsilon$ be a minimizing sequence for the variational problem*

$$J(v) \to \min \quad \textit{in } K \tag{2.2}$$

*such that*

$$|\lim J(v_\varepsilon)| < \infty \tag{2.3}$$

*and*

$$\int_\Omega |Dv_\varepsilon|\, dx + \int_\Omega |v_\varepsilon|\, dx \leqslant c_3 \tag{2.4}$$

*uniformly in $\varepsilon$. Then a subsequence of the $v_\varepsilon$'s converges in $L^1(\Omega)$ to some function $u \in BV(\Omega)$ which minimizes $J$.*

PROOF. The theorem has more or less been demonstrated in [7; Appendix II], but for convenience we shall repeat the rather short proof.

From [12; Theorem XVI], the Sobolev imbedding theorem, and [11; Theorem 2.1.3] we conclude from (2.4) that the $v_\varepsilon$'s are precompact in any $L^p(\Omega)$, $1 \leqslant p < n/(n-1)$. Hence, let us suppose for simplicity that $v_\varepsilon \to u$ in $L^1(\Omega)$. Assume by contradiction that $J(u)$ is strictly greater than $\lim J(v_\varepsilon)$. Then, there exists a positive constant $\gamma$ and a number $\varepsilon_0$ such that

$$J(v_\varepsilon) \leqslant J(u) - \gamma \qquad \forall \varepsilon \leqslant \varepsilon_0 . \tag{2.5}$$

In view of (1.8) and (2.1) we have the estimate

$$\int_{\partial\Omega} |j(x, v_\varepsilon) - j(x, u)| \, d\mathcal{H}_{n-1} \leqslant \int_{\Omega_\delta} |D(v_\varepsilon - u)| \, dx + c_\delta \cdot \int_\Omega |v_\varepsilon - u| \, dx \tag{2.6}$$

where $\Omega_\delta$ is a boundary strip of width $\delta$, and $\delta$ is sufficiently small.

Thus, we deduce

$$\begin{aligned} \int_{\Omega - \bar{\Omega}_\delta} (1 + |Dv_\varepsilon|^2)^{\frac{1}{2}} \, dx + \int_\Omega \int_0^{v_\varepsilon} H(x, t) \, dt \, dx \leqslant \int_\Omega (1 + |Du|^2)^{\frac{1}{2}} \, dx + \\ + \int_\Omega \int_0^u H(x, t) \, dt \, dx + \int_{\Omega_\delta} |Du| \, dx + c_\delta \cdot \int_\Omega |u - v_\varepsilon| \, dx - \gamma . \end{aligned} \tag{2.7}$$

If $\varepsilon$ tends to zero, then we obtain in view of the lower semicontinuity of the integrals on the left side of (2.7) (cf. [8; formula (64)]

$$\begin{aligned} \int_{\Omega - \bar{\Omega}_\delta} (1 + |Du|^2)^{\frac{1}{2}} + \int_\Omega \int_0^u H(x, t) \, dt \, dx \leqslant \int_\Omega (1 + |Du|^2)^{\frac{1}{2}} \, dx + \\ + \int_\Omega \int_0^u H(x, t) \, dt \, dx + \int_{\Omega_\delta} |Du| \, dx - \gamma . \end{aligned} \tag{2.8}$$

To complete the proof, we let $\delta$ converge to zero which gives the contradiction.

The interior regularity of solutions to the variational problem (1.6) follows from the theorem below which has been proved in [8; Theorem 6 and Lemma 4].

THEOREM 2.2. *Let $w$ be a locally bounded solution in $BV(\Omega)$ of the variational problem*

$$(2.9)\qquad \int_\Omega (1+|Dv|^2)^{\frac12}\,dx + \int_\Omega \int_0^v H(x,t)\,dt\,dx \to \min \qquad \text{in } K_2\cap\{v|_{\partial\Omega} = w|_{\partial\Omega}\}\,,$$

*where $H\in C^{0,1}(\mathbf{R}^n\times\mathbf{R})$ satisfies $\partial H/\partial t > 0$. Then $w$ is locally Lipschitz in $\Omega$ provided that $\psi\in C^{0,1}(\bar\Omega)$, and we have the interior gradient estimate*

$$(2.10)\qquad |Dw|_{\Omega'} \leqslant c_4\left(|w|_{\Omega''},\ |D\psi|_{\Omega''},\ \left|\frac{\partial}{\partial x}H(x,u(x))\right|_{\Omega''}\right) \qquad \forall \Omega'\subset\subset\Omega''\subset\subset\Omega\,.$$

*Furthermore, if we assume $\psi$ to be of class $H^{2,p}(\Omega)$, $n<p\leqslant\infty$, then $w$ belongs to $H^{2,p}_{\mathrm{loc}}(\Omega)$. Precisely, we have the estimate*

$$(2.11)\qquad \|Au\|_{p,\Omega'} \leqslant \|A\psi\|_{p,\Omega'} + 2\cdot\|H(x,u)\|_{p,\Omega'} \qquad \forall \Omega'\subset\Omega\,,$$

*where $A$ is the minimal surface operator in divergence form.*

## 3. – Existence of a Lagrange multiplier.

In this section we shall show the existence of a real number $\lambda$ such that the variational problem

$$(3.1)\qquad J_\lambda(v) = J(v) + \lambda\cdot\int_\Omega v\,dx \to \min \qquad \text{in } K_2$$

has a solution $u_\lambda\in K_2$ such that

$$(3.2)\qquad \int_\Omega (u_\lambda - \psi)\,dx = V\,,$$

where the volume $V$ is prescribed. Thus, $u_\lambda$ also solves

$$(3.3)\qquad J(v)\to\min \qquad \text{in } K_2\cap\left\{\int_\Omega (v-\psi)\,dx = V\right\}.$$

THEOREM 3.1. *Suppose that $\Omega$, $\psi$, $H$ and $j$ satisfy the conditions stated in Section 1. Then, the variational problem*

$$(3.4)\qquad J(v)\to\min \qquad \text{in } K_2\cap\left\{\int_\Omega (v-\psi)\,dx = V\right\}$$

*has a solution* $u \in C^{0,1}(\Omega) \cap L^\infty(\Omega)$ *for any prescribed volume* $V$. *Moreover, u also solves the variational problem*

$$J_\lambda(v) = J(v) + \lambda \cdot \int_\Omega v\, dx \to \min \quad \text{in } K_2, \tag{3.5}$$

*where* $\lambda$ *is a suitable unique Lagrange multiplier. The mappings*

$$h_1 \colon \mathbb{R}_+ \to L^1(\Omega), \quad h_1(V) = u, \tag{3.6}$$

*and*

$$h_2 \colon \mathbb{R}_+ \to \mathbb{R}, \quad h_2(V) = \lambda, \tag{3.7}$$

*are continuous and nondecreasing resp. nonincreasing. Furthermore, if* $\psi$ *is supposed to be of class* $H^{2,p}(\Omega)$, $n < p \leqslant \infty$, *then u satisfies*

$$u \in H^{2,p}_{\mathrm{loc}}(\Omega). \tag{3.8}$$

PROOF. For $\varepsilon > 0$ set

$$H_\varepsilon(x, t) = H(x, t) + \varepsilon \cdot t. \tag{3.9}$$

Similarly, we define $J_\varepsilon$ and

$$J_{\varepsilon,\mu}(v) = J_\varepsilon(v) + \mu \cdot \int_\Omega v\, dx. \tag{3.10}$$

Then, for arbitrary $\mu \in \mathbb{R}$ we shall demonstrate the following lemma.

LEMMA 3.1. *Let* $\varepsilon$, $0 < \varepsilon < 1$, *be given. Then, under the preceding assumptions, the variational problem*

$$J_{\varepsilon,\mu}(v) \to \min \quad \text{in } K_2 \tag{3.11}$$

*has a unique solution* $u_{\varepsilon,\mu} \in C^{0,1}(\Omega) \cap L^\infty(\Omega)$ *such that the estimates*

$$(\mu - c_5) \cdot \int_\Omega (u_{\varepsilon,\mu} - \psi)\, dx \leqslant c_6 \tag{3.12}$$

*and*

$$-c_7 \leqslant (\alpha + \varepsilon) \cdot \int_\Omega (u_{\varepsilon,\mu} - \psi)\, dx + \mu \cdot |\Omega| \tag{3.13}$$

*are valid, where the positive constants are independent from $\varepsilon$ and $\mu$, and where $|\Omega|$ denotes the Lebesgue measure of $\Omega$.*

PROOF OF LEMMA 3.1. In order to prove the existence of a solution to (3.11) let $v_\iota$ be a minimizing sequence of the variational problem. Then, taking the estimates

$$\int_0^{v_\iota} H(x,t)\,dt \geq H_0 \cdot v_\iota \tag{3.14}$$

and

$$|j(x, v_\iota) - j(x, 0)| \leq (1-a)\cdot |v_\iota| \tag{3.15}$$

into account we deduce from Lemma 1.2

$$\begin{aligned} a\cdot\int_\Omega |Dv_\iota|\,dx + \frac{\varepsilon}{2}\int_\Omega |v_\iota|^2\,dx + \mu\int_\Omega v_\iota\,dx - (c_0 + |H_0|)\cdot\int_\Omega |v_\iota|\,dx + \\ + \int_{\partial\Omega} j(x,0)\,d\mathcal{H}_{n-1} \leq J_\varepsilon(\psi) + \mu\cdot\int_\Omega \psi\,dx\,, \end{aligned} \tag{3.16}$$

where we have set $c_0 = c_{R/2}$.

Thus, we conclude that the sequence

$$\int_\Omega |Dv_\iota|\,dx + \int_\Omega |v_\iota|\,dx \tag{3.17}$$

is uniformly bounded. Hence, the existence of a solution $u_{\varepsilon,\mu}$ follows from Theorem 2.1. Moreover, since the functional $J_{\varepsilon,\mu}$ is strictly convex the solution is unique. The Lipschitz regularity and the boundedness of $u_{\varepsilon,\mu}$ are consequence of the Theorem 2.2 and Lemma 1.1.

To derive the estimate (3.12), we observe that the inequality (3.16) is satisfied by $u_{\varepsilon,\mu}$, too; hence the result.

On the other hand, let $\eta \in H^{1,1}(\Omega)$, $\eta \geq 0$, and $\delta > 0$ be given. Then, $v_\delta = u_{\varepsilon,\mu} + \delta\eta$ belongs to $K_2$ and we obtain from the minimum property of $u_{\varepsilon,\mu}$

$$J_{\varepsilon,\mu}(u_{\varepsilon,\mu}) \leq J_{\varepsilon,\mu}(v_\delta)\,, \tag{3.18}$$

or, if we set

$$\varphi(\delta) = J_{\varepsilon,\mu}(u_{\varepsilon,\mu} + \delta\eta) - \int_{\partial\Omega} j(x, u_{\varepsilon,\mu} + \delta\eta)\,d\mathcal{H}_{n-1}\,, \tag{3.19}$$

$$\varphi(0) \leq \varphi(\delta) + \delta\cdot\int_{\partial\Omega}\eta\,d\mathcal{H}_{n-1}\,. \tag{3.20}$$

Therefore, we finally conclude that the inequality

$$(3.21)\quad \int_\Omega D^i u_{\varepsilon,\mu}\cdot[1+|Du_{\varepsilon,\mu}|^2]^{-\frac{1}{2}}\cdot D^i\eta\,dx+\int_\Omega H(x,u_{\varepsilon,\mu})\cdot\eta\,dx+ \\ +\varepsilon\int_\Omega u_{\varepsilon,\mu}\cdot\eta\,dx+\mu\cdot\int_\Omega\eta\,dx+\int_{\partial\Omega}\eta\,d\mathcal{H}_{n-1}\geqslant 0 \qquad \forall\eta\in H^{1,1}(\Omega)\cap\{\eta\geqslant 0\}$$

is valid. Now, the estimate (3.13) follows from inserting $\vartheta\eta=1$ in the preceding inequality.

Let us remark that we needed the assumption (1.1$a$) only for this estimate.

Thus, if we define for fixed $\varepsilon$

$$(3.22)\qquad V(\mu)=\int_\Omega(u_{\varepsilon,\mu}-\psi)\,dx$$

we deduce from (3.12) and (3.13)

$$(3.23)\qquad \lim_{\mu\to\infty}V(\mu)=0$$

and

$$(3.24)\qquad \lim_{\mu\to-\infty}V(\mu)=+\infty\,.$$

The existence of a Lagrange multiplier now follows from the fact that $V$ depends continuously on $\mu$.

LEMMA 3.2. *Let the assumptions of Lemma 3.1 be satisfied. Then, for fixed $\varepsilon$, the mapping*

$$(3.25)\qquad h_3\colon \mathbb{R}\to L^1(\Omega)\,,\qquad h_3(\mu)=u_{\varepsilon,\mu}\,,$$

*is continuous.*

PROOF OF LEMMA 3.2. Let $\mu_\iota$ be a convergent sequence, $\lim\mu_\iota=\mu_0$, and let $u_{\varepsilon,\mu}$ resp. $u_{\varepsilon,\mu_0}$ be the solutions to the variational problem (3.11) with the respective functionals $J_{\varepsilon,\mu_\iota}$ and $J_{\varepsilon,\mu_0}$. Then, the $u_{\varepsilon,\mu_\iota}$'s form a minimizing sequence for the variational problem

$$(3.26)\qquad J_{\varepsilon,\mu_0}(v)\to\min\quad\text{in } K_2\,,$$

such that the integrals

$$(3.27)\qquad \int_\Omega|Du_{\varepsilon,\mu_\iota}|\,dx+\int_\Omega|u_{\varepsilon,\mu_\iota}|\,dx$$

are uniformly bounded (cf. formula (3.11)). The rest of the proof now follows immediately in view of the Theorem 2.1 and the uniqueness of the solution.

Thus, for fixed $\varepsilon$ and $V$ there exists a parameter $\lambda_\varepsilon$ such that the solution $u_{\varepsilon,\lambda_\varepsilon}$ of the variational problem

$$J_{\varepsilon,\lambda_\varepsilon}(v) \to \min \quad \text{in } K_2 \tag{3.28}$$

satisfies

$$\int_\Omega (u_{\varepsilon,\lambda_\varepsilon} - \psi)\,dx = V\,. \tag{3.29}$$

Hence, we obtain

$$J_\varepsilon(u_{\varepsilon,\lambda_\varepsilon}) \leqslant J_\varepsilon(v) \qquad \forall v \in K_2 \cap \left\{\int_\Omega (v-\psi)\,dx = V\right\}. \tag{3.30}$$

Moreover, from Lemma 3.1 and Lemma 1.1 we deduce that $\lambda_\varepsilon$ and $u_{\varepsilon,\lambda_\varepsilon}$ are uniformly bounded for fixed $V$. Hence, the integrals

$$\int_\Omega |Du_{\varepsilon,\lambda_\varepsilon}|\,dx + \int_\Omega |u_{\varepsilon,\lambda_\varepsilon}|\,dx \tag{3.31}$$

are uniformly bounded.

Thus, letting $\varepsilon$ go to zero, a subsequence of the $\lambda_\varepsilon$'s converges to some real number $\lambda$. The respective solutions $u_{\varepsilon,\lambda_\varepsilon}$ then form a minimizing sequence for the variational problems

$$J(v) \to \min \quad \text{in } K_2 \cap \left\{\int_\Omega (v-\psi)\,dx = V\right\} \tag{3.32}$$

and

$$J_\lambda(v) \to \min \quad \text{in } K_2\,. \tag{3.33}$$

Hence, we conclude from Theorem 2.1 that a subsequence of the $u_{\varepsilon,\lambda_\varepsilon}$'s converges in $L^1(\Omega)$ to some function $u_\lambda \in BV(\Omega)$ which solves both variational problems. Furthermore, the solution of the variational problem (3.32) is uniquely determined in the class $H^{1,1}(\Omega)$, since the difference of two solutions must be a constant, which has to be zero in view of the side conditions. Thus, the first part of Theorem 3.1 is proved.

It remains to prove the properties of the mappings $h_1$ and $h_2$, since the interior regularity of $u$ follows from the estimates for $u_{\varepsilon,\lambda_\varepsilon}$.

First of all, let us observe that both mappings are continuous, which follows from the fact that they are compact and the solution $u$ as well as the Lagrange multiplier $\lambda$ are uniquely determined.

The monotonicity of $h_1$ and $h_2$ will be a consequence of the following lemma

LEMMA 3.3. *Let $u_\lambda$ and $u_{\lambda^*}$ be solutions of the variational problem* (3.33) *with respect to the data $\lambda$, $\psi$, $j$ and $\lambda^*$, $\psi^*$, $j^*$, where, in contrast to condition* (1.2) *$j$ resp. $j^*$ are not forced to be strict contractions. They are only supposed to be uniformly Lipschitz in $t$. Moreover, we assume that at least one of the solutions $u_\lambda$, $u_{\lambda^*}$ is unique. Then, we obtain*

$$u_\lambda \leqslant u_{\vartheta\lambda^*}\,, \tag{3.34}$$

*provided that the relations*

$$\lambda \geqslant \lambda^*\,, \tag{3.35}$$

*and*

$$\psi \leqslant \psi^* \tag{3.36}$$

*are valid, and where, furthermore, the difference $j(x,\cdot)-j^*(x,\cdot)$ is supposed to be nondecreasing a.e. in $\partial\Omega$, which can formally be written as*

$$\frac{\partial j}{\partial t}-\frac{\partial j^*}{\partial t}\geqslant 0\,. \tag{3.37}$$

Suppose the lemma to be valid. Then, the solution $u_\varepsilon$ of the perturbed problem

$$J_{\varepsilon,\lambda}(v) \to \min \quad \text{in } K_2\,, \tag{3.38}$$

where we have replaced $H$ by $H_\varepsilon(x,t) = H(x,t) + \varepsilon\cdot t$, is unique. Furthermore, the solution coincides with the one of the variational problem

$$J_\varepsilon(v) \to \min \quad \text{in } K_2 \cap \left\{\int_\Omega (v-\psi)\,dx = V\right\} \tag{3.39}$$

if $\lambda$ is equal to the Lagrange multiplier $\lambda_\varepsilon$. For fixed $\varepsilon > 0$, let the function

$$h_{2,\varepsilon}\colon\ V \to \lambda_\varepsilon \tag{3.40}$$

describe the dependece between $V$ and $\lambda_\varepsilon$, and define

$$h_{1,\varepsilon}\colon\ V \to u_\varepsilon \tag{3.41}$$

and

$$h_{3,\varepsilon}\colon\ \lambda_\varepsilon \to u_\varepsilon \tag{3.42}$$

similarly.

Then, we deduce that $h_{2,\varepsilon}$ is nonincreasing for $h_{3,\varepsilon}$ has this property; hence, $h_{1,\varepsilon} = h_{3,\varepsilon}\circ h_{2,\varepsilon}$ is nondecreasing.

In the limit case, $\varepsilon \to 0$, the functions $h_{1,\varepsilon}$ and $h_{2,\varepsilon}$ converge to the functions $h_1$ and $h_2$ which can be seen by using a compactness argument and the uniqueness in $H^{1,1}(\Omega)$ of the solution to the variational problem (3.32).

Thus, to complete the proof of the Theorem 3.1, we have merely to demonstrate Lemma 3.3.

PROOF OF LEMMA 3.3. Suppose that $u_\lambda$ is the unique solution. Then, we have

$$J_\lambda(u_\lambda) < J_\lambda(\min(u_\lambda, u_{\lambda^*})) \quad \text{or} \quad u_\lambda = \min(u_\lambda, u_{\lambda^*}) \tag{3.43}$$

and

$$J_{\lambda^*}(u_{\lambda^*}) \leqslant J_{\lambda^*}(\max(u_\lambda, u_{\lambda^*}))\,. \tag{3.44}$$

Combining these relations and using the fact that

$$j(x, u_\lambda) - j(x, \min(u_\lambda, u_{\lambda^*})) \geqslant j^*(x, \max(u_\lambda, u_{\lambda^*})) - j^*(x, u_{\lambda^*}) \tag{3.45}$$

or equivalently

$$j(x, u_\lambda) - j^*(x, \max(u_\lambda, u_{\lambda^*})) \geqslant j(x, \min(u_\lambda, u_{\lambda^*})) - j^*(x, u_{\lambda^*})\,, \tag{3.46}$$

which can easily be checked distinguishing the cases $u_\lambda \leqslant u_{\lambda^*}$ and $u_\lambda > u_{\lambda^*}$, in view of (3.37), we deduce from (3.43) that

$$u_\lambda = \min(u_\lambda, u_{\lambda^*})\,, \tag{3.47}$$

hence the result.

REMARK 3.1. Let $j(x,t) = |t - \varphi(x)|$ and $j^*(x,t) = |t - \varphi^*(x)|$ with $\varphi, \varphi^* \in L^1(\partial\Omega)$. Then, the condition (3.37) is satisfied provided that

$$\varphi \leqslant \varphi^* \quad \mathcal{H}_{n-1} - a.e.\,, \tag{3.48}$$

for we have

$$\frac{\partial j}{\partial t} - \frac{\partial j^*}{\partial t} = \begin{cases} 0 & \text{if } t < \varphi\,, \\ 2 & \text{if } \varphi < t < \varphi^*, \\ 0 & \text{if } \varphi^* < t\,. \end{cases} \tag{3.49}$$

## 4. – The case where $H$ satisfies a certain growth condition.

In this section we shall assume that besides the preceding conditions $H$ satisfies the relations

$$\lim_{t\to\infty} \inf_{\Omega} H(x, t) = +\infty \tag{4.1}$$

and

$$\lim_{t\to-\infty} \sup_{\Omega} H(x, t) = -\infty . \tag{4.2}$$

Then, we can prove the following generalization of Theorem 3.1.

THEOREM 4.1. *Suppose that $H$ satisfies the growth conditions* (4.1) *and* (4.2). *Then, the results of Theorem* 3.1 *remain valid if we replace the condition* (1.2) *by the more general assumption*

$$|j(x, r) - j(x, s)| \leqslant |r - s| \quad \mathcal{H}_{n-1} - a.e. . \tag{4.3}$$

*Moreover, there exists a positive number $V^*$ such that a solution $u \in H^{1,1}(\Omega)$ of the variational problem*

$$J(v) \to \min \quad in\ K_2 \cap \left\{ \int_\Omega (v - \psi)\, dx = V \right\} \tag{4.4}$$

*satisfies*

$$u > \psi , \tag{4.5}$$

*provided that $V \geqslant V^*$.*

PROOF. Let us remark that the solution $u$ of the variational problem (3.4) is absolutely bounded in terms of $a$, $\lambda$, and $V$ (cf. Lemma 1.1), whereas $|\lambda|$ is estimated in terms of $V$ (cf. Lemma 3.1). Thus, to prove the first part of Theorem 4.1, we have only to show that $|u|$ is bounded independently of $a$, using an approximation argument afterwards (cf. Theorem 2.1).

LEMMA 4.1. *Suppose that $H$ satisfies the conditions* (4.1) *and* (4.2), *and let $u \in H^{1,1}(\Omega)$ be a solution of the variational problem* (3.5). *Then, $u$ is absolutely bounded by some constant $m$, which only depends on $H$, $R$, $\lambda$, $n$, and* $\sup_\Omega \max(\psi, 0)$.

PROOF OF LEMMA 4.1. We shall only show the existence of an upper bound, since the lower bound could be established by similar considerations.

First of all, let us assume that

$$\psi \in C^2(\bar{\Omega}). \tag{4.6}$$

Then, $u$ belongs to $H^{2,\infty}_{\text{loc}}(\Omega)$ and is a solution of the variational inequality

$$\langle Au + H(x, u) + \lambda, v - u \rangle \geq 0 \qquad \forall v \in K_{\Omega'}, \tag{4.7}$$

$$K_{\Omega'} = \{v \in H^{1,\infty}(\Omega') : v \geq \psi,\ v|_{\partial\Omega'} = u|_{\partial\Omega'}\},$$

where $A$ is the minimal surface operator and $\Omega'$ any compact subdomain of $\Omega$. From (4.7) and the regularity of $u$ we immediately deduce

$$Au + H(x, u) + \lambda = \begin{cases} \text{nonnegative a.e.} & \text{in } \Omega \\ 0 & \text{in } \{u > \psi\}. \end{cases} \tag{4.8}$$

Now, let $B_R$ be a ball of radius $R$ such that $B_R \subset\subset \Omega$, and let $B_{R_0}$, $B_{R_0} \subset\subset \Omega$, be a concentric ball of radius $R_0$, $R < R_0$, where we assume that the center of the balls lies in the origin. Let $\delta_0$ be the lower hemisphere

$$\delta_0 = -[R_0^2 - |x|^2]^{\frac{1}{2}}. \tag{4.9}$$

Then, we have $\delta_0 \in C^2(B_R)$ and

$$A\delta_0 = -\frac{n}{R_0}. \tag{4.10}$$

Furthermore, let $M$ be a positive constant such that

$$M - R_0 \geq \sup_{\Omega} \psi \tag{4.11}$$

and

$$\inf_{\Omega} H(x, M - R_0) + \lambda \geq \frac{n}{R_0}. \tag{4.12}$$

Then, $\delta = \delta_0 + M$ satisfies the inequality

$$A\delta + H(x, \delta) + \lambda \geq 0 \qquad \text{in } B_R. \tag{4.13}$$

Combining the relations (4.8) and (4.13) we obtain

$$\int_{B_R} \{A\delta - Au + H(x, \delta) - H(x, u)\} \cdot \max(u - \delta, 0)\, dx \geq 0. \tag{4.14}$$

On the other hand, we know that $|Du|$ is uniformly bounded in $B_R$. Thus, we deduce

$$|Du|\cdot[1+|Du|^2]^{-\frac{1}{2}}\leqslant L<1 \quad \text{in } B_R . \tag{4.15}$$

Since we have $R_0$ still at our disposal, we choose $R_0$ near $R$ such that

$$D\delta\cdot\nu\cdot[1+|D\delta|^2]^{-\frac{1}{2}}=\frac{R}{R_0}\geqslant L \quad \text{on } \partial B_R , \tag{4.16}$$

where $\nu$ is the outward normal vector to $\partial B_R$.

Partial integration in (4.14) then leads to the desired result

$$\max(u-\delta, 0)=0 , \tag{4.17}$$

in view of (4.16) and the strong monotonicity of the operator $A+H(x,\cdot)$. Hence, we obtain

$$u\leqslant M+R_0 \tag{4.18}$$

or finally

$$u\leqslant M+R \quad \text{in } B_R . \tag{4.19}$$

As $\Omega$ satisfies an ISC of radius $R$, it can be completely covered by balls of radius $R$. Hence the estimate (4.19) holds uniformly in $\Omega$.

If $\psi$ is merely Lipschitz, we approximate $\psi$ by smooth functions $\psi_\varepsilon$. Let $u_\varepsilon$ be the respective solutions of (3.5) which satisfy the estimate (4.19). Then, since the solution $u$ is unique, the $u_\varepsilon$'s converge uniformly on compact subdomains of $\Omega$ to $u$, hence the result.

REMARK 4.1. Concus and Finn [2] have been the first who used the ISC to get a bound for the solution to the capillarity problem.

In order to prove the second part of Theorem 4.1, let us observe that the *free* problem

$$J(v)\to\min \quad \text{in } BV(\Omega) \tag{4.20}$$

has a solution $u_0\in C^{0,1}(\Omega)\cap L^\infty(\Omega)$ as follows from the preceding considerations (we may formally set $\psi=-\infty$), provided that $H$ satisfies the growth conditions. Let $M_0$ be sufficiently large such that

$$u^*=u_0+M_0>\psi , \tag{4.21}$$

Then we conclude from (3.6) that we may choose