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A Degree for a Class of Acyclic-Valued Vector Fields in Banach Spaces.

M. FURI - M. MARTELLI (*)

1. - Introduction.

Let S be the unit sphere of a Banach space E . We say that an upper semicontinuous multivalued map $\varphi: S \multimap E$ is an *admissible vector field* if the following conditions are satisfied:

- i) φ is singularity free, i.e. $0 \notin \text{Im}(\varphi)$;
- ii) φ is a compact vector field, that is $\varphi(x) = x - F(x)$, where $F: S \multimap E$ is a compact multivalued map ($\overline{\text{Im}(F)}$ is compact);
- iii) $\varphi(x)$ is compact and acyclic for every $x \in F$.

We want to define a function χ (called characteristic) from the set Φ of all admissible vector fields into \mathbf{Z}_2 with the following properties:

- 1) *Normalization*: $\chi(\tau) = 1$, where $\tau: S \hookrightarrow E$ is the inclusion;
- 2) *Homotopy*: if φ_0 and φ_1 are homotopic (in a sense that will be defined later) then $\chi(\varphi_0) = \chi(\varphi_1)$;
- 3) *Solvability*: if $\chi(\varphi) \neq 0$ then φ is essential, i.e. it does not admit an admissible extension to the closed unit ball D of E ;
- 4) *Antipodality*: if $\varphi(x) = -\varphi(-x)$ for all $x \in S$ then $\chi(\varphi) = 1$.

The problem of defining a characteristic for admissible vector fields was previously studied by A. Granas - J. W. Jaworowski [9] and S. Williams [13]. The characteristic of A. Granas - J. W. Jaworowski satisfies all of the above properties but the space E considered is finite dimensional. S. Williams removed the condition that E is finite dimensional but obtained only the properties 2 and 3.

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We would like to recall that A. Granas [7] obtained a characteristic, satisfying all the properties for convex valued admissible vector fields defined in a non necessarily finite dimensional Banach space.

2. - Notations and definitions.

Throughout this paper we use the reduced Vietoris-Čech homology theory with compact carriers and coefficients in \mathbf{Z}_2 (see [6]). We denote by $\tilde{H}_p(\mathcal{X}, \mathcal{A})$ the p -th homology group of the topological pair (X, A) and we say that X is acyclic if $\tilde{H}_p(X) = \tilde{H}_p(X, \emptyset) = 0$ for every integer p .

Two admissible vector fields φ_0 and φ_1 (see Introduction) are said to be homotopic, $\varphi_0 \sim \varphi_1$, if there exists an upper semicontinuous compact multi-valued map $H: S \times I \rightarrow E$, where I is the unit interval $[0, 1]$, with the following properties:

- a) $H_t: S \rightarrow E$ defined by $H_t(x) = H(x, t)$ is an admissible vector field for every $t \in I$;
- b) $H_0 = \varphi_0$ and $H_1 = \varphi_1$.

Such a map is called an admissible homotopy joining φ_0 and φ_1 .

We shall use frequently the well-known

VIETORIS MAPPING THEOREM (see [1]). *Let X and Y be compact metric spaces and $f: X \rightarrow Y$ be continuous and such that $f^{-1}(y)$ is acyclic for every $y \in Y$ (i.e. f is inverse acyclic). Then $f_*: \tilde{H}_p(X) \rightarrow \tilde{H}_p(Y)$ is an isomorphism for all p .*

3. - The definition of characteristic.

Let $\varphi: S \rightarrow E$ be an admissible vector field and denote by F its compact part, i.e. $F(x) = x - \varphi(x)$. Put $\varrho(\varphi) = d(0, \varphi(S)) = \inf \{\|y\| : y \in \varphi(S)\}$. Obviously $\varrho(\varphi) > 0$ since $\varphi(S)$ is closed and $0 \notin \varphi(S)$. Let $p: F(S) \rightarrow E$ be a map such that $\text{Im}(p)$ is contained in a finite dimensional subspace of E and $\|p(x) - x\| < \varrho(\varphi)$ for all $x \in F(S)$; that is p is a finite dimensional $\varrho(\varphi)$ -approximation of the inclusion $i: F(S) \hookrightarrow E$. The existence of such a map is ensured by the compactness of $F(S)$ (see [8]). Let $E_{n+1} \supset \text{Im}(p)$ and denote by Γ_n the graf of F/S^n ($S^n = S \cap E_{n+1}$). We define $\hat{p}: \Gamma_n \rightarrow E$ by $\hat{p}(x, y) = x - p(y)$. We shall prove that $\hat{p}(\Gamma_n) \subset E_{n+1} \setminus \{0\}$ and so \hat{p} induces a map

$$\hat{p}_*: \tilde{H}_n(\Gamma_n) \rightarrow \tilde{H}_n(E_{n+1} \setminus \{0\}) \cong \mathbf{Z}_2.$$

Since the projection $\pi: \Gamma_n \rightarrow S^n$ (defined by $\pi(x, y) = x$) is inverse acyclic, by Vietoris Mapping Theorem we have $\tilde{H}_n(\Gamma_n) \cong \mathbf{Z}_2$. Therefore \hat{p}_* is either an isomorphism or the zero map. We put $\chi(\varphi, p, E_{n+1}) = 1$ in the first case and $\chi(\varphi, p, E_{n+1}) = 0$ in the second one. We shall prove that $\chi(\varphi, p, E_{n+1})$ is independent of $E_{n+1} \supset \text{Im}(p)$ and p . This allows us to define a characteristic $\chi(\varphi)$ of φ by putting $\chi(\varphi) = \chi(\varphi, p, E_{n+1})$ where p is an arbitrary finite dimensional $\varrho(\varphi)$ -approximation of the inclusion $i: F(S) \hookrightarrow E$ and $E_{n+1} \supset \text{Im}(p)$.

We need some preliminary Lemmas.

LEMMA 1. *Let $\hat{p}: \Gamma_n \rightarrow E$ be as above. Then $\hat{p}(\Gamma_n) \subset E_{n+1} \setminus \{0\}$.*

PROOF. Clearly $\hat{p}(\Gamma_n) \subset E_{n+1}$. Moreover $\hat{p}(x, y) = x - p(y) = (x - y) + (y - p(y))$. But $\|x - y\| \geq \varrho(\varphi)$ (since $(x - y) \in \text{Im}(\varphi)$) and $\|y - p(y)\| < \varrho(\varphi)$. Therefore $0 \notin \hat{p}(\Gamma_n)$. Q.E.D.

LEMMA 2. *Let (X, A) and (Y, B) be compact metric pairs and let $f: (X, A) \rightarrow (Y, B)$ be an inverse acyclic map such that $f^{-1}(B) = A$. Then*

$$f_*: \tilde{H}_p(X, A) \rightarrow \tilde{H}_p(Y, B)$$

is an isomorphism for all p .

PROOF. Consider the following commutative diagram

$$\begin{array}{ccccccccc} \cdots & \rightarrow & \tilde{H}_q(A) & \rightarrow & \tilde{H}_q(X) & \rightarrow & \tilde{H}_q(X, A) & \rightarrow & \tilde{H}_{q-1}(A) & \rightarrow & \tilde{H}_{q-1}(X) & \rightarrow \cdots \\ & & \downarrow f'_* & & \downarrow f'_* & & \downarrow f_* & & \downarrow f'_* & & \downarrow f'_* & \\ \cdots & \rightarrow & \tilde{H}_q(B) & \rightarrow & \tilde{H}_q(Y) & \rightarrow & \tilde{H}_q(Y, B) & \rightarrow & \tilde{H}_{q-1}(B) & \rightarrow & \tilde{H}_{q-1}(Y) & \rightarrow \cdots \end{array}$$

where f' and f'' are the maps defined by f . Since by Vietoris Mapping Theorem the maps f'_* and f''_* are isomorphisms the Five Lemma implies that f_* is an isomorphism. Q.E.D.

Let $E_n \subset E_{n+1}$ be two subspaces of E with dimensions n and $n + 1$ respectively. Call E_{n+1}^+ and E_{n+1}^- the two closed halfspaces of E_{n+1} such that $E_n = E_{n+1}^+ \cap E_{n+1}^-$ and put $S_+^n = E_{n+1}^+ \cap S$ and $S_-^n = E_{n+1}^- \cap S$.

Clearly $S^n = S \cap E_{n+1} = S_+^n \cup S_-^n$ and $S^{n-1} = S \cap E_n = S_+^n \cap S_-^n$. Denote by $\Gamma_n, \Gamma_n^+, \Gamma_n^-, \Gamma_{n-1}$ the graphs of $F/S^n, F/S_+^n, F/S_-^n, F/S^{n-1}$ respectively. We have $\Gamma_n = \Gamma_n^+ \cup \Gamma_n^-$ and $\Gamma_{n-1} = \Gamma_n^+ \cap \Gamma_n^-$.

LEMMA 3. *$(\Gamma_n, \Gamma_n^+, \Gamma_n^-)$ is a proper topological triad and the Mayer-Vietoris map $\Delta: \tilde{H}_n(\Gamma_n) \rightarrow \tilde{H}_{n-1}(\Gamma_{n-1})$ is an isomorphism.*

PROOF. Let us show first that $(\Gamma_n, \Gamma_n^+, \Gamma_n^-)$ is a proper topological triad,

i.e. the inclusions

$$\begin{aligned} (\Gamma_n^+, \Gamma_n^+ \cap \Gamma_n^-) &\xrightarrow{i} (\Gamma_n^+ \cup \Gamma_n^-, \Gamma_n^-) \\ (\Gamma_n^-, \Gamma_n^+ \cap \Gamma_n^-) &\xrightarrow{j} (\Gamma_n^+ \cup \Gamma_n^-, \Gamma_n^+) \end{aligned}$$

are excisions. Consider the following commutative diagram:

$$\begin{array}{ccc} \tilde{H}_*(\Gamma_n^+, \Gamma_{n-1}^-) & \xrightarrow{i_*} & \tilde{H}_*(\Gamma_n, \Gamma_n^-) \\ \downarrow \pi'_* & & \downarrow \pi''_* \\ \tilde{H}_*(S_+^n, S^{n-1}) & \xrightarrow{\tau_*} & \tilde{H}_*(S^n, S_-^n) \end{array}$$

where π' and π'' are the maps defined by the projection $\pi: \Gamma_n \rightarrow S^n$ and $\tau: (S_+^n, S^{n-1}) \hookrightarrow (S^n, S_-^n)$ is the inclusion.

By Lemma 2 π'_* and π''_* are isomorphisms. Therefore i_* is an isomorphism since, as it is known, τ is an excision. The proof for j is similar.

Consider now the Mayer-Vietoris sequence of $(\Gamma_n, \Gamma_n^+, \Gamma_n^-)$:

$$\dots \rightarrow \tilde{H}_q(\Gamma_n^+) \oplus \tilde{H}_q(\Gamma_n^-) \rightarrow \tilde{H}_q(\Gamma_n) \xrightarrow{\Delta} \tilde{H}_{q-1}(\Gamma_{n-1}) \rightarrow \tilde{H}_{q-1}(\Gamma_n^+) \oplus \tilde{H}_{q-1}(\Gamma_n^-) \rightarrow \dots$$

Since by Vietoris Mapping Theorem Γ_n^+ and Γ_n^- are acyclic it follows that Δ is an isomorphism. Q.E.D.

In the following two Propositions we prove that $\chi(\varphi, p, E_n)$ is independent of $E_n \supset \text{Im}(p)$ and p .

PROPOSITION 1. $\chi(\varphi, p, E_n)$ is independent of the finite dimensional subspace $E_n \supset \text{Im}(p)$.

PROOF. It is enough to prove that $\chi(\varphi, p, E_{n+1}) = \chi(\varphi, p, E_n)$ where $E_n \subset E_{n+1}$. Let $\hat{p}': \Gamma_n \rightarrow E_{n+1} \setminus \{0\}$ and $\hat{p}'': \Gamma_{n-1} \rightarrow E_n \setminus \{0\}$ be the two maps defined by p . The following diagram is commutative by the naturality of Mayer-Vietoris map Δ :

$$\begin{array}{ccc} \tilde{H}_n(\Gamma_n) & \xrightarrow{\Delta} & \tilde{H}_{n-1}(\Gamma_{n-1}) \\ \downarrow \hat{p}'_* & & \downarrow \hat{p}''_* \\ \tilde{H}_n(E_{n+1} \setminus \{0\}) & \xrightarrow{\Delta} & \tilde{H}_{n-1}(E_n \setminus \{0\}) \end{array}$$

It is known that the bottom arrow is an isomorphism. The same is true for the top one by Lemma 3. Thus the Proposition is proved. Q.E.D.

According to Proposition 1 we can define now $\chi = \chi(\varphi, p, E_n)$ where E_n is any finite dimensional subspace of E containing $\text{Im}(p)$.

PROPOSITION 2. Let p_0 and p_1 be two finite dimensional $\varrho(\varphi)$ -approximations of $i: F(S) \hookrightarrow E$.

Then $\chi(\varphi, p_0) = \chi(\varphi, p_1)$.

PROOF. For any $t \in [0, 1]$ define $p_t(y) = (1 - t)p_0(y) + tp_1(y)$. Since there exists E_{n+1} containing $\text{Im}(p_0)$ and $\text{Im}(p_1)$ we have $\text{Im}(p_t) \subset E_{n+1}$ for any $t \in [0, 1]$. It is easy to see that p_t is a $\varrho(\varphi)$ -approximation of $i: F(S) \hookrightarrow E$. Thus by Lemma 1 the map $\hat{p}_t(x, y) = x - p_t(y)$ is such that $0 \notin \text{Im}(\hat{p}_t)$ for all $t \in [0, 1]$. Therefore $\{\hat{p}_t\}$ is a homotopy from Γ_n into $E_{n+1} \setminus \{0\}$ joining p_0 and p_1 . Q.E.D.

Let φ be an admissible vector field. We define its characteristic:

$$\chi(\varphi) \stackrel{\text{def}}{=} \chi(\varphi, p, E_n)$$

where p is a finite dimensional $\varrho(\varphi)$ -approximation of $i: F(S) \hookrightarrow E$ and $E_n \supset \text{Im}(p)$. The above two Propositions insure that $\chi(\varphi)$ is well defined.

4. - The four properties of the characteristic.

In this section we shall prove that our definition of the characteristic of an admissible vector field satisfies the four properties we have mentioned in the Introduction.

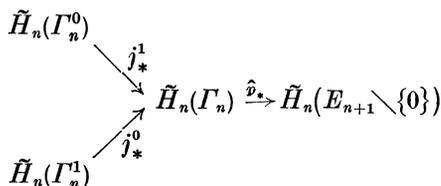
THEOREM 1 (Normalization). *Let $\tau: S \hookrightarrow E$ be the inclusion. Then $\chi(\tau) = 1$.*

PROOF. It is trivial. Q.E.D.

THEOREM 2 (Homotopy). *Let φ_0 and φ_1 be two homotopic admissible vector fields. Then $\chi(\varphi_0) = \chi(\varphi_1)$.*

PROOF. Let $H: S \times I \dashrightarrow E$ be an admissible homotopy joining φ_0 and φ_1 . Denote by $F_t: S \dashrightarrow E$ the map $F_t(x) = F(x, t)$, where F is the compact part of H (i.e. $F(x, t) = x - H(x, t)$). Let $p: \text{Im}(F) \rightarrow E$ be a finite dimensional $\varrho(\varphi)$ -approximation of $i: \text{Im}(F) \hookrightarrow E$ and take $E_{n+1} \supset \text{Im}(p)$. Denote by $\Gamma_n, \Gamma_n^0, \Gamma_n^1$ the graphs of $F/S^n \times I, F_0/S^n$ and F_1/S^n respectively.

Consider the following diagram:



where $\hat{p}(x, t, y) = x - p(y)$ and $j^0(x, y) = (x, 0, y), j^1(x, y) = (x, 1, y)$.

Clearly the characteristic of $\varphi_k, k = 0, 1$, is defined by the composite map $\hat{p}_* \circ j_*^k$; in other words $\chi(\varphi_k) = 1$ if $\hat{p}_* \circ j_*^k$ is an isomorphism and $\chi(\varphi_k) = 0$

otherwise. Therefore the Theorem is proved if we show that j_*^1 and j_*^0 are isomorphisms.

Consider the following commutative diagram:

$$\begin{array}{ccc}
 \Gamma_n^k & \xrightarrow{j^k} & \Gamma_n \\
 \downarrow \pi^k & & \downarrow \pi \\
 S^n & \xleftarrow{q} & S^n \times I
 \end{array}$$

where $\pi^k(x, y) = x$, $\pi(x, t, y) = (x, t)$ and $q(x, t) = x$. By Vietoris Mapping Theorem π_*^k , π_* , and q_* are isomorphisms, therefore also j_*^k is an isomorphism.

Q.E.D.

Let $\varphi: S \rightarrow E$ be an admissible vector field. By an admissible extension of φ we mean a multivalued map $\tilde{\varphi}$ from the closed unit ball D of E into E such that:

- i) $\tilde{\varphi}$ is singularity free;
- ii) $\tilde{\varphi}(x) = x - \tilde{F}(x)$, where $\tilde{F}: D \rightarrow E$ is a compact multivalued map;
- iii) $\tilde{\varphi}(x)$ is compact and acyclic for every $x \in D$;
- iv) $\tilde{\varphi}|_S = \varphi$.

We shall call φ inessential if it admits an admissible extension and essential otherwise (see [8]).

THEOREM 3 (Solvability). *Let φ be an admissible vector field. Then if $\chi(\varphi) \neq 0$, φ is essential.*

PROOF. Assume that φ is inessential. We shall prove that $\chi(\varphi) = 0$. Consider an admissible extension $\tilde{\varphi}$ of φ . Let p be a finite dimensional $\varrho(\tilde{\varphi})$ -approximation of $i: \tilde{F}(D) \hookrightarrow E$ and $E_{n+1} \supset \text{Im}(p)$. Denote by $\tilde{\Gamma}_n$ and Γ_n the graphs of $\tilde{\varphi}|_{D_{n+1}}$ and $\tilde{\varphi}|_{S^n}$ respectively, where $D_{n+1} = D \cap E_{n+1}$ and $S^n = S \cap E_{n+1}$.

We have the following commutative diagram:

$$\begin{array}{ccc}
 \tilde{H}_n(\Gamma_n) & & \\
 \downarrow j_* & \searrow \hat{p}_* & \\
 & & \tilde{H}_n(E_{n+1} \setminus \{0\}) \\
 & \nearrow \tilde{p}_* & \\
 \tilde{H}_n(\tilde{\Gamma}_n) & &
 \end{array}$$

where j is the inclusion of Γ_n into $\tilde{\Gamma}_n$, $p(x, y) = x - p(y)$ and $\hat{p} = \tilde{p} \circ j$. By

Victoris Mapping Theorem \tilde{I}_n is acyclic. So \hat{p}_* , which defines the characteristic of φ , is the zero map. Therefore $\chi(\varphi) = 0$. Q.E.D.

We recall that a multivalued map $\varphi: S \multimap E$ is odd if $\varphi(x) = -\varphi(-x)$, for any $x \in S$. In order to prove that the characteristic of an odd admissible vector field is one we need the following result due to J. W. Jaworowski [11].

Let X be a metric space such that $\tilde{H}_p(X) = 0$ if $0 \leq p \leq n-1$ and $\tilde{H}_n(X) \cong \mathbf{Z}_2$. Let $\sigma: X \rightarrow X$ be a continuous involution (i.e. $\sigma \circ \sigma$ is the identity on X) and $f: X \rightarrow S^n$ a continuous map such that $f(x) \neq f(\sigma(x))$ for any $x \in X$. Then $f_: \tilde{H}_n(X) \rightarrow \tilde{H}_n(S^n)$ is an isomorphism.*

THEOREM 4 (Antipodality) *Let $\varphi: S \multimap E$ be an odd admissible vector field. Then $\chi(\varphi) = 1$.*

PROOF. Let $q: F(S) \rightarrow E$ be a finite dimensional $\varrho(\varphi)$ -approximation of $i: F(S) \hookrightarrow E$ and $E_{n+1} \supset \text{Im}(q)$. Since $F(S)$ is symmetric with respect to the origin we can define $p(y) = (q(y) - q(-y))/2$ for any $y \in F(S)$. It is easy to see that $p(y) = -p(-y)$, $\|y - p(y)\| < \varrho(\varphi)$ and $\text{Im}(p) \subset E_{n+1}$.

Let I_n and \hat{p} be as usual. Since F is odd we can define an involution σ on I_n by $\sigma(x, y) = (-x, -y)$. We have $\hat{p}(\sigma(x, y)) = \hat{p}(-x, -y) = -x + p(y) \neq x - p(y) = \hat{p}(x, y)$. Then by Jaworowski's Theorem \hat{p}_* is an isomorphism. Q.E.D.

5. - Some consequences.

In this section we give some applications of the characteristic. In particular we give a version of the Sweeping Theorem and we extend to acyclic valued maps the Theorems of Birkoff-Kellogg [3] and Rothe [12].

We need the following Lemma.

LEMMA 4. *The characteristic of the admissible vector field $\varphi(x) = x - x_0$, $x_0 \notin S$ is one if $\|x_0\| < 1$ and zero otherwise.*

PROOF. If $\|x_0\| < 1$ then $H(x, t) = x - tx_0$ defines a homotopy between the inclusion $\tau: S \hookrightarrow E$ and φ . Thus $\chi(\varphi) = 1$ by Theorems 1 and 2.

If $\|x_0\| > 1$ the map $\tilde{\varphi}: D \rightarrow E$ defined by $\tilde{\varphi}(x) = x - x_0$ is an admissible extension of φ . Then $\chi(\varphi) = 0$ by Theorem 3. Q.E.D.

We remark that two admissible vector fields of the form $\varphi_0(x) = x - x_0$ and $\varphi_1(x) = x - x_1$ have the same characteristic if and only if x_0 and x_1 belong to the same component of $E \setminus S$.

Let $F: S \times I \multimap E$ be an upper semicontinuous compact acyclic valued map. Define $\Psi: S \times I \multimap E$ by $\Psi(x, t) = x - F(x, t)$ and put $\psi_t(x) = \Psi(x, t)$.

THEOREM 5 (On the Sweeping). *Let Ψ be as above and assume that $\psi_0(x) = x$. If x_0, x_1 belong to different components of $E \setminus S$ and to the same component of $E \setminus \psi_1(S)$ then either $x_0 \in \text{Im}(\Psi)$ or $x_1 \in \text{Im}(\Psi)$.*

PROOF. Assume that $x_0, x_1 \notin \text{Im}(\Psi)$. We shall prove that if they belong to the same component of $E \setminus \psi_1(S)$ then they belong to the same component of $E \setminus S$.

Since $\psi_1(S)$ is closed we can find a continuous map $\alpha: I \rightarrow E \setminus \psi_1(S)$ such that $\alpha(0) = x_0$ and $\alpha(1) = x_1$. Define

$$\sigma'_i(x) = \Psi(x, 1) - \alpha(t),$$

$$\sigma''_i(x) = \Psi(x, t) - x_0,$$

$$\sigma'''_i(x) = \Psi(x, t) - x_1.$$

Clearly $\sigma'_0 \sim \sigma'_1$, $\sigma''_0 \sim \sigma''_1$ and $\sigma'''_0 \sim \sigma'''_1$.

Moreover $\sigma'_1 = \sigma'''_1$ and $\sigma'_0 = \sigma'''_0$. Therefore $\sigma''_0 \sim \sigma''_0$. Since $\sigma''_0(x) = x - x_0$ and $\sigma'''_0(x) = x - x_1$ it follows that x_0, x_1 belong to the same component of $E \setminus S$ by Lemma 4. Q.E.D.

COROLLARY 1. *Let $\Psi: S \times I \rightarrow E$ be as in Theorem 5. If $y \in E$ belongs to the unbounded component of $E \setminus \Psi_1^{(S)}(S)$ and $\|y\| < 1$ then $y \in \text{Im}(\Psi)$.*

PROOF. There exists $z \in E$ such that $\|z\| > 1$, $z \notin \text{Im}(\Psi)$ and z belongs to the unbounded component of $E \setminus \psi_1(S)$. It follows that y and z belong to different components of $E \setminus S$ and to the same component of $E \setminus \psi_1(S)$. Therefore by Theorem 5 $y \in \text{Im}(\Psi)$. Q.E.D.

Let $F: S \rightarrow E$ be an upper semicontinuous compact acyclic valued map. We shall say that F has an invariant direction if the equation $x \in \lambda F(x)$ has a solution for some $\lambda > 0$ and $x \in S$.

We give now a multivalued version of the Birkoff-Kellogg Theorem [3].

THEOREM 6. *Let $F: S \rightarrow E$ be as above. Assume that E is infinite dimensional and $0 \notin F(S)$. Then F has an invariant direction.*

PROOF. Define $\varphi_\lambda(x) = x - \lambda F(x)$. We want to prove that $0 \in \varphi_\lambda(S)$ for some $\lambda > 0$. Since zero belongs to the bounded component of $E \setminus S$ it is enough to show, by Corollary 1, that there exists $\lambda_0 > 0$ such that zero belongs to the unbounded component of $E \setminus \varphi_{\lambda_0}(S)$. We shall prove in fact that there exists $\lambda_0 > 0$ and $x_0 \in S$ such that the half line $A(x_0) = \{tx_0: t \geq 0\}$ does not intersect $\varphi_{\lambda_0}(S)$.

Let $\pi: E \setminus \{0\} \rightarrow S$ defined by $\pi(x) = x/\|x\|$. By assumption $\overline{F(S)}$ is a compact subset of $E \setminus \{0\}$, so $\pi(\overline{F(S)}) \subset S$ is compact. Therefore, since $\dim E = +\infty$, there exists $x_0 \in S$ such that $x_0 \notin -\pi(\overline{F(S)})$. It follows that $A(x_0) \cap \{-\overline{F(S)}\} = \emptyset$. Since $E \setminus A(x_0)$ is an open neighborhood of the compact set $-\overline{F(S)}$ there exists $\varepsilon > 0$ such that $A(x_0) \cap B(-F(S), \varepsilon) = \emptyset$, where $B(-F(S), \varepsilon) = \{x \in E: \text{there exists } y \in -F(S) \text{ such that } \|x - y\| < \varepsilon\}$. Let λ_0 be such that $\lambda_0 \varepsilon > 1$. Since $\lambda_0 A(x_0) = A(x_0)$ we have $A(x_0) \cap B(-\lambda_0 F(S), \lambda_0 \varepsilon) = \emptyset$. But $\varphi_{\lambda_0}(S) \subset S - \lambda_0 F(S) \subset B(-\lambda_0 F(S), \lambda_0 \varepsilon)$ and so $A(x_0) \cap \varphi_{\lambda_0}(S) = \emptyset$. Q.E.D.

By a positive cone in a Banach space E we mean a convex subset Q of E such that $Q \cap \{-Q\} = \{0\}$ and $\lambda Q = Q$ for every $\lambda > 0$.

THEOREM 7. *Let $F: S \cap Q \rightarrow Q$ be an upper semicontinuous compact acyclic valued map. Assume that E is infinite dimensional and $0 \notin F(Q)$. Then F has an invariant direction.*

PROOF. The set $S \cap Q$ is an AR , since it is a retract of the convex set $Q \setminus \{0\}$ (see [10]). Therefore there exists a retraction $r: S \rightarrow S \cap Q$. Consider the map $F \circ r: S \rightarrow Q$ and apply Theorem 6. Q.E.D.

We remark that the above result could be proved, using the Eilenberg-Montgomery Theorem [5], also in the finite dimensional case since $Q \setminus B(0, \varepsilon)$ is an AR for every $\varepsilon > 0$.

The following result is an extension of the well-known Rothe's Theorem [12].

THEOREM 8. *Let $F: D \rightarrow E$ be an upper semicontinuous compact acyclic valued map such that $x \in S$ and $\lambda > 1$ implies $\lambda x \notin F(x)$. Then F has a fixed point.*

PROOF. We can assume that F is fixed point free on S . The two admissible vector fields $\varphi_0(x) = x$ and $\varphi_1(x) = x - F(x)$ are homotopic via the map $\Phi: S \times I \rightarrow E$ defined by $\Phi(x, t) = x - tF(x)$. In fact, $x \in S$ and $0 < t < 1$ implies $x \notin tF(x)$ and so $0 \notin x - tF(x)$. Therefore φ_1 is essential since $\chi(\varphi_1) = \chi(\varphi_0) = 1$. Q.E.D.

COROLLARY 2 (Eilenberg-Montgomery [5]). *Let X be a compact metric AR and $F: X \rightarrow X$ be an upper semicontinuous acyclic valued map. Then F has a fixed point.*

PROOF. By a well-known result of Kuratowski (see [4]) we can regard X as a closed subset of a Banach space E . Since X is an AR there exists a retraction $r: E \rightarrow X$. Let $\varrho > 0$ be such that $X \subset D_\varrho$ where $D_\varrho = \{x \in E: \|x\| \leq \varrho\}$. Since $F \circ r(D_\varrho) \subset X \subset D_\varrho$ by Theorem 8 $F \circ r$ has a fixed point x . On the other hand x must belong to X and so it is a fixed point of F . Q.E.D.

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