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A Fubini-Type Theorem for Finitely Additive Measure Spaces.

WILLIAM D. L. APPLING (*)

1. – Introduction.

If $\boldsymbol{F}$ is a field of subsets of a set U, then $\mathfrak{p}_F$ denotes the set of all functions from $\boldsymbol{F}$ into $\exp(\mathbf{R})$, $\mathfrak{p}_{FAB}$ denotes the set of all functions from $\boldsymbol{F}$ into $\mathbf{R}$ that are bounded and finitely additive, and $\mathfrak{p}_{FA}^+$ denotes the set of all nonnegative-valued elements of $\mathfrak{p}_{FAB}$ Throughout this paper all integrals will be limits, for refinements of (finite) subdivisions, of the appropriate sums (see section 2). Furthermore, L and G, with appropriate symbols affixed, when necessary, to denote the particular field of sets under consideration, will denote, respectively, the « sum supremum » and « sum infimum » functional (see section 2).

Suppose each of $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ is a field of subsets of the sets U_1 and U_2, respectively, μ_1 and μ_2 are in $\mathfrak{p}_{F_1A}^+$ and $\mathfrak{p}_{F_2A}^+$, respectively, and $\{U_1\times U_2, \boldsymbol{F}_3, \mu_3\}$ is the associated product space, i.e., $\boldsymbol{F}_3$ is the smallest field of subsets of $U_1\times U_2$ including $\{X'\times X'': X' \text{ in } \boldsymbol{F}_1, X'' \text{ in } \boldsymbol{F}_2\}$, and μ_3 is the extension to $\boldsymbol{F}_3$ of the function ν defined on the immediately previously mentioned set by

$$\nu(X'\times X'') = \mu_1(X')\mu_2(X'') .$$

Suppose α is a function from $\boldsymbol{F}_3$ into $\exp(\mathbf{R})$. Suppose β is a function with domain

$$\{(x, I, y, J): I \text{ in } \boldsymbol{F}_1,\ J \text{ in } \boldsymbol{F}_2,\ (x, y) \text{ in } I\times J\}$$

such that if I is in $\boldsymbol{F}_1$, J is in $\boldsymbol{F}_2$ and (x, y) is in $I\times J$, then $\beta(x, I, y, J)\subseteq \subseteq\alpha(I\times J)$.

In this paper we prove two theorems, the second of which is a consequence and a generalization of the first and is an analogue, for finitely additive measure spaces, of Fubini's Theorem.

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We first treat (section 3) the « bounded » case; a very simple consequence of very simple inequalities involving L-sums and G-sums with respect to $\boldsymbol{F}_1$, $\boldsymbol{F}_2$ and $\boldsymbol{F}_3$:

THEOREM 3.1. Suppose α has bounded range union, and the integral (section 2)

$$\int_{U_1\times U_2}\alpha\mu_3$$

exists. Then if Q is either L or G, then each of the integrals

$$\int_{U_1}\Big[\int_{U_2}Q(\beta(x, I, \ldots)\mu_2)\Big]\mu_1(I)$$

and

$$\int_{U_2}\Big[\int_{U_1}Q(\beta(\ldots, y, J)\mu_1)\Big]\mu_2(J)$$

exists and is

$$\int_{U_1\times U_2}\alpha\mu_3 .$$

Note, for example, that one consequence of the above theorem is the equation:

$$0=\int_{U_1}\Big[\int_{U_2}L(\beta(x, I, \ldots)\mu_2)-\int_{U_2}G(\beta(x, I, \ldots)\mu_2)\Big]\mu_1(I),$$

which from basic considerations about upper and lower integrals (section 2) implies that if for some ω,

$$\omega(x, \ldots)=\beta(x, I, \ldots)=\beta(x, U_1, \ldots), \quad x \text{ in } I, \quad I \text{ in } \boldsymbol{F}_1,$$

then the set of all x for which

$$\int_{U_2}\omega(x, \ldots)\mu_2$$

does not exist (if any) has μ_1 outer measure 0.

We then extend Theorem 3.1 to the « summable » (section 2) case (section 4):

THEOREM 4.1. Suppose α is μ_3-summable and s_{μ_3} is the μ_3-summability operator (section 2). Suppose that if $p \leqslant 0 \leqslant q$, then $\alpha_{p,q}$ denotes $\max\{\min\{\alpha, q\}, p\}$ and $\beta_{p,q}$ denotes $\max\{\min\{\beta, q\}, p\}$. Then, if Q is L or G, then

$$\int_{U_1}\left|s_{\mu_3}(\alpha)(V\times U_2)/\mu_1(V) - \int_{U_2} Q(\beta_{p,q}(x, V, \ldots)\mu_2)\right|\mu_1(V) =$$
$$\int_{U_1}\left|s_{\mu_3}(\alpha)(V\times U_2) - \int_V\left[\int_{U_2} Q(\beta_{p,q}(x, I, \ldots)\mu_2)\mu_1(I)\right]\right| \leqslant$$
$$\int_{U_1\times U_2}\left|s_{\mu_3}(\alpha)(W) - \int_W \alpha_{p,q}(W')\mu_3(W')\right| \to 0\,, \qquad \min\{-p, q\}\to\infty\,,$$

and

$$\int_{U_2}\left|s_{\mu_3}(\alpha)(U_1\times Y)/\mu_2(Y) - \int_{U_1} Q(\beta_{p,q}(\ldots, y, Y)\mu_1)\right|\mu_2(Y) =$$
$$\int_{U_2}\left|s_{\mu_3}(\alpha)(U_1\times Y) - \int_Y\left[\int_{U_1} Q(\beta_{p,q}(\ldots, y, J)\mu_1)\right]\mu_2(J)\right| \leqslant$$
$$\int_{U_1\times U_2}\left|s_{\mu_3}(\alpha)(W) - \int_W \alpha_{p,q}(W')\mu_3(W')\right| \to 0\,, \qquad \min\{-p, q\}\to\infty\,.$$

We close this introduction with some remarks, partly heuristic, to clarify certain respects in which Theorem 4.2 is an analogue of Fubini's Theorem. First let us observe (section 2) that the notion of set function summability, which we shall discuss in section 2, is an analogue of Lebesgue integrability; the number

$$s_\mu(\alpha)(V)\,,$$

for a μ-summable set function α corresponding to the Lebesgue integral

$$\int_V f d_\mu$$

for a Lebesgue integrable point function f. With this in mind, then, we see that the conclusion of a summability analogue of Fubini's Theorem would ideally have the form, given the hypothesis of Theorem 4.2,

$$\text{«}\ s_{\mu_1}(s_{\mu_2}(\beta(x, I, \ldots))(U_2))(U_1) = s_{\mu_3}(\alpha)(U_1\times U_2) = s_{\mu_2}(s_{\mu_1}(\beta(\ldots, y, J))(U_1))(U_2)\ \text{»}\,.$$

Naturally, in a fashion analogous to Fubini's Theorem, $s_{\mu_2}(\beta(x, I, \ldots))(U_2)$ and $s_{\mu_1}(\beta(\ldots, y, J))(U_1)$ are not necessarily defined for all x, I and y, J, respectively, in the sense that it is not necessarily true that $\int\limits_{U_1}\beta_{p,q}(\ldots, y, J)\mu_1$ and $\int\limits_{U_2}\beta_{p,q}(x, I, \ldots)\mu_2$ exist for all p, q with $p \leqslant 0 \leqslant q$ and y, J and x, I, and it is not necessarily true that $\lim\int\limits_{U_1} Q(\beta_{p,q}(\ldots, y, J)\mu_1)$ as $\min\{-p, q\} \to \infty$ and $\lim\int\limits_{U_2} Q(\beta_{p,q}(x, I, \ldots)\mu_2)$ as $\min\{-p, q\} \to \infty$ exist for all y, J and x, I. We seek a conclusion « of the form »

$$« \, \boldsymbol{s}_{\mu_1}\Big(\lim_{-p,q}\int\limits_{U_2} Q(\beta_{p,q}(x, I, \ldots)\mu_2)\Big)(U_1) = \boldsymbol{s}_{\mu_3}(\alpha)(U_1 \times U_2) =$$

$$\boldsymbol{s}_{\mu_2}\Big(\lim_{-p,q}\int\limits_{U_1} Q(\beta_{p,q}(\ldots, y, J)\mu_1)\Big)(U_2) \, ».$$

With this in mind, let us examine the left side of the immediately preceding « equation »; similar remarks will clearly hold for the right side. If by « $\lim\limits_{-p,q}\int\limits_{U_2} Q(\beta_{p,q}(x, I, \ldots)\mu_2)$ » we mean an element $\mathfrak{H}$ of $\mathfrak{p}_{F_1}$ with respect to which and the function « sequence »

$$\Big\{\Big((p, q), \Big\{\Big(I, \Big\{\int\limits_{U_2} Q(\beta_{p,q}(x, I, \cdot, \cdot)\mu_2) : \ x \text{ in } I\Big\}\Big) : \ I \text{ in } F_1\Big\}\Big) : \ p \leqslant 0 \leqslant q\Big\}$$

the hypothesis of a certain dominated convergence theorem [3] (see section 2) holds, so that $\mathfrak{H}$ is μ_1 summable and

$$\boldsymbol{s}_{\mu_1}\Big(\Big|\mathfrak{H} - \int\limits_{U_2} Q(\beta_{p,q}(x, I, \ldots)\mu_2)\Big|\Big)(U_1) \to 0 \,, \qquad \min\{-p, q\} \to \infty \,,$$

then we are done, inasmuch as it follows (section 4) that the function

$$\boldsymbol{s}_{\mu_3}(\alpha)(\cdot \times U_2)/\mu_1$$

satisfies such conditions and (see section 2) has the further property that

$$\boldsymbol{s}_{\mu_1}(\boldsymbol{s}_{\mu_3}(\alpha)(\cdot \times U_2)/\mu_1)(U_1) = \boldsymbol{s}_{\mu_3}(\alpha)(U_1 \times U_2) \,.$$

2. – Preliminary theorems and definitions.

We refer the reader to [1] for the notions of subdivision, refinement, integral, $\sum$-boundedness, sum supremum functional and sum infimum functional. The statement « $\mathfrak{E} \ll \mathfrak{D}$ » shall mean $\mathfrak{E}$ is a refinement of $\mathfrak{D}$. The sum supremum functional and sum infimum functional will be denoted respectively, by L and G, and when there is no possibility of confusion with respect to which field of subsets of a given set each is defined, identifying subscripts will be omitted. We also refer the reader to [1] for the basic inequalities for « L-sums » and « G-sums » and the resulting existence assertions for the corresponding « upper integrals » and « lower integrals ». The reader is referred to [1] for a statement of Kolmogoroff's differential equivalence theorem [4] and its implications about the existence and equivalence of the integrals that we shall use. We further refer the reader to [1] for certain refinement-sum inequalities involving various set functions, as well as certain integral existence assertions that follow from these inequalities. In this paper, when the existence of an integral or its equivalence to an integral is an easy consequence of the above mentioned material, the integral need only be written or the equivalence assertion made, and the proof left to the reader.

Let us note that in most of the references cited, the definitions and theorems referred to are usually for « single-valued » set functions. Nevertheless, these definitions and theorems carry over for the « many-valued » set functions that we consider in this paper with only minor modifications, and we therefore take the liberty of stating all cited definitions and theorems in « many-valued » form.

We shall also take certain notational liberties, when there is no danger of misunderstanding, particularly with functions defined in terms of integrals, e.g.,

$$\int_{U_2} \mathfrak{K}(I,\cdot)$$

can, in one expression, denote the value for I of the function

$$\left\{\left(J, \int_{U_2} \mathfrak{K}(J,\cdot)\right) : J \text{ in } \boldsymbol{F}\right\},$$

and in another expression, such as

$$s_{\mu_1}\left(\mathfrak{H} - \int_{U_2} \mathfrak{K}(I,\cdot)\right)(U_1),$$

denote the above defined function itself. Furthermore, throughout this paper we shall, in various expressions involving integrals, again when misunderstanding can be avoided, not write the « variable of integration », e.g.,

$$\int_V \alpha(I) \;\; vs. \int_V \alpha\,.$$

We now make some remarks about the notion of set function summability [1]. Suppose $\{U, \boldsymbol{F}, \mu\}$ is a finitely additive measure space. If α is in $\mathfrak{p}_{\boldsymbol{F}}$, then the statement that α is μ-summable is equivalent to α's being of the form $\beta_1 - \beta_2$, such that if γ is β_1 or β_2, then γ is a function from $\boldsymbol{F}$ into a collection of nonnegative number sets such that for some number z and all $K \geqslant 0$,

$$\int_U \min\{\gamma(I), K\}\mu(I)$$

exists and does not exceed z. We state a previous characterization theorem of the author [2].

THEOREM 2.A.1. If α is in $\mathfrak{p}_{\boldsymbol{F}}$, then α is μ-summable iff there is an element ϑ of $\mathfrak{p}_{\boldsymbol{FAB}}$, absolutely continuous with respect to μ, such that if $p \leqslant 0 \leqslant q$, then

$$\int_U \max\{\min\{\alpha(I), q\}, p\}\mu(I)$$

exists, and

$$\int_U \left|\vartheta(I) - \int_I \max\{\min\{\alpha(J), q\}, p\}\mu(J)\right| \to 0\,, \qquad \min\{-p, q\} \to \infty\,.$$

It is easy to see that for each α in $\mathfrak{p}_{\boldsymbol{F}}$ which is μ-summable, the ϑ associated with α by Theorem 2.A.1 is unique, and we shall denote this ϑ by $\boldsymbol{s}_\mu(\alpha)$.

We state a condensation of a few of the results of [1].

THEOREM 2.A.2. If each of ϱ and β is in $\mathfrak{p}_{\boldsymbol{F}}$ and is μ-summable, then so is $\varrho + \beta$, $c\varrho$ for each c in $\mathbb{R}$, $\min\{\varrho, \beta\}$ and $\max\{\varrho, \beta\}$. Furthermore,

$$\boldsymbol{s}_\mu(\varrho + \beta) = \boldsymbol{s}_\mu(\varrho) + \boldsymbol{s}_\mu(\beta)\,,$$

$$\boldsymbol{s}_\mu(c\varrho) = c\boldsymbol{s}_\mu(\varrho)$$

or each c in $\mathbb{R}$, and for each V in $\boldsymbol{F}$,

$$\int_V \min\{\boldsymbol{s}_\mu(\varrho)(I), \boldsymbol{s}_\mu(\beta)(I)\} = \boldsymbol{s}_\mu(\min\{\varrho, \beta\})(V)$$

and

$$\int_V \max\{\boldsymbol{s}_\mu(\varrho)(I), \boldsymbol{s}_\mu(\beta)(I)\} = \boldsymbol{s}_\mu(\max\{\varrho, \beta\})(V)\,.$$

We now discuss the dominated convergence theorem mentioned in the introduction. In a previous paper [3] the author proved the following:

THEOREM 2.A.3. Suppose $\mathfrak{H}$ is in $\mathfrak{p}_F$, $\{\beta_i\}_{i=1}^{\infty}$ is a sequence of elements of $\mathfrak{p}_F$, $\varkappa$ is in $\mathfrak{p}_{FA}^{+}$ and absolutely continuous with respect to μ such that if n is a positive integer, then β_n is μ-summable and $\varkappa - \boldsymbol{s}_\mu(|\beta_n|)$ is nonnegative-valued. Suppose if $0 < \min\{c, d\}$, then there is a positive integer N such that if n is a positive integer $\geqslant N$, then there is $\mathfrak{D}_n \ll \{U\}$ such that if $\mathfrak{E} \ll \mathfrak{D}_n$, for each I in $\mathfrak{E}$, $h(I)$ is in $\mathfrak{H}(I)$ and $b_n(I)$ is in $\beta_n(I)$ and

$$\mathfrak{E}^* = \{I\colon\ I \text{ in } \mathfrak{E},\ |h(I) - b_n(I)| \geqslant c\},$$

then

$$\sum_{\mathfrak{E}^*} \mu(I) < d\,.$$

Then $\mathfrak{H}$ is μ-summable and

$$\boldsymbol{s}_\mu(|\mathfrak{H} - \beta_n|)(U) \to 0\,, \qquad n \to \infty\,.$$

It is easy to see that Theorem 2.A.3 can be put in the following extended form, and it is this form to which we shall refer in our closing remarks following the proof of Theorem 4.1.

THEOREM 2.A.4. Suppose $\mathfrak{H}$ is in $\mathfrak{p}_F$, S is a set with partial ordering « $\leqslant^*$ », with respect to which S is directed, for each x in S, β_x is in $\mathfrak{p}_F$, $\varkappa$ is an element of $\mathfrak{p}_{FA}^{+}$ absolutely continuous with respect to μ such that if x is in S, then β_x is μ-summable and $\varkappa - \boldsymbol{s}_\mu(|\beta_x|)$ is nonnegative-valued. Suppose if $0 < \min\{c, d\}$, then there is an X in S such that if y is in S and $X \leqslant^* y$, then there is $\mathfrak{D}_y \ll \{U\}$ such that if $\mathfrak{E} \ll \mathfrak{D}_y$, for each I in $\mathfrak{E}$, $h(I)$ is in $\mathfrak{H}(I)$ and $b_y(I)$ is in $\beta_y(I)$ and

$$\mathfrak{E}^* = \{I\colon\ I \text{ in } \mathfrak{E},\ |h(I) - b_y(I)| \geqslant c\},$$

then

$$\sum_{\mathfrak{E}^*} \mu(I) < d\,.$$

Then $\mathfrak{H}$ is μ-summable and

$$\boldsymbol{s}_\mu(|\mathfrak{H} - \beta_z|)(U) \to 0\,,$$

the limit, of course, with respect to $\leqslant^*$.

We state a theorem, the proof of which is quite routine and which will be left to the reader, that we will cite at the end of the paper.

THEOREM 2.1. Suppose S is a set with partial ordering « $\leqslant^*$ » with respect to which S is directed. Suppose that for each x in S, $\mathfrak{B}_x$ is an element of $\mathfrak{p}_F$ with range union a subset of the nonnegative numbers, such that $\int_U \mathfrak{B}_x\mu$ exists. Suppose

$$\int_U \mathfrak{B}_x\mu \to 0 ,$$

limit with respect to $\leqslant^*$. Then, if $0<\min\{c, d\}$, then there is X in S such that if y is in S and $X\leqslant^* y$, then there is $\mathfrak{D}_y \ll \{U\}$ such that if $\mathfrak{E} \ll \mathfrak{D}_y$, for each I in E, $b_y(I)$ is in $\mathfrak{B}_y(I)$ and

$$\mathfrak{E}^* = \{I\colon\ I \text{ in } \mathfrak{E},\ b_y(I) \geqslant c\} ,$$

then

$$\sum_{\mathfrak{E}^*} \mu(I) < d .$$

Finally, we end this section with a theorem [3] that substantiates the final assertion of the introduction.

THEOREM 2.A.5. If μ is in $\mathfrak{p}_{FA}^+$, ϑ is in $\mathfrak{p}_{FAB}$ and absolutely continuous with respect to μ, then ϑ/μ is μ-summable and $\mathbf{s}_\mu(\vartheta/\mu) = \vartheta$.

3. – The bounded case.

In this section we prove Theorem 3.1, as stated in the introduction.

PROOF OF THEOREM 3.1. Suppose $0<c$. There is a subdivision $\mathfrak{D}$ of $U_1\times U_2$ such that if $\mathfrak{E} \ll \mathfrak{D}$, then

$$\max\left\{\left|\int_{U_1\times U_2}\alpha\mu_3 - \sum_{\mathfrak{E}} L(\alpha\mu_3)(W)\right|, \left|\int_{U_1\times U_2}\alpha\mu_3 - \sum_{\mathfrak{E}} G(\alpha\mu_3)(W)\right|\right\} < c .$$

There is a subdivision $\mathfrak{D}_1$ of U_1 and $\mathfrak{D}_2$ of U_2 such that

$$\mathfrak{D}_3 = \{I\times J\colon\ I \text{ in } \mathfrak{D}_1 \text{ and } J \text{ in } \mathfrak{D}_2\} \ll \mathfrak{D} .$$

First, suppose I is in F_1, x is in I and J is in F_2. We see that

$$L(\beta(x, I, \ldots)\mu_2)(J)\,\mu_1(I) = \mu_1(I) \sup\Big\{\sum_{\mathfrak{H}} b(x, I, y, V)\mu_2(V):$$

$$\mathfrak{H} \ll \{J\},\ V \text{ in } \mathfrak{H},\ y \text{ in } V,\ b(x, I, y, V) \text{ in } \beta(x, I, y, V)\Big\} =$$

$$\sup\Big\{\sum_{\mathfrak{H}} b(x, I, y, V)\mu_1(I)\mu_2(V): \text{ same as before}\Big\} \leqslant$$

$$\sup\Big\{\sum_{\mathfrak{K}}\sum_{\mathfrak{H}} b(x', I', y, V)\mu_1(I')\mu_2(V): \mathfrak{K} \ll \{I\},\ I' \text{ in } \mathfrak{K},\ x' \text{ in } I',\ \mathfrak{H} \ll \{J\},$$

$$V \text{ in } \mathfrak{H},\ y \text{ in } V,\ b(x', I', y, V) \text{ in } \beta(x', I', y, V)\Big\} \leqslant L(\alpha\mu_3)(I\times J),$$

so that

$$L(\beta(x, I, \ldots)\mu_2)(J)\mu_1(I) \leqslant L(\alpha\mu_3)(I\times J).$$

In a similar fashion,

$$G(\alpha\mu_3)(I\times J) \leqslant G(\beta(x, I, \ldots)\mu_2)(J)\mu_1(I).$$

Now, suppose $\mathfrak{E}_1 \ll \mathfrak{D}_1$, and for each I in $\mathfrak{E}_1$, x is in I.

$$\sum_{\mathfrak{E}_1}\sum_{\mathfrak{D}_2} G(\alpha\mu_3)(I\times J) \leqslant \sum_{\mathfrak{E}_1}\sum_{\mathfrak{D}_2} G(\beta(x, I, \ldots)\mu_2)(J)\mu_1(I) =$$

$$\sum_{\mathfrak{E}_1}\Big\{\sum_{\mathfrak{D}_2} G(\beta(x, I, \ldots)\mu_2)(J)\Big\}\mu_1(I) \leqslant$$

$$\sum_{\mathfrak{E}_1}\Big\{\int_{U_2} G(\beta(x, I, \ldots)\mu_2)\Big\}\mu_1(I) \leqslant \sum_{\mathfrak{E}_1}\Big\{\int_{U_2} L(\beta(x, I, \ldots)\mu_2)\Big\}\mu_1(I) \leqslant$$

$$\sum_{\mathfrak{E}_1}\Big\{\sum_{\mathfrak{D}_2} L(\beta(x, I, \ldots)\mu_2)(J)\Big\}\mu_1(I) =$$

$$\sum_{\mathfrak{E}_1}\sum_{\mathfrak{D}_2} L(\beta(x, I, \ldots)\mu_2)(J)\mu_1(I) \leqslant \sum_{\mathfrak{E}_1}\sum_{\mathfrak{D}_2} L(\alpha\mu_3)(I\times J).$$

Since

$$\{I\times J:\ I \text{ in } \mathfrak{E}_1,\ J \text{ in } \mathfrak{D}_2\} \ll \mathfrak{D}_3 \ll \mathfrak{D},$$

it follows that

$$\max\Big\{\Big|\int_{U_1\times U_2}\alpha\mu_3 - \sum_{\mathfrak{E}_1}\sum_{\mathfrak{D}_2} L(\alpha\mu_3)(I\times J)\Big|, \Big|\int_{U_1\times U_2}\alpha\mu_3 - \sum_{\mathfrak{E}_1}\sum_{\mathfrak{D}_2} G(\alpha\mu_3)(I\times J)\Big|\Big\} < c,$$

so that

$$\max\left\{\left|\int_{U_1\times U_2}\alpha\mu_3-\sum_{\mathfrak{E}_1}\left\{\int_{U_2}L(\beta(x,I,\cdot,\cdot)\mu_2)\right\}\mu_1(I)\right|,\right.$$
$$\left.\left|\int_{U_1\times U_2}\alpha\mu_3-\sum_{\mathfrak{E}_1}\left\{\int_{U_2}G(\beta(x,I,\cdot,\cdot)\mu_2)\right\}\mu_1(I)\right|\right\}<c\,.$$

Therefore, if Q is L or G, then

$$\int_{U_1}\left\{\int_{U_2}Q(\beta(x,I,\ldots)\mu_2)\right\}\mu_1(I)$$

exists and is

$$\int_{U_1\times U_2}\alpha\mu_3\,.$$

In a similar fashion, it follows that if Q is L or G, then

$$\int_{U_2}\left[\int_{U_1}Q(\beta(\ldots,y,J)\mu_1)\right\}\mu_2(J)$$

exists and is

$$\int_{U_1\times U_2}\alpha\mu_3\,.$$

We close this section by stating the following corollary which we shall use in section 4.

COROLLARY 3.1. The proof, and hence the statement of Theorem 3.1 remain valid of U_1 and U_2 are replaced throughout, respectively, with V_1 in $\boldsymbol{F}_1$ and V_2 in $\boldsymbol{F}_2$.

4. – The summable case.

In this section we prove Theorem 4.1, as stated in the introduction.

PROOF OF THEOREM 4.1. Let $\boldsymbol{s}$ denote $\boldsymbol{s}_{\mu_3}(\alpha)$. The function

$$\{(V,\boldsymbol{s}(V\times U_2)):\ V\text{ in }\boldsymbol{F}_1\}$$

is clearly bounded and finitely additive; also, if V is in $\boldsymbol{F}_1$ and $p \leqslant 0 \leqslant q$, then

$$\left|\int_V\Big[\int_{U_2} Q(\beta_{p,q}(x, I, \ldots)\mu_2)\Big]\,\mu_1(I)\right| = \left|\int_{U_1\times U_2}\alpha_{p,q}(W)\mu_3(W)\right| \leqslant$$

$$\int_{V\times U_2}|\alpha_{p,q}(W)|\mu_3(W) \leqslant \int_{V\times U_2}\min\{|\alpha(W)|, \max\{-p, q\}\}\mu_3(W) \leqslant$$

$$\boldsymbol{s}_{\mu_3}(|\alpha|)(V\times U_2)\,.$$

Furthermore, if $\mathfrak{D} \ll \{U_1\}$, then

$$\sum_{\mathfrak{D}}\left|\int_V\Big[\int_{U_2} Q(\beta_{p,q}(x, I, \ldots)\mu_2)\Big]\mu_1(I) - \boldsymbol{s}(V\times U_2)\right| =$$

$$\sum_{\mathfrak{D}}\left|\int_{V\times U_2}\alpha_{p,q}\mu_3 - \boldsymbol{s}(V\times U_2)\right| \leqslant \int_{U_1\times U_2}\left|\int_W\alpha_{p,q}\mu_3 - \boldsymbol{s}(W)\right|.$$

Therefore

$$\int_{U_1}\left|\int_V\Big[\int_{U_2} Q(\beta_{p,q}(x, I, \ldots)\mu_2)\Big]\mu_1(I) - \boldsymbol{s}(V\times U_2)\right| \leqslant \int_{U_1\times U_2}\left|\int_W\alpha_{p,q}\mu_3 - \boldsymbol{s}(W)\right|.$$

Furthermore, by Theorem 2.A.1,

$$\int_{U_1\times U_2}|\alpha_{p,q}\mu_3 - \boldsymbol{s}(W)| \to 0\,, \qquad \min\{-p, q\} \to \infty\,.$$

Finally

$$\int_{U_1}\left|\int_V\Big[\int_{U_2} Q(\beta_{p,q}(x, I, \ldots)\mu_2)\Big]\mu_1(I) - \boldsymbol{s}(V\times U_2)\right| =$$

$$\int_{U_1}\left|\Big(\int_{U_2} Q(\beta_{p,q}(x, V, \ldots)\mu_2)\Big)\mu_1(V) - \boldsymbol{s}(V\times U_2)\right| =$$

$$\int_{U_1}\left|\int_{U_2} Q(\beta_{p,q}(x, V, \ldots)\mu_2) - [\boldsymbol{s}(V\times U_2)/\mu_1(V)]\right|\mu_1(V)\,.$$

A similar argument proves the remainder of the theorem.

We close by showing that the function $\boldsymbol{s}(\cdot\times U_2)/\mu_1$ and function « sequence »,

$$\left\{\Big((p, q), \Big\{\Big(I, \Big\{\int_{U_2} Q(\beta_{p,q}(x, I, \cdot, \cdot)\mu_2):\ x \text{ in } I\Big\}\Big):\ I \text{ in } \boldsymbol{F}_1\Big\}\Big):\ p \leqslant 0 \leqslant q\right\},$$

satisfy the hypothesis of Theorem 2.A.3. As was shown above, if V is in $\boldsymbol{F}_1$ and $p \leqslant 0 \leqslant q$, then

$$\left| \int_V \left[\int_{U_2} Q_2(\beta_{p,q}(x, I, \ldots)\mu_2) \right] \mu_1(I) \right| \leqslant \boldsymbol{s}(|\alpha|)(V \times U_2) .$$

Furthermore, the function $\boldsymbol{s}_{\mu_2}(|\alpha|)(\cdot \times U_2)$ is clearly absolutely continuous with respect to μ_1. Finally, the limiting assertion

$$\int_{U_1} \left| \boldsymbol{s}(V \times U_2)/\mu_1(V) - \int_{U_2} Q_2(\beta_{p,q}(x, V, \ldots)\mu_2) \right| \mu_1(V) \to 0 , \quad \min\{-p, q\} \to \infty ,$$

together with the fact that the relation $\leqslant^*$ on the set

$$\{(p, q) \colon p \leqslant 0 \leqslant q\} ,$$

given by

$$(p, q) \leqslant^* (p', q') \quad \text{iff } p' \leqslant p \text{ and } q \leqslant q' ,$$

is a partial ordering with respect to which the above set is directed, show that the hypothesis and therefore the conclusion of Theorem 2.A.4 are satisfied with respect to the function « sequence »

$$\left\{ \left((p, q), \left\{ \left(I, \left\{ \left| \boldsymbol{s}(I \times U_2)/\mu_1(I) - \int_{U_2} Q_2(\beta_{p,q}(x, I, \cdot, \cdot)\mu_2) \right| \colon x \text{ in } I \right\} \right) \colon I \text{ in } \boldsymbol{F}_1 \right\} \right) \colon p \leqslant 0 \leqslant q \right\} .$$

Thus the remainder of the hypothesis of Theorem 2.A.3 is satisfied for the function and function « sequence » given at the beginning of the paragraph and for the partial ordering given above, and, as was made clear in the introduction, we are done. Similar remarks hold for the « right side » of the heuristic « equation » of the introduction.

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