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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3e série, tome 27, nº 2 (1973), p. 345-357

<http://www.numdam.org/item?id=ASNSP_1973_3_27_2_345_0>

ON THE HOLDER-CONTINUITY OF SOLUTIONS OF A NONLINEAR PARABOLIC VARIATIONAL INEQUALITY

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Parabolic variational inequalities have been studied extensively by Brezis [2], Browder [4], Lions [6], Lions-Stampacchia [7] and others. The existence of a weak solution is shown and when the elliptic operator involved is strongly monotone, the solution is unique.

Using the penalisation method, Lions [6] has shown the regularity of solutions of some linear parabolic inequalities. For nonlinear parabolic inequalities, the regularity of solutions with respect to time has been obtained by Brezis [2] and the regularity with respect to both space and time by the writer in [8].

The purpose of this paper is to show the Holder-continuity of solutions u of:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \sum\limits_{j=1}^{n} \dfrac{\partial}{\partial x_j} (a_j(x, u, Du)) = f \text{ on the region where } u(x, t) \geq 0, \\ u(x, t) = 0 \text{ elsewhere, } \; u(x, t) = 0 \text{ on } \partial G \times [0, T], \; u(x, 0) = 0 \text{ and} \\ \text{« continuity » of } u, \partial u / \partial x_j \text{ at the two regions.} \end{cases}$$

Moreover, if $a_j(x, u, Du) = a_j(x) D_j u$ for $j = 1, \dots, n$, it will be shown that $u \in L^p(0, T; W^{2,p}(G))$ for any p, $2 \leq p < \infty$.

To prove the result, we use Lions' penalisation method, a time discretisation of the penalized equation and a nonlinear singular perturbution of the latter equation.

The notations and the main results of the paper are given in Section 1. Proofs are carried out in Section 2.

Pervenuto alla Redazione il 31 Gennaio 1972.

SECTION 1: Let G be a bounded open subset of R_n with a smooth boundary ∂G. Set: $D_j = i^{-1}\, \partial/\partial x_j$, $j = 1, \dots, n$ and for each n-tuple $\alpha = (\alpha_1, \dots, \alpha_n)$ of non-negative integers, we write:

$$D^\alpha \prod_{j=1}^{n} D_j^\alpha j \quad \text{with } |\alpha| = \sum_{j=1}^{n} \alpha_j .$$

$W^{m,p}(G)$ is the real reflexive separable Banach space:

$$W^{m,p}(G) = \{u : u \text{ in } L^p(G),\ D^\alpha u \text{ in } L^p(G),\ |\alpha| \leq m\}$$

with the norm:

$$\| n \|_{m,p} = \Big\{ \sum_{|\alpha| \leq m} \| D^\alpha u \|^p_{L^p(G)} \Big\}^{1/p},\ 2 \leq p < \infty .$$

$W_0^{m,p}(G)$ is the completion of C_0^∞, the family of all infinitely differentiable functions with compact suport in G, in the $\| \cdot \|_{m,p}$ — norm. The pairing between $W_0^{m,p}(G)$ and its dual $W^{-m,q}(G)$ is denoted by $(.,.)$. Set: $H = L^2(G)$ and $\| u \|_s$ is the $L^s(G)$-norm of u for $1 < s < \infty$.

$C^\lambda(G)$ is the space of all Holder-continuous functions of any compact subset of G, with Holder-exponent λ, $0 < \lambda < 1$.

Let $[0, T]$ be a compact interval of the real line E. The derivative of u with respect to t will be denoted by u'.

$L^p(0, T; W^{m,p}(G))$ is the space of all equivalence classes of functions $n(t)$ from $[0, T]$ to $W^{m,p}(G)$ which are L^p-integrale. It is a real reflexive separable Banach space with the norm:

$$\| u \|_{L^p(0,T;\, W^{m,p}(G))} = \Big\{ \int_0^T \| u(t) \|^p_{m,p}\, dt \Big\}^{1/p} .$$

$C^\lambda(0, T; C^{2\lambda}(G))$ is the space of all Holder-continuous fonctions on any compact subsets of $G \times [0, T]$ with Holder-exponent λ with respect to t and with exponent 2λ with respect to x. $0 < 2\lambda < 1$.

We consider nonlinear partial differential operators on G of the form:

$$A(u) = \sum_{|\alpha| \leq 1} D^\alpha A_\alpha(x, x, Du).$$

ASSUMPTION (I): (i) *Let* $\zeta = \{\zeta_\alpha : |\alpha| \leq 1\}$, *then each* $A_\alpha(x, \zeta)$ *is continuously differentiable in* x *and in* ζ.

(ii) *There exists a positive constant C such that*:

$$| A_\alpha (x, \zeta) | + | D_x A_\alpha (x, \zeta) | + (1 + | \zeta |) \sum_{|\beta| \leq 1} | A_{\alpha\beta}(x, \zeta) | \leq C | \zeta |$$

where $A_{\alpha\beta} = \partial A_\alpha / \partial \zeta_\beta$.

(iii) $$\sum_{|\alpha|, |\beta| \leq 1} A_{\alpha\beta}(x, \zeta) \eta_\alpha \eta_\beta \geq c \sum_{|\alpha| \leq 1} \eta_\alpha^2 .$$

c is a positive constant.

(iv) $$\sum_{|\alpha| \leq 1} A_\alpha (x, \zeta) \zeta_\alpha \geq 0.$$

Let $K = \{u : u \text{ in } L^2(G),\ u \geq 0 \text{ a. e. on } G\}$. It is clear that K is a closed convex subset of both H and $W_0^{1,2}(G)$.

The main results of the paper are the following two theorems.

THEOREM 1: *Let A be an elliptic operator satisfying Assumption (I). Suppose that* $f \in L^\infty (0, T; L^\infty (G))$, $f' \in L^2 (0, T; W^{-1,2} (G))$ *with* $f(0) = 0$. *Then there exists a unique solution u in* $L^2(0, T; W_0^{1,2} (G)) \cap L^2(0, T; W^{2,2}(G))$ *with* u' *in* $L^2 (0, T; W^{1,2} (G))$, $u(t)$ *in K* a. c. *and* $u(0) = 0$ *such that*:

$$\int_0^T (u' + Au - f, v - u)\, dt \leq 0$$

for all v in $L^2 (0, T; W^{1,2}(G))$, v' *in* $L^2 (0, T; W^{-1,2}(G))$, $v(t)$ *in K and* $v(0) = 0$.

Moreover: $u \in C^\lambda ([0, T]; C^{2\lambda} (\mathrm{cl} G))$, $D_j u \in C^\gamma (0, T; C^{2\gamma} (G))$, *for any j, with* $0 < 2\lambda,\ 2\gamma < 1$.

When A is a linear elliptic operator, we have a stronger result.

THEOREM 2: *Let* $Au = \sum_{|\alpha|, |\beta| \leq 1} D^\alpha (a_{\alpha\beta}(x) D^\beta u)$ *be a positively strongly uniformly elliptic operator on G with coefficients* $a_{\alpha\beta}(x)$ *in* $C^1 (\mathrm{cl} G)$. *Suppose that* $f \in L^\infty (0, T; L^\infty (G))$, $f' \in L^2 (0, T; W^{-1,2} (G))$ *and* $f(0) = 0$. *Then there exists a unique* $u \in L^2 (0, T; W_0^{1,2} (G)) \cap L^p (0, T; W^{2,p} (G))$, $1 < p < \infty$, $u(t)$ *in K* a. e., $u' \in L^p(0, T; L^p(G)) \cap L^2 (0, T; W^{1,2}(G)$ *and* $u(0) = 0$ *such that*:

$$\int_0^T (u' + Au - f, v - u)\, dt \geq 0$$

for all v in $L^2 (0, T; W_0^{1,2} (G))$, v' *in* $L^2 (0, T; W^{-1,2}(G))$, $v(t)$ *in K and* $v(0) = 0$.

Moreover: $u \in C^{\lambda}([0, T]; C^{2\lambda}(\mathrm{cl}G))$, $D_j u \in C^{\gamma}([0, T]; C^{2\gamma}(G))$ *for any* j, *with* $0 < 2\lambda$, $2\gamma < 1$.

Theorem 1 is a consequence of Theorem 3 which will be proved in Section 2.

THEOREM 3: *Suppose all the hypotheses of Theorem* 1 *are satisfied. Then for each* ε, $0 < \varepsilon < 1$, *there exists a unique solution* u_ε *of the equation:*

$$\varepsilon(u_\varepsilon' + Au_\varepsilon) - u_\varepsilon^- = \varepsilon f,\ u_\varepsilon(x, t) = 0 \ \text{on} \ \partial G \times [0, T], u_\varepsilon(x, 0) = 0.$$

Moreover

$$\| u_\varepsilon \|_{L^\infty(0, T; L^\infty(G))} + \| u_\varepsilon \|_{L^2(0, T; W_0^{1,2}(G))} + \| u_\varepsilon' \|_{L^2(0, T; W^{1,2}(G))} \leq C.$$

C is a constant independent of ε.

PROOF OF THEOREM 1 USING THEOREM 3: We shall make use of the of the following crucial estimate of Theorem 3:

$$\| u_\varepsilon \|_{L^\infty(0, T; L^\infty(G))} \leq C.$$

1) Since $u_\varepsilon \in L^\infty(0, T; L^\infty(G)) \cap L^2(0, T; W_0^{1,2}(G))$, we have:

$$u_\varepsilon^- \in L^\infty(0, T; L^\infty(G)) \cap L^2(0, T; W_0^{1,2}(G)).$$

Thus: $| u_\varepsilon^- |^{2s-2} u_\varepsilon^- \in L^2(0, T; W_0^{1,2}(G))$ for any positive integer s. It follows from conditions (ii) and (iii) of Assumption (I) that:

$$\int_0^T (Au_\varepsilon, -| u_\varepsilon^- |^{2s-2} u_\varepsilon^-)\, dt = -\int_0^T Au_\varepsilon^-, | u_\varepsilon^- |^{2s-2} u_\varepsilon^-)\, dt \geq 0.$$

Since $u_\varepsilon' \in L^2(0, T; H)$ and $u_\varepsilon^- \in L^\infty(0, T; L^\infty(G))$ with $u_\varepsilon(0) = 0$, we have:

$$2s \int_0^T (u_\varepsilon', -| u_\varepsilon^- |^{2s-2} u_\varepsilon^-)_H\, dt = \int_0^T \int_{\substack{G \\ u_\varepsilon \geq 0}} \frac{d}{dt}(u_\varepsilon^{2s})\, dx\, dt = \int_{\substack{G \\ u_\varepsilon \geq 0}} | u_\varepsilon(x, T) |^{2s}\, dx \geq 0.$$

Hence:

$$\varepsilon^{-1} \| u_\varepsilon^- \|_{L^{2s}(0, T; L^{2s}(G))}^{2s} \leq C \| f \|_{L^\infty(0, T; L^\infty(G))} \| u_\varepsilon^- \|_{L^{2s}(0, T; L^{2s}(G))}^{2s-1}.$$

Therefore :

$$\varepsilon^{-1} \| u_\varepsilon^- \|_{L^{2s}(0,T;L^{2s}(G))} \leq C \| f \|_{L^\infty(0,T;L^\infty(G))} .$$

C is a constant independent of s and of ε.

Since u_ε^- lies in $L^\infty(0, T; L^\infty(G))$, we may let $s \to +\infty$ and the above inequality gives :

$$\varepsilon^{-1} \| u_\varepsilon^- \|_{L^\infty(0,T;L^\infty(G))} \leq C \| f \|_{L^\infty(0,T;L^\infty(G))} .$$

2) From the weak compactness oe the unit ball in a reflexive Banach space, we obtain by taking subsequences if necesary: $u_\varepsilon \to u$ weakly in $L^2(0, T; W_0^{1,2}(G))$, $u_\varepsilon' \to u'$ weakly in $L^2(0, T; W^{1,2}(G))$ $Au_\varepsilon \to h$ weakly in $L^2(0, T; W^{-1,2}(G))$, $\varepsilon^{-1} u_\varepsilon^- \to g$ in the weak*-topology of $L^\infty(0, T; L^\infty(G))$ and $u_\varepsilon^- \to 0$ in $L^2(0, T; H)$).

Thus : $u' + h + g = f$, $u(0) = 0$ and $u \in K$.

Condition (iii) of Assumption (I) implies that A is monotone. Moreover :

$$\int_0^T (Au_\varepsilon, u_\varepsilon - u)\, dt = \int_0^T (f + \varepsilon^{-1} u_\varepsilon^- - u_\varepsilon', u_\varepsilon - u)_H\, dt.$$

Aubin's theorem [1] gives :

$$\lim\sup \int_0^T (Au_\varepsilon, u_\varepsilon - u)\, dt \leq 0.$$

By a standard argument of the theory of monotone operatcrs, we get $h = Au$ and

$$\int_0^T (Au, u)\, dt \leq \lim\inf \int_0^T (Au_\varepsilon, u_\varepsilon)\, dt.$$

We have :

$$\int_0^T (u_\varepsilon' + Au_\varepsilon - f - \varepsilon^{-1} u_\varepsilon^-, v - u_\varepsilon)\, dt = 0.$$

Let v be an element of $L^2(0, T; W_0^{1,2}(G))$, v' in $L^2(0, T; W^{-1,2}(G))$, with $v(t)$ in K and $v(0) = 0$. Then $v^- = 0$. and we have:

$$\int_0^T (u'_\varepsilon + Au_\varepsilon - f, v - u_\varepsilon)\, dt \geq 0.$$

Let $\varepsilon \to 0$ and we get:

$$\int_0^T (u' + Au - f, v - u)\, dt \geq 0.$$

3) So: $u' + Au = f - g \in L^\infty(0, T; L^\infty(G))$, $u(x, t) = 0$ on $\partial G \, x \, [0, T]$ and $u(x, 0) = 0$. It follows from Theorem 6.4 of Ladyzenskaya Solonnikov and Uralceva [5] (page 460), that $u \in C^\lambda[0, T]; C^{2\lambda}(\mathrm{cl}G))$, $D_j u \in C^\gamma(0, T; C^{2\gamma}(G))$. $0 < 2\lambda, 2\gamma < 1$.

4) All the other assertions of Theorem 1 have been proved in [8],

PROOF OF THEOREM 2: From the proof of Theorem 1, we know that there exists a unique u in $L^2(0, T; W_0^{1,2}(G))$ with u' in $L^2(0, T; W^{1,2}(G))$, $u(t)$ in K and $u(0) = 0$ snch that:

$$\int_0^T (u' + Au - f, v - u)\, dt \geq 0$$

for all v in $L^2(0, T; W_0^{1,2}(G))$, v' in $L^2(0, T; W^{-1,2}(G))$, $v(t)$ in K and $v(0) = 0$.

Moreover u satisfies the equation: $u' + Au = f - g \in L^\infty(0, T; L^\infty(G))$. By a well-known result of the theory of linear parabolic equations of order 2 (Cf. e. g. Theorem 9.1 of [5], p. 341-342), we have:

$$u \in L^p(0, T; W^{2,p}(G)),\ u' \in L^p(0, T; L^p(G)) \quad \text{for any } p,\ 1 < p < \infty.$$

SECTION 2: The proof of Theorem 3 is long and will be carried out in this section. We shall give an outline of the proof before going into the details.

Consider the equation:

$$u_{h\varepsilon}^k - u_{\varepsilon h}^{k-1} + hAu_{h\varepsilon}^k - h\varepsilon^{-1}(u_{\varepsilon h}^k)^- = hf^k,\ u_{\varepsilon h}^0 = 0.$$

It is obtained by a discretisation of the time-variable of the equation of Theorem 3.

Let $A_2 v$ be the nonlinear elliptic operator:

$$A_2 v = \sum_{j=1}^{n} D_j(|D_j v|^{p-2} D_j v) \qquad \text{with } p > n.$$

1) First, we shall consider the equation:

$$u^k_{\varepsilon h\eta} - u^{k-1}_{\varepsilon h\eta} + \eta A_2 u^k_{\varepsilon h\eta} + hAu^k_{\varepsilon h\eta} - \varepsilon^{-1} h(u^k_{\varepsilon h\eta})^- = hf^k, u^0_{\varepsilon h\eta} = 0; \ \eta > 0.$$

It has a uniqne solution $u^k_{\varepsilon h\eta}$ in $W_0^{1,p}(G)$ and since $p > n$, $u^k_{\varepsilon h\eta}$ is in $C(\mathrm{clG})$.

2) $\| u^k_{\varepsilon h\eta} \|_{L^\infty(G)} \leq C$. C is a constant independent of ε, h, k and η. Then let $\eta \to 0$.

3) The final step is standard.

Set: $Bv = -v^-$. Denote by $f^k = h^{-1} \int_{kh}^{(k+1)h} f(t)\, dt$ with $h > 0$.

Lemma 1: *Let $h = T/N$ and suppose all the hypotheses of Theorem 3 are satisfied. Then for each k, $1 \leq k \leq N$, there exists a unique solution $u^k_{\varepsilon h\eta}$ in $W_0^{1,p}(G)$ of the equation:*

$$u^k_{\varepsilon h\eta} - u^{k-1}_{\varepsilon h\eta} + \eta h A_2 u^k_{\varepsilon h\eta} + hAu^k_{\varepsilon h\eta} + \varepsilon^{-1} hBu^k_{\varepsilon h\eta} = hf^k, u^0_{\varepsilon h\eta} = 0.$$

Moreover:

$$\| u^n_{\varepsilon h\eta} \|_{L^\infty(G)} + \eta h \sum_{k=1}^{n} \| u^k_{\varepsilon h\eta} \|^p_{1,p} + h \sum_{k=1}^{n} \| u^k_{\varepsilon h\eta} \|^2_{1,2} \leq C.$$

C is a constant independent of h, ε, η and n.

Proof: It is clear that $u + \eta h A_2 u + h\varepsilon^{-1} Bu$ is a monotone hemicontinuous, coercive operator mapping bounded sets of $W_0^{1,p}(G)$ into bounded sets of $W^{-1,q}(G)$. It follows from the theory of monotone operators that for each k, there exists a unique solution $u^k_{\varepsilon h\eta}$ of:

$$u^k_{\varepsilon h\eta} - u^{k-1}_{\varepsilon h\eta} + \eta h A_2 u^k_{\varepsilon h\eta} + hAu^k_{\varepsilon h\eta} + h\varepsilon^{-1} Bu^k_{\varepsilon h\eta} = hf^k, u^0_{\varepsilon h\eta} = 0, \quad k = 1, \dots, N.$$

So:

$$\frac{1}{2} \| u^k_{\varepsilon h\eta} \|^2_H + \eta h \| u^k_{\varepsilon h\eta} \|^p_{1,p} + h \| u^k_{\varepsilon h\eta} \|^2_{1,2} \leq Ch \| f^k \|_{L^\infty(G))} + \frac{1}{2} \| u^{k-1}_{\varepsilon h\eta} \|^2_H.$$

Taking the summation from $k = 1$ to n, we obtain:

$$\sum_{k=1}^{n} \eta h \| u^k_{\varepsilon h\eta} \|^p_{1,p} + h \| u^k_{\varepsilon h\eta} \|^2_{1,2} \leq Ch \sum_{k=1}^{n} \| f^k \|_{L^\infty(G)} \leq C_2 T.$$

C_2 is a constant independent of ε, η, h and n.

2) We show the crucial estimate: $\| u^n_{\varepsilon h\eta} \|_{L^\infty(G))}$.

Since $p > n$, the Sobolev imbedding theorem gives: $W^{1,p}_0(G) \subset C(\mathrm{clG})$. Thus $| u^k_{\varepsilon h\eta} |^{s-2} u^k_{\varepsilon h\eta}$ is in $W^{1,p}_0(G)$ for any positive integer $s \geq 2$. Therefore:

$$\| u^k_{\varepsilon h\eta} \|^s_s + \eta h (A_2 u^k_{\varepsilon h\eta}, | u^k_{\varepsilon h\eta} |^{s-2} u^k_{\varepsilon h\eta}) + h (A u^k_{\varepsilon h\eta}, | u^k_{\varepsilon h\eta} |^{s-2} u^k_{\varepsilon h\eta})$$

$$+ h\varepsilon^{-1} (B u^k_{\varepsilon h\eta}, | u^k_{\varepsilon h\eta} |^{s-2} u^k_{\varepsilon h\eta}) \leq Ch \| f^k \|_{L^\infty(G)} \| u^k_{\varepsilon h\eta} \|^{s-1}_s + \| u^{k-1}_{\varepsilon h\eta} \|_s \| u^k_{\varepsilon h\eta} \|^{s-1}_s.$$

Consider the second term of the left hand side of the inequality:

$$(A_2 u^k_{\varepsilon h\eta}, | u^k_{\varepsilon h\eta} |^{s-2} u^k_{\varepsilon h\eta}) = \sum_j (s-1) \int_G | u^k_{\varepsilon h\eta} |^{s-2} | D_j u^k_{\varepsilon h\eta} |^p \, dx \geq 0.$$

On the other hand:

$$(A u^k_{\varepsilon h\eta}, | u^k_{\varepsilon h\eta} |^{s-2} u^k_{\varepsilon h\eta}) = \sum_{|\alpha| \leq 1} (s-1) \int_G A_\alpha (x, u^k_{\varepsilon h\eta}, D u^k_{\varepsilon h\eta}) | u^k_{\varepsilon h\eta} |^{s-2} D^\alpha (u^k_{\varepsilon h\eta}) \, dx$$

It follows from condition (iv) of Assumption (I) that the above expression is positive.

It is tıivial to check that: $(B u^k_{\varepsilon h\eta}, | u^k_{\varepsilon h\eta} |^{s-2} u^k_{\varepsilon h\eta}) \geq 0$.

Therefore:

$$\| u^k_{\varepsilon h\eta} \|_s \leq Ch \| f^k \|_{L^\infty(G)} + \| u^{k-1}_{\varepsilon h\eta} \|_s.$$

Taking the summation from $k = 1$ to n, we obtain:

$$\| u^n_{\varepsilon h\eta} \|_s \leq Ch \sum_{k=1}^{n} \| f^k \|_{L^\infty(G))} \leq C_2 T.$$

We know that $u^n_{\varepsilon h\eta}$ is in $C(\mathrm{clG})$, thus letting $s \to +\infty$, we have:

$$\| u^n_{\varepsilon h\eta} \|_{L^\infty(G)} \leq C_2 T.$$

The lemma is proved.

LEMMA 2: *Let $h = T/N$ and suppose all the hypotheses of Theorem 3 are satisfied. Then for each k, $1 \leq k \leq N$, there exists a unique solution $u_{\varepsilon h}^k$ in $W_0^{1,2}(G)$ of:*

$$u_{\varepsilon h}^k - u_{\varepsilon h}^{k-1} + hAu_{\varepsilon h}^k + h\varepsilon^{-1} Bu_{\varepsilon h}^k = hf^k, \; u_{\varepsilon h}^0 = 0.$$

Moreover:

$$\| u_{\varepsilon k}^n \|_{L^\infty(G)} + \sum_{k=1}^n h \| u_{\varepsilon h}^k \|_{1,2}^2 \leq C.$$

PROOF: From Lemma 1, we know that for each k, $k = 1, \dots, N$, there exists a unique solution $u_{\varepsilon h\eta}^k$ in $W_0^{1,p}(G)$ of the equation:

$$u_{\varepsilon h\eta}^k - u_{\varepsilon h\eta}^{k-1} + \eta h A_2 u_{\varepsilon h\eta}^k + hAu_{\varepsilon h\eta}^k + h\varepsilon^{-1} Bu_{\varepsilon h\eta}^k = hf^k, \; u_{\varepsilon h\eta}^0 = 0.$$

Moreover:

$$\| u_{\varepsilon h\eta}^n \|_{L^\infty(G)} + \sum_{k=1}^n \eta h \| u_{\varepsilon h\eta}^k \|_{1,p}^p + h \sum_{k=1}^n \| u_{\varepsilon h\eta}^k \|_{1,2}^2 \leq C.$$

C is a constant independent of ε, η, h and n.

Let $\eta \to 0$. The weak compactness of the unit ball in a reflexive Banach space gives: $h^{\frac{1}{2}} u_{\varepsilon h\eta}^k \to h^{\frac{1}{2}} u_{\varepsilon h}^k$ weakly in $W_0^{1,2}(G)$, $(h\eta)^{1/p} u_{\varepsilon h\eta}^k \to 0$ weakly in $W_0^{1,p}(G)$, $Au_{\varepsilon h\eta}^k \to g_{\varepsilon h}^k$ weakly in $W^{-1,2}(G)$ and $u_{\varepsilon k\eta}^k \to u_{\varepsilon h}^k$ in the weak*-topology of $L^\infty(0, T; L^\infty(G))$.

It follows from the Sobolev imbedding theorem that $u_{\varepsilon h\eta}^k \to u_{\varepsilon h}^k$ in $L^2(G)$ and thus $Bu_{\varepsilon h\eta}^k \to Bu_{\varepsilon h}^k$ weakly in $L^2(G)$.

We obtain:

$$u_{\varepsilon h}^k - u_{\varepsilon h}^{k-1} + h\, g_{\varepsilon h}^k + h\varepsilon^{-1} Bu_{\varepsilon h}^k = hf^k, \; u_{\varepsilon h}^0 = 0.$$

Since A is monotone, it is easy to show that $g_{\varepsilon h}^k = Au_h^k$.

All the other assertions of the lemma follow trivially from the above arguments.

PROOF OF THEOREM 3: Let $u_{\varepsilon h}^k$, $k = 1, \dots, N$, be the solution of:

$$u_{\varepsilon h}^k - u_{\varepsilon h}^{k-1} + hAu_{\varepsilon h}^k + h\varepsilon^{-1} Bu_{\varepsilon h}^k = hf^k \text{ with } u_{\varepsilon h}^0 = 0.$$

1) Set: $u_{\varepsilon h}(t) = u^k_{\varepsilon h}$ when $kh \leq t < (k+1)h$, $k = 0, \dots, N-1$ and $h = T/N$. Then from Lemma 2, we obtain:

$$\| u_{\varepsilon h} \|_{L^\infty(0, T; L^\infty(G))} + \| u_{\varepsilon h} \|_{L^2(0, T; W_0^{1,2}(G))} \leq C$$

C is a constant independent of both ε and h.

It is easy to show that:

$$\sum_{k=1}^{n} \| h^{-1} (u^k_{\varepsilon h} - u^{k-1}_{\varepsilon h}) \|^2_{W^{-1,2}(G)} \leq M(\varepsilon).$$

$M(\varepsilon)$ is independent of h and n.

2) From the weak compactness of the unit ball in a reflexive Banach space, we get by taking subsequences if necessary: $u_{\varepsilon k} \to u_\varepsilon$ weakly in $L^2(0, T; W_0^{1,2}(G))$, $u_{\varepsilon h} \to u_\varepsilon$ in the weak*-topology of $L^\infty(0, T; L^\infty(G))$, $Au_{\varepsilon h} \to g_\varepsilon$ weakly in $L^2(0, T; W^{-1,2}(G))$ and $h^{-1}(u_{\varepsilon h}(t) - u_{\varepsilon h}(t-h)) \to u'_\varepsilon$ weakly in $L^2(0, T; W^{-1,2}(G))$.

Since the injection mapping of $W^{1,2}(G)$ into $L^2(G)$ is compact, the discrete analogue of Aubin's theorem [1] gives: $u_{\varepsilon h} \to u_\varepsilon$ in $L^2(0, T; H)$. Hence: $Bu_{\varepsilon h} \to Bu_\varepsilon$ weakly in $L^2(0, T; H)$.

Thus:

$$u'_\varepsilon + g_\varepsilon = \varepsilon^{-1} Bu_\varepsilon = f.$$

3) We show thet $u_\varepsilon(0) = 0$.

Le $v \in W_0^{1,2}(G)$ and $\varphi \in C([0, T]$. Set: $\varphi_h(t) = \varphi(nh)$ with $nh \leq t\,(n+1)h$. Then:

$$h^{-1}(u^n_{\varepsilon h} - u^{n-1}_{\varepsilon h}, v)\,\varphi_h(t) + (Au^n_{\varepsilon h}, v)\,\varphi_h(t) + \varepsilon^{-1}(Bu^n_{\varepsilon h}, v)\,\varphi_h(t) = (f^n, v)\,\varphi_h(t)$$

Let $h \to 0$ and we get:

$$\int_0^T (u'_\varepsilon + g_\varepsilon + \varepsilon^{-1} Bu_\varepsilon - f, v)\,\varphi\, dt = 0$$

for all v in $W_0^{1,2}(G)$ and all φ in $C([0, T])$.

A standard argument gives:

$$\int_0^T (u'_\varepsilon + g_\varepsilon + \varepsilon^{-1} Bu_\varepsilon - f, \varphi)\, dt = 0$$

for all φ in $L^2(0, T; W_0^{1,2}(G))$.

On the other hand:

$$\sum_{n=1}^{N} -(u_{\varepsilon h}, v)_H(\varphi(nh) - \varphi(nh-h)) + h(Au_{\varepsilon h}, v)\varphi(uh-h) + h\varepsilon^{-1}(Bu_{\varepsilon h}, v)\varphi(nh-h)$$

$$- h(f^n, v)\varphi(nh-h) = -(u_{\varepsilon h}(T), v)_H \varphi(T).$$

Take $\varphi \in C([0, T])$ with $\varphi(T) = 0$ and let $h \to 0$. We obtain:

$$-\int_0^T (u_\varepsilon, v)\varphi' \, dt + \int_0^T (g_\varepsilon + \varepsilon^{-1} Bu_\varepsilon - f, v)\varphi \, dt = 0.$$

So:

$$-\int_0^T (u_\varepsilon, \varphi') \, dt + \int_0^T (g_\varepsilon + \varepsilon^{-1} Bu_\varepsilon - f, \varphi) \, dt = 0$$

for all φ in $L^2(0, T; W_0^{1,2}(G))$ with φ' in $L^2(0, T; W^{-1,2}(G))$ and $\varphi(T) = 0$. Therefore: $(u_\varepsilon(0), \varphi(0))_H = 0$ for all φ in $L^2(0, T; W^{1,2}(G))$ with φ' in $L^2(0, T; W^{-1,2}(G))$ and $\varphi(T) = 0$.

Since the set $\{\varphi(0) : \varphi$ in $L^2(0, T; W_0^{1,2}(G))$, φ' in $L^2(0, ; W^{-1,2}(G))$ and $\varphi(T) = 0\}$ is dense in H, we have: $u_\varepsilon(0) = 0$.

4) We show that $g_\varepsilon = Au_\varepsilon$.

An elementary computation gives:

$$\frac{1}{2} \| u_{\varepsilon h}^N(T) \|_H^2 + \int_0^T (Au_{\varepsilon h} + \varepsilon^{-1} Bu_{\varepsilon h} - f, u_{\varepsilon h}) \, dt \leq 0.$$

So:

$$\frac{1}{2} \| u_{\varepsilon h}(T) \|_H^2 + \limsup \int_0^T (Au_{\varepsilon h}, u_{\varepsilon h}) \, dt \leq \int_0^T (f - \varepsilon^{-1} Bu_\varepsilon, u_\varepsilon) \, dt.$$

On the other hand: $u_\varepsilon' + g_\varepsilon + \varepsilon^{-1} Bu_\varepsilon = f$.

Thus:

$$\frac{1}{2} \| u_\varepsilon(T) \|_H^2 + \limsup \int_0^T (Au_{\varepsilon h}, u_{\varepsilon h}) \, dt \leq \int_0^T (u_\varepsilon' + g_\varepsilon, u_\varepsilon) \, dt.$$

$$\leq \frac{1}{2} \| u_\varepsilon(T) \|_H^2 + \int_0^T (g_\varepsilon, u_\varepsilon) \, dt.$$

Hence:

$$\limsup \int_0^T (Au_{\varepsilon h}, u_{\varepsilon h})\, dt \leq \int_0^T (g_\varepsilon, u_\varepsilon)\, dt.$$

Since A is monotone, the above inequality implies that $g_\varepsilon = Au_\varepsilon$. It is clear that the solution is unique.

5) It remains to show that $\| u'_\varepsilon \|_{L^2(0,T;W^{1,2}(G))} \leq C$.
C is a constant independent of ε.

The proof has been carried out in [8]. To show it, we note that u_ε is the restriction to $[0, T]$ of v_ε where v_ε is the unique solution of a global boundary-value problem:

$$v'_\varepsilon + Av_\varepsilon + \varepsilon^{-1} Bv_\varepsilon = \widehat{f} \qquad \text{on } E \times G, \quad v_\varepsilon = 0 \text{ on } E \times \partial G.$$

$\widehat{f} = \zeta(t) f$ where $\zeta \in C_0^1(E)$, $\zeta(t) = 1$ for t in $[0, T]$, $\zeta(t) = 0$ for $t \leq -1$ and $t \geq 2T$. $f(t)$ is extended to E with $f(t) = 0$ for $t \leq 0$ and $f(t) = f(T)$ for $t \geq T$.

The method of difference quotients applied to v_ε gives the desired estimate. Since:

$$Au_\varepsilon + \varepsilon^{-1} Bu_\varepsilon = f - u'_\varepsilon \quad \text{is now in} \quad L^2(0, T; L^2(G)),$$

by using again the method of difference quotients and some standard results of the theory of elliptic operators, it is not difficult to show that:

$$\| u_\varepsilon \|_{L^2(0,T;W^{2,2}(G))} \leq C.$$

C is independent of ε. Cf. [8].

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Added in proof:

H. BREZIS: *Problèmes unilatéraux.* J. Math. Pures et Appl. 51 (1972), 1-68.