ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

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Duality on complex spaces

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3e série, tome 27, nº 2 (1973), p. 187-263
<http://www.numdam.org/item?id=ASNSP_1973_3_27_2_187_0>

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DUALITY ON COMPLEX SPACES

by ALDO ANDREOTTI and ARNOLD KAS

This paper has grown out of a seminar held at Stanford by us on the subject of Serre duality [20]. It should still be regarded as a seminar rather than as an original piece of research. The method of presentation is as elementary as possible with the intention of making the results available and understandable to the non-specialist. Perhaps the main feature is the introduction of Čech homology; the duality theorem is essentially divided into two steps, first duality between cohomology and homology and then an algebraic part expressing the homology groups in terms of well-known functors. We have wished to keep these steps separated as it is only in the first part that the theory of topological vector spaces is of importance.

Malgrange [12] first obtained for manifolds an extension of Serre duality making use of the theory of division of distributions. Here a proof not involving the use of that theory is given. The proof is not very different from one given by Suominen [28]. In his proof, although homology is not used explicitly, it is there between the lines.

For complex spaces, the duality theorem has appeared recently in a paper of Ramis and Ruget [17] in the language of derived categories. Here, however, for any coherent sheaf $\mathcal{F}$, we introduce a sequence of coherent sheaves $\mathcal{D}^q(\mathcal{F})$ and, dually, a sequence of « co-coherent » cosheaves $\mathcal{H}_k^q(\mathcal{F})$ which enable us to write some spectral sequences converging respectively to the homology or cohomology groups with values in the dual cosheaf $\mathcal{F}_*$ of $\mathcal{F}$ or, respectively, with values in $\mathcal{F}$. The connection of these objects with the dualising complex of Ruget and Ramis is given at the end and was suggested to us by C. Banica and O. Stanasila; to them we also want to express our warmest thanks for the tedious task of revising this paper.

Pervenuto alla Redazione l'11 Marzo 1970 e in forma definitiva il 25 Ottobre 1971.

SITI

The idea of using homology is already in a paper of Kultze [11]. The use of the open mapping theorem of Raikov [16], Schwartz [19] and Martineau [13] simplify the exposition at some point.

The second part of this paper (§ 9) has been revised after completion of the manuscript to present an exposition as self-contained as possible.

Results of duality between the separated groups of cohomology and homology can be obtained under less strict assumptions using an improved form of the duality lemma (cf. [5] and [17]).

CHAPTER 1. PRELIMINARIES

§ 1. Dual Families of Supports.

1. *Dual Families of Supports.* *a*) Let X be a locally compact and paracompact space. By a *family of supports* Φ on X we mean a collection of closed subsets of X such that

(i) if $S \in \Phi$ then any closed subset of S is in Φ

(ii) every finite union of subsets of Φ is in Φ.

By the *dual family of* Φ we mean the family Ψ of all closed subsets of X with the property

$$C \in \Psi \text{ if and only if } C \cap S \text{ is compact } \forall\, S \in \Phi.$$

It is clear that Ψ is also a family of supports.

b) A family of supports Φ is called a *paracompactifying* family if moreover it satisfies the following condition

(iii) every $S \in \Phi$ has a closed neighborhood $U(S) \in \Phi$.

The dual family Ψ of a paracompactifying family is not necessarily a paracompactifying family of supports.

EXAMPLE. Let $X = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1 \text{ or } x^2 + y^2 = 1 \text{ and } x \geq 0\}$.

Set $\Gamma = \{x^2 + y^2 = 1,\ x \geq 0\}$ and let Φ be the family of supports defined as follows:

$$\Phi = \{C \subset X \mid C \text{ closed},\ C \cap \Gamma = \emptyset\}.$$

It is a paracompactifying family of supports. The set $S = \{x^2 + y^2 \leq 1,\ x \geq 0\}$ belongs to the dual family Ψ but no neighborhood of S is in Ψ.

c) We will restrict our consideration to dual families of supports obtained by the following procedure.

Consider a 2 point compactification $\widehat{X} = X \cup \{0\} \cup \{1\}$ of the space X. By this we mean a topological structure on $X \cup \{0\} \cup \{1\}$ which is Hausdorff compact with a countable basis for open sets and agrees on X with the natural topology of X.

Consider on $\widehat{X}$ the Urysohn function $\varphi : \widehat{X} \longrightarrow \mathbb{R}$ with

$$\{0\} = \{\widehat{x} \in \widehat{X} \mid \varphi(\widehat{x}) = 0\}, \qquad \{1\} = \{\widehat{x} \in \widehat{X}\} \mid \varphi(\widehat{x}) = 1\}$$

and satisfying the condition $0 \leq \varphi \leq 1$ throughout $\widehat{X}$.

We then consider the families of supports

$$(*) \qquad \begin{aligned} \Phi &= \{C \subset X \mid C \text{ closed}, \sup_C \varphi < 1\} \\ \Psi &= \{C \subset X \mid C \text{ closed}, \inf_C \varphi > 0\}. \end{aligned}$$

Then Φ and Ψ are both paracompactifying families of supports on X each one the dual of the other.

EXAMPLES. (1) If $X \cup \{0\}$ is the Alexandroff compactification of X and if $\widehat{X}$ is the disjoint union of $X \cup \{0\}$ and $\{1\}$ then Φ is the family of all closed subsets of X while Ψ is the family of all compact subsets of X.

(2) Let Z be a topological space $g : Z \longrightarrow \mathbb{R}$ a proper continuous function on Z and $X = \{z \in Z \mid 0 < g(z) < 1\}$ then taking for $\widehat{X}$ the space obtained from $\{0 \leq g \leq 1\}$ by collapsing $\{g = 0\}$ into $\{0\}$ and $\{g = 1\}$ into $\{1\}$ we obtain as dual families of supports

$$\Phi = \{C \in X \mid \sup_C g < 1\}, \quad \psi = \{C \in X \mid \inf_C g > 0\}.$$

Note 1. Without explicit mention in the sequel any family Φ or Ψ of supports will be assumed of this form (*).

Note 2. Closed support will be denotes with the suffix c, compact supports will be denoted with the suffix k.

§ 2. Preliminaries on Topological Vector Spaces.

2. *Fréchet-Schwartz spaces.* These spaces have been explicitly introduced in [8, 12, 2]. We recall briefly their definition and their main pro-

perties. A locally convex topological vector space F is called a space of Fréchet-Schwartz (FS) if it satisfies the following assumptions:

(i) F is metrisable, its topology being defined by a sequence $\{p_n\}_{n \in \mathbb{N}}$ of seminorms on which we can make the assumption

$$p_n(x) \leq p_{n+1}(x) \quad \forall\, x \in F \quad \forall\, n \in \mathbb{N},$$

(ii) F is complete (i. e., F is a Fréchet space),

(iii) Given $\varepsilon > 0$ and $n \geq 1$ we can find a finite number of points $a_1, \dots, a_k$ in F such that

$$\{x \in F \mid p_n(x) < 1\} \subset \bigcup_{i=1}^{k} \{x \in F \mid p_{n-1}(x - a_i) < \varepsilon\}.$$

PROPOSITION 1. *Let F be a space of Fréchet-Schwartz, then*

(a) *every bounded subset of F is relatively compact,*

(b) *every closed subspace of F is a space of Fréchet-Schwartz,*

(c) *every quotient of F by a closed subspace is a space of Fréchet-Schwartz.* For the proof see [2], [8].

PROPOSITION 2. *Let F' be the strong dual of a space F of Fréchet-Schwartz* (DFS) *then F' is an inductive limit of a sequence of Banach spaces $B_0 \subset B_1 \subset \dots$,*

$$F' = \varinjlim B_n, \quad \text{(and the injective maps } B_n \to B_{n+1} \text{ are compact).}$$

This proposition is due to Sebastiao e Silva [22].

PROPOSITION 3. *Let F' be the strong dual of a space F of Fréchet-Schwartz. Let Z be a closed subspace of F'. Then*

(a) *Z is topologically isomorphic to the strong dual of F/Z^0* (1);

$$Z = \text{Hom cont}\,(F/Z^0, \mathbb{C}).$$

(b) *F'/Z is topologically isomorphic to the strong dual of Z^0*

$$F'/Z = \text{Hom cont}\,(Z^0, \mathbb{C}).$$

In particular both Z and F'/Z are strong duals of spaces of Fréchet-Schwartz.

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(1) $Z^0 = \{x \in F \mid \langle x, z \rangle = 0 \ \forall\, z \in Z\}$.

PROOF. (*a*) Let $\mathfrak{z} \in Z$, then $\mathfrak{z}$ is a continuous linear map $\mathfrak{z}: F \to C$ with the property that $\mathfrak{z} \mid Z^0 = 0$ thus $\mathfrak{z}$ defines an element $\dot{\mathfrak{z}}$ of $(F/Z^0)' = \mathrm{Hom\,cont}(F/Z^0, \mathbb{C})$ and we have

$$Z = Z^{00} = \{h \in F' \mid h \mid Z^0 = 0\} = (F/Z^0)'.$$

If $\{\mathfrak{z}_\alpha\}_{\alpha \in A} \subset Z$ converges zero this means that, for every closed convex bounded set $B \subset F$, $\sup \mid \mathfrak{z}_\alpha(B) \mid \to 0$. Let b be a convex bounded set of F/Z^0. There exists a closed convex bounded set $B \subset F$ whose image is b. This follows from the fact that F is Fréchet-Schwartz so that bounded sets are relatively compact. Then it follows that $\sup \mid \dot{\mathfrak{z}}_\alpha(b) \mid \to 0$, i. e., the map $Z \to (F/Z^0)'$ is continuous. Conversely if for every b closed convex bounded in F/Z^0 $\sup \mid \dot{\mathfrak{z}}_\alpha(b) \mid \to 0$ then for every bounded closed convex set B in F we have also $\sup \mid \mathfrak{z}_\alpha(B) \mid \to 0$ since the image of B in F/Z_0 is bounded.

(*b*) The natural map $\sigma: F' \to (Z^0)' = \mathrm{Hom\,cont}(Z^0, \mathbb{C})$ which associates to $h \in F'$, $h = h \mid Z^0$ is continuous for the strong topologies and surjective. Its kernel is $Z^{00} = Z$, thus a continuous one-to-one surjective map

$$\tau: F'/Z \to (Z^0)'.$$

The first of these spaces as a quotient of a space $\mathcal{LF}$ is a space $\mathcal{LF}$ and the second is an $\mathcal{LF}$ space as strong dual of Fréchet Schwartz. Therefore τ is also an isomorphism topologically (cf. [8] p. 271).

3. *Open Mapping Theorem.* *a*) A *Souslin space* is a topological space which is the continuous image of a complete, matric, separable space. Closed subsets and continuous images of Souslin spaces are also Souslin spaces. For Souslin topological vector spaces one has the following useful theorems.

THEOREM 1. *Let E be any Hausdorff-topological vector space.*

Let F be any locally convex Souslin topological vector space.

Let $v: F \to E$ be a continuous linear map. Then if $\mathrm{Im}\, v$ *is non meager* (*i. e., of* 2^d *category*) *then v is surjective and open.*

This theorem is useful if we know that v is surjective and E is itself of 2^d category (for instance, E a Fréchet space) (cfr. [13]).

EXAMPLE. Let H be a separable Hilbert space. Take $F = H$ with its natural Hilbert-space topology and $E = H$ with the weak topology. Take for v the identity map. Obviously v is continuous but not open. Thus H with its weak topology is of 1st category. This fact is of easy direct verification.

b) A second type of open mapping theorem is the following.

THEOREM 2. *Let E be a locally convex topological vector space which is an inductive limit of Banach spaces.*

Let F be a locally convex Souslin topological vector space.

Let $u: F \to E$ be a continuous linear map. Then if u is surjective u is also open.

This theorem is particularly useful in the category of locally convex topological vector spaces which are both Souslin and inductive limits of Banach spaces. In this category of spaces surjective and open surjective are synomymous (cf. [24]).

§ 3. Čech Homology.

4. *Precosheaves.* *a*) A *precosheaf* on a topological space X is a covariant functor

$$U \to D(U)$$

from the category of open sets $U \subset X$ to the category of abelian groups. If $U \subset V$ is an inclusion of open subsets in X, we thus have a homomorphism

$$\iota_V^U : D(U) \to D(V),$$

with the conditions that, if $U \subset V \subset W$ are open sets, we must have

$$\iota_W^U = \iota_W^V \iota_V^U .$$

Given two precosheaves on X, $\mathcal{D} = \{D(U), i_V^U\}$, $\Delta = \{\Delta(U), j_V^U\}$ a *homomorphism* $h: \mathcal{D} \to \Delta$ is a collection of group-homomorphisms

$$h_U : D(U) \to \Delta(U)$$

such that for $U \subset V$ we get a commutative diagram

$$\begin{tikzcd} D(U) \arrow[r, "h_U"] \arrow[d, "i_V^U"] & \Delta(U) \arrow[d, "j_V^U"] \\ D(V) \arrow[r, "h_V"] & \Delta(V). \end{tikzcd}$$

EXAMPLE. If $\mathcal{F}$ is a sheaf of abelian groups, setting $D(U) = \Gamma_k(U, \mathcal{F})$ (where Γ_k denotes compactly supported sections) with natural injection maps $i_V^U \colon D(U) \to D(V)$ for $U \subset V$, we obtain a precosheaf.

REMARK. We may introduce in an analogous manner precosheaves of sets, of rings, etc. Also, we may wish not to use the whole class of open sets of X but a privileged subclass $\mathcal{W}$ provided it satisfy the following property:

« for every open covering $\mathcal{U}$ of X we can find a refinement by sets of $\mathcal{W}$ ».

b) Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open covering of X. We say that $\mathcal{U}$ is *locally finite* if each compact subset meets a finite number of U_j's: this implies that the nerve $\mathcal{N}(\mathcal{U})$ of this covering is a locally finite complex.

Let $\mathcal{D} = \{D(U), i_V^U\}$ be a precosheaf on X, we set

$$C_q(\mathcal{U}, \mathcal{D}) = \coprod_{(i_0, \dots, i_q)} D(U_{i_0} \cap \dots \cap U_{i_q})$$

and define

$$\partial_{q-1} \colon C_q(\mathcal{U}, \mathcal{D}) \to C_{q-1}(\mathcal{U}, \mathcal{D}), \qquad \text{for} \quad q \geq 1$$

by

$$(*) \qquad \partial_{q-1}(g_{i_0 \dots i_q}) = \coprod_{h=0}^{q} (-1)^h \, i^{U_{i_0} \cap \dots \cap U_{i_q}}_{U_{i_0} \cap \dots \cap \hat{U}_{i_h} \cap \dots \cap U_{i_q}} \, g_{i_0 \dots i_q}$$

and equivalently by

$$(**) \qquad (\partial_{q-1} g)_{i_0 \dots i_{q-1}} = \sum_{h=0}^{q} \sum_{\gamma} (-1)^h \, i^{U_{i_0} \cap \dots \cap U_{i_{q-1}} \cap U_\gamma}_{U_{i_0} \cap \dots \cap U_{i_{q-1}}} \, g_{i_0 \dots i_{h-1} \gamma i_h \dots i_{q-1}}$$

for any $g = \{g_{i_0 \dots i_q}\} \in C_q(\mathcal{U}, \mathcal{D})$. From (**) it is clear that the definition is meaningful even if $\mathcal{U}$ is not locally finite

REMARK. Let $\Phi(\mathcal{N}(\mathcal{U})) = \{(i_0, \dots, i_q) \mid U_{i_0} \cap \dots \cap U_{i_q} \neq \emptyset\}$ be the set of symplices of the nerve of $\mathcal{U}$. Then $C_q(\mathcal{U}, \mathcal{D})$ is the chain group of the complex $\mathcal{N}(\mathcal{U})$ with coefficients in the local system $\mathcal{D} = \{D(U_{i_0} \cap \dots \cap U_{i_q})\}$. With loose notations $C_q(\mathcal{U}, \mathcal{D}) = \{\Sigma\, g_{i_0 \dots i_q}(i_1, \dots, i_q)\}$ and the boundary ∂_{q-1} is the usual operator:

$$\partial(\Sigma\, g_{i_0 \dots i_q}(i_0, \dots, i_q)) = \Sigma \sum_h (-1)^h \left(i^{i_0 \dots i_q}_{i_0 \dots \hat{i}_h \dots i_q} \, g_{i_0 \dots i_q} \right) (i_0 \dots \hat{i}_h \dots i_q).$$

From the previous remark we derive the folloving conclusion:

(i) For every $q \geq 0$ $\partial_q \circ \partial_{q+1} = 0$. We define $\partial_{-1} = 0$ so that we get a chain complex:

$$\dots \xrightarrow{\partial_q} C_q(\mathcal{U}, \mathcal{D}) \xrightarrow{\partial_{q-1}} C_{q-1}(\mathcal{U}, \mathcal{D}) \xrightarrow{\partial_{q-1}} \dots \xrightarrow{\partial_2} C_2(\mathcal{U}, \mathcal{D})$$

$$\xrightarrow{\partial_1} C_1(\mathcal{U}, \mathcal{D}) \xrightarrow{\partial_0} C_0(\mathcal{U}, \mathcal{D}) \xrightarrow{\partial_{-1}} 0.$$

Its homology will be denoted by $H_q(\mathcal{U}, \mathcal{D})$, *for* $q = 0, 1, \dots$.

Note that we have an augmentation $\varepsilon : C_0(\mathcal{U}, \mathcal{D}) \to \mathcal{D}(X)$ given by $\varepsilon\{g_i\} = \Sigma\, \iota_X^{U_i} g_i$ and which vanishes on $\operatorname{Im} \partial_0$.

(ii) If $\mathcal{V} = \{V_j\}_{j \in J}$ is a refinement of the covering $\mathcal{U} = \{U\}_{i \in I}$, for any refinement function $\tau : J \to I$ $(V_j \subset U_{\tau(j)})$ we get a simplicial map

$$\tau : \mathcal{N}(\mathcal{V}) \to \mathcal{N}(\mathcal{U}).$$

Correspondingly we get a chain map and therefore a homomorphism

$$\tau_* : H_q(\mathcal{V}, \mathcal{D}) \to H_q(\mathcal{U}, \mathcal{D}).$$

This map τ_* *is independent of the choice of the refinement function.* Indeed if $\tau' : J \to I$ is another such function the simplicial maps τ and τ' are $\mathcal{N}(\mathcal{U})$-near (i. e., $\forall j$ $\tau(j)$ and $\tau'(j)$ lie in a simplex of $\mathcal{N}(\mathcal{U})$). Thus they are homotopic and therefore have the same effect on homology.

Explicitly the homotopy operator $k : C_q(\mathcal{V}, \mathcal{D}) \to C_{q+1}(\mathcal{U}, \mathcal{D})$ is given by

$$k(g(j_0, \dots, j_q)) = \Sigma\, (-1)^h\, \iota_*^* g(\tau(j_0), \dots, \tau(j_h)\, \tau'(j_h), \dots, \tau'(j_q)).$$

c) Given an exact sequence of precoscheaves and homomorphisms

$$0 \to \mathcal{D}' \to \mathcal{D} \to \mathcal{D}'' \to 0 \tag{1}$$

for any covering $\mathcal{U}$ we obtain an exact sequence of complexes.

$$0 \to C_*(\mathcal{U}, \mathcal{D}') \to C_*(\mathcal{U}, \mathcal{D}) \to C_*(\mathcal{U}, \mathcal{D}'') \to 0$$

and therefore an exact homology sequence:

$$\dots \to H_1(\mathcal{U}, \mathcal{D}') \to H_1(\mathcal{U}, \mathcal{D}) \to H_1(\mathcal{U}, \mathcal{D}'') \xrightarrow{\partial}$$

$$\to H_0(\mathcal{U}, \mathcal{D}') \to H_0(\mathcal{U}, \mathcal{D}) \to H_0(\mathcal{U}, \mathcal{D}'') \to 0.$$

(*d*) We can define

$$H_q(X, \mathcal{D}) = \varprojlim H_q(\mathcal{U}, \mathcal{D})$$

the limit being taken over all locally finite coverings. We will call $H_q(X, \mathcal{D})$ the q-th homology group of X with coefficients in the precosheaf $\mathcal{D}$.

Note that we can take the limit over a family of coverings which is cofinal to the family of all coverings of X.

Given an exact sequence of precosheaves (1) we deduce from it a sequence of order 2 (the composition of two successive maps is zero) which, however, may no longer be exact:

$$\ldots \to H_1(X, \mathcal{D}') \to H_1(X, \mathcal{D}) \to H_1(X, \mathcal{D}'') \xrightarrow{\partial}$$

$$\xrightarrow{\partial} H_0(X, \mathcal{D}') \to H_0(X, \mathcal{D}) \to H_0(X, \mathcal{D}'') \to 0.$$

5. *Homology with Supports.* *a*) We assume that X is a locally compact paracompact space with a countable basis of open sets. This assumption will always be made in the following sections.

Let $\mathcal{U}$ be a locally finite covering and let $\mathcal{D} = \{D(U), i_V^U\}$ be a precosheaf. Instead of the complex introduced in the previous section we can define a new complex by setting

$$C_q^c(\mathcal{U}, \mathcal{D}) = \prod_{(i_0, \ldots, i_q)} D(U_{i_0} \ldots U_{i_q})$$

and define the boundary operator

$$\partial_{q-1} : C_q^c(\mathcal{U}, \mathcal{D}) \to C_{q-1}^c(\mathcal{U}, \mathcal{D})$$

by the formula analogous to (*) or (**) of the previous section. From its very expression (**) this definition is meaningful and we obtain a chain complex

$$\ldots \to C_q^c(\mathcal{U}, \mathcal{D}) \xrightarrow{\partial_{q-1}} C_{q-1}^c(\mathcal{U}, \mathcal{D}) \to \ldots \xrightarrow{\partial_1} C_1^c(\mathcal{U}, \mathcal{D}) \xrightarrow{\partial_0} C_0^c(\mathcal{U}, \mathcal{D}) \to 0$$

whose homology will be denoted by $H_q^c(\mathcal{U}, \mathcal{D})$ for $q = 0, 1, \ldots$.

b) Let Φ and Ψ denote dual families of supports as defined in section 1, by means of a continuous function $\varphi : X \to (0, 1)$:

$$\Phi = \{C \subset X \mid C \text{ closed}, \sup_C \varphi < 1\}, \quad \Psi = \{C \subset X \mid C \text{ closed}, \inf_C \varphi > 0\}.$$

A locally finite covering $\mathcal{U} = \{U_i\}_{i \in I}$ of X will be called *adapted* to the given pair of dual families of supports if the following conditions are satisfied

(i) $\forall\, i \in I$ $\overline{U}_i$ is compact

(ii) If $\inf\limits_{U_i} \varphi < 1 - \dfrac{1}{n}$ then $\sup\limits_{U_i} \varphi < 1 - \dfrac{1}{n+1}$ for $n = 1, 2, \ldots$ (adapted to Φ).

(iii) If $\sup\limits_{U_i} \varphi > \dfrac{1}{n}$ then $\inf\limits_{U_i} \varphi > \dfrac{1}{n+1}$ for $n = 1, 2, \ldots$ (adapted to Ψ).

Locally finite adapted coverings are cofinal to all coverings.

Let $\mathcal{F}$ be a presheaf on X and $\mathcal{D}$ a precosheaf on X. We consider for each $q > 0$ to groups

$$C^q(\mathcal{U}, \mathcal{F}) = \prod_{(i_0, \ldots, i_q)} \mathcal{F}(U_{i_0} \cap \ldots \cap U_{i_q});$$

$$C_q^c(\mathcal{U}, \mathcal{D}) = \prod_{(i_0, \ldots, i_q)} D(U_{i_0} \cap \ldots \cap U_{i_q}).$$

For $f \in C^q(\mathcal{U}, \mathcal{F})$ and for $g \in C_q^c(\mathcal{U}, \mathcal{D})$ we define

$$\operatorname{supp}(f) = \overline{\bigcup_{f_{i_0 \ldots i_q} \neq 0} U_{i_0} \cap \ldots \cap U_{i_q}}\ (^2);$$

$$\operatorname{supp}(g) = \overline{\bigcup_{g_{i_0 \ldots i_q} \neq 0} U_{i_0} \cap \ldots \cap U_{i_q}}$$

and

$$C_\Phi^q(\mathcal{U}, \mathcal{F}) = \{f \in C^q(\mathcal{U}, \mathcal{F}) \mid \operatorname{supp}(f) \in \Phi\};$$

$$C_q^\Psi(\mathcal{U}, \mathcal{D}) = \{g \in C_q^c(\mathcal{U}, \mathcal{D}) \mid \operatorname{supp}(g) \in \Psi\}.$$

If the covering $\mathcal{U}$ is adapted to the families of supports Φ and Ψ (as we will always assume) then the coboundary and respectively the boundary

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(2) Given $f \in C^q(\mathcal{U}, \mathcal{F})$, $\operatorname{supp}(f)$ as it is defined here may be different (actually larger) than the support of f as defined in Godement ([6], p. 208). However, if $\operatorname{supp}(f) \in \Phi$ in our sense then $\operatorname{supp}(f) \in \Phi$ in the sense of Godement and conversely. This is due to the fact that $\mathcal{U}$ is adapted to Φ.

operators induces maps

$$\delta_q : C^q_\Phi(\mathcal{U}, \mathcal{F}) \to C^{q+1}_\Phi(\mathcal{U}, \mathcal{F}); \quad \partial_{q-1} : C^\psi_q(\mathcal{U}, \mathcal{D}) \to C^\psi_{q-1}(\mathcal{U}, \mathcal{D}).$$

Therefore we get new complexes and corresponding cohomology and homology groups that we will denote by

$$H^q_\Phi(\mathcal{U}, \mathcal{F}) \text{ and respectively } H^\psi_q(\mathcal{U}, \mathcal{D}).$$

We define $H^q_\Phi(X, \mathcal{F}) = \varinjlim_{\mathcal{U}} H^q(\mathcal{U}, \mathcal{F})$ and $H^\psi_q(X, \mathcal{D}) = \varprojlim H^\psi_q(\mathcal{U}, \mathcal{D})$ the limits to be taken over all adapted coverings.

REMARK. If Φ is the family of all closed sets and thus Ψ that of compact sets then we obtain again the definitions of $H^q(\mathcal{U}, \mathcal{F})$ and $H_q(\mathcal{U}, \mathcal{D})$.

c) Let $\mathcal{U}$ be adapted to Φ and Ψ and set for $s = 1, 2, \ldots$

$$\mathcal{U}^s = \left\{ U_i \in \mathcal{U} \mid \sup_{U_i} \varphi < 1 - \frac{1}{s} \right\}$$

$$\mathcal{U}_s = \left\{ U_i \in \mathcal{U} \mid \inf_{U_i} \varphi > \frac{1}{s} \right\}.$$

We can define a sequence of *injections*

$$j^s : C^q(\mathcal{U}^s, \mathcal{F}) \to C^q(\mathcal{U}^{s+1}, \mathcal{F})$$

$$j_s : C^c_q(\mathcal{U}_s, \mathcal{D}) \to C^c_q(\mathcal{U}_{s+1}, \mathcal{D})$$

by identifying the left-hand group to a direct factor of the right-hand group:

$$j^s(f)_{i_0 \ldots i_q} = \begin{cases} f_{i_0 \ldots i_q} & \text{if } U_{i_0}, \ldots, U_{i_q} \in \mathcal{U}^s \\ 0 & \text{otherwise} \end{cases}$$

and similarly for j_s.

We get in this way that

$$C^q_\Phi(\mathcal{U}, \mathcal{F}) = \varinjlim_s C^q(\mathcal{U}^s, \mathcal{F})$$

$$C^\psi_q(\mathcal{U}, \mathcal{D}) = \varinjlim_s C^c_q(\mathcal{U}_s, \mathcal{D}).$$

d) In a similar manner we can define maps

$$r^{s+1}: C_q(\mathcal{U}^s, \mathcal{D}) \leftarrow C_q(\mathcal{U}^{s+1}, \mathcal{D})$$

by setting

$$r^{s+1}(f)_{i_0 \dots i_q} = f_{i_0 \dots i_q} \quad \text{if} \quad U_{i_0}, \dots, U_{i_q} \in \mathcal{U}^s$$

and we obtain in this way that

$$C_q^{\psi}(\mathcal{U}, \mathcal{D}) = \varprojlim_s C_q(\mathcal{U}^s, \mathcal{D}).$$

An analogous argument can be given for cohomology if we use the cohomology groups with compact supports

$$C_k^q(\mathcal{U}, \mathcal{F}) = \coprod_{i_0 \dots i_q} \mathcal{F}(U_{i_0} \cap \dots \cap U_{i_q}).$$

6. *Cosheaves.* *a*) Let X be a topological space and let $\mathcal{W}$ be a privileged class of open sets on X.

A precosheaf $\mathcal{D} = \{D(U), i_V^U\}$ defined on $\mathcal{W}$ is called a *cosheaf* if for any open set $\Omega \in \mathcal{W}$ and any covering $\mathcal{U}_\Omega \subset \mathcal{W}$ of Ω we have an exact sequence:

$$C_1(\mathcal{U}_\Omega, \mathcal{D}) \xrightarrow{\partial_0} C_0(\mathcal{U}_\Omega, \mathcal{D}) \xrightarrow{\varepsilon_{\mathcal{U}_\Omega}} D(\Omega) \to 0$$

where $\varepsilon_{\mathcal{U}}$ is defined as follows:

$$\varepsilon_{\mathcal{U}_\Omega}(f) = \Sigma\, i_\Omega^{U_i} f_i$$

and $D(\emptyset) = 0$. For a cosheaf we have therefore that for every $\Omega \in \mathcal{W}$

$$D(\Omega) = H_0(\mathcal{U}_\Omega, \mathcal{D})$$

for any choice of the covering $\mathcal{U}_\Omega$.

b) A cosheaf $\mathcal{D} = \{D(U), i_V^U\}$ is called a *fine cosheaf* if for any locally finite covering $\mathcal{U} = \{U_i\}_{i \in I} \subset \mathcal{W}$ and any open set $V \in \mathcal{W}$ a system of homomorphisms

$$\varrho^V_{V \cap U_i}: D(V) \to D(V \cap U_i)$$

is given such that

(i) for any $f \in D(V)$, $\varrho^V_{V \cap U_i}(f) = 0$ but all for finitely many U_i's.

(ii) $\Sigma\, i_V^{V \cap U_i} \circ \varrho_{V \cap U_i}^V =$ identity on $D(V)$.

(iii) for $V \subset W$ both open and in $\mathcal{W}$ we have a commutative diagram

$$\begin{tikzcd} D(W) \arrow[r, "\varrho_{W \cap U_i}^W"] & D(W \cap U_i) \\ D(V) \arrow[u, "i_W^V"] \arrow[r, "\varrho_{V \cap U_i}^V"'] & D(V \cap U_i) \arrow[u, "i_{W \cap U_i}^{V \cap U_i}"'] \end{tikzcd}$$

Note that condition (i) is automatically satisfied if X is locally compact and $\mathcal{W}$ a family of relatively compact open sets.

PROPOSITION 4. *Let X be locally compact paracompact, let Ψ be a family of supports and let $\mathcal{U} \subset \mathcal{W}$ be any adapted locally finite covering to the family of supports Ψ* (cf. n. 5).

For any fine cosheaf $\mathcal{D} = \{D(U), i_V^U\}$ we have

$$H_q^\psi(\mathcal{U}, \mathcal{D}) = 0 \quad \textit{for any } q > 0.$$

PROOF. Let $h = (h_{\alpha_0 \dots \alpha_q}) \in C_q^\psi(\mathcal{U}, \mathcal{D})$ we define

$$g = (g_{\beta_0, \dots \beta_{q+1}}) \in C_{q+1}^\psi(\mathcal{U}, \mathcal{D})$$

by

$$g_{\beta_0 \dots \beta_{q+1}} = \Sigma\, (-1)^j\, \varrho_{\beta_j} h_{\beta_0 \dots \hat{\beta}_j \dots \beta_{q+1}}$$

where ϱ_β stands for $\varrho_{\cap U_\beta}$. We get, with slightly loose notations,

$$\begin{aligned} (\partial g)_{\alpha_0 \dots \alpha_q} &= \sum_\gamma \Sigma\, (-1)^i\, g_{\alpha_0 \dots \alpha_{i-1}\, \gamma \alpha_i \dots \alpha_q} \\ &= \sum_\gamma \Big\{ \varrho_\gamma h_{\alpha_0 \dots \alpha_q} + \sum_{1 \le j \le i-1} (-1)^{i+j}\, \varrho_{\alpha_j} h_{\alpha_0 \dots \hat{\alpha}_j \dots \alpha_{i-1}\, \gamma \dots \alpha_q} \\ &\qquad - \sum_{1 \le i-1 < j} (-1)^{i+j}\, \varrho_{\alpha_j} h_{\alpha_0 \dots \alpha_{i-1}\, \gamma \dots \hat{\alpha}_j \dots \alpha_q} \Big\} \\ &= \sum_\gamma \varrho_\gamma h_{\alpha_0 \dots \alpha_q} - \Sigma\, (-1)^j\, \varrho_{\alpha_j} (\partial h)_{\alpha_0 \dots \hat{\alpha}_j \dots \alpha_q}. \end{aligned}$$

Therefore

$$\partial g = h \quad \text{if} \quad \partial h = 0, \quad \text{by virtue of (ii).}$$

c) Let $\mathcal{S}$ denote a sheaf on X. We set for U open in X

$$D(U) = \Gamma_k(U, \mathcal{S})$$

where Γ_k denotes sections with compact support. For $U \subset V$ we get natural injection maps

$$i_V^U : D(U) \longrightarrow D(V).$$

In this way to any sheaf $\mathcal{S}$ on X we associate a precosheaf $\mathcal{D}(\mathcal{S}) \equiv \mathcal{D} = \{D(U), i_V^U\}$. This precosheaf has the property:

(*) for every U open the map

$$i_X^U : D(U) \longrightarrow D(X)$$

is injective.

A precosheaf having property (*) is called a *flabby* precosheaf (cf. [3]).

PROPOSITION 5. (*a*) *If $\mathcal{S}$ is a soft sheaf the associated precosheaf $\mathcal{D}(\mathcal{S})$ is a cosheaf (which is also flabby).*

(*b*) *For any family of supports Ψ and any adapted locally finite covering $\mathcal{U}$ to Ψ we have*

$$H_q^{\Psi}(\mathcal{U}, \mathcal{D}(\mathcal{S})) = 0 \quad \textit{for any } q > 0.$$

PROOF. For every set $U_{i_0} \cap \ldots \cap U_{i_q}$ we define the sheaf

$$\mathcal{S}_{i_0 \ldots i_q} = \begin{cases} \mathcal{S} \text{ on } U_{i_0} \cap \ldots \cap U_{i_q} \\ 0 \text{ outside } U_{i_0} \cap \ldots \cap U_{i_q}. \end{cases}$$

These sheaves are soft. We get an exact sequence of sheaves

$$\rightarrow \coprod_{(i_0 \ldots i_q)} \mathcal{S}_{i_0 \ldots i_q} \xrightarrow{\partial} \coprod_{(i_0 \ldots i_{q-1})} \mathcal{S}_{i_0 \ldots i_{q-1}} \xrightarrow{\partial}$$

$$\rightarrow \coprod_{i_0} \mathcal{S}_{i_0} \xrightarrow{\varepsilon} \mathcal{S} \rightarrow 0 \text{ (3)}$$

where the boundary operators ∂ are defined by the usual formula

$$\partial(s_{i_0 \ldots i_q}) = \coprod (-1)^h i_h(s_{i_0 \ldots i_q})$$

(3) By $\coprod \mathcal{S} \ldots$ we mean the sheaf whose stalk at each point is the direct sum of the stalks of the sheaves $\mathcal{S} \ldots$.

where $i_h : \mathcal{S}_{i_0 \dots i_q} \to \mathcal{S}_{i_0 \dots \hat{i}_h \dots i_q}$ is the natural inclusion. The sheaves of the sequence are all soft sheaves and the sequence is exact since at each point $x \in X$ the stalks give the homology of a k-simplex with coefficients in $\mathcal{S}_x$, k being the number of sets U_i containing x. Applying to the sequence the functor Γ_ψ we get an exact sequence of groups. Now we remark that

$$\Gamma_\psi (X, \coprod \mathcal{S}_{i_0 \dots i_q}) = C_q^\psi (\mathcal{U}, \mathcal{D}(\mathcal{S}))$$

since on each $U_{i_0} \cap \dots \cap U_{i_q}$, being relatively compact, one must have

$$\Gamma (X, \mathcal{S}_{i_0 \dots i_q}) = \Gamma_k (U_{i_0} \cap \dots \cap U_{i_q}, \mathcal{S}).$$

This proves the second assertion. In particular we get $H_0^\psi (\mathcal{U}, \mathcal{D}(\mathcal{S})) = \Gamma_\psi (X, \mathcal{S})$. If Ψ is the family of compact supports we obtain therefore the exactness of

$$C_1 (\mathcal{U}, \mathcal{D}(\mathcal{S})) \to C_0 (\mathcal{U}, \mathcal{D}(\mathcal{S})) \to \Gamma_k (X, \mathcal{S}) = \mathcal{D}(\mathcal{S})(X)$$

which shows that the precosheaf $\mathcal{D}(\mathcal{S})$ is a cosheaf.

The following is a remarkable theorem due to Bredon [3]·

THEOREM 3. *Every flabby cosheaf is of the form $\mathcal{D}(\mathcal{S})$ for a unique sof sheaf $\mathcal{S}$.*

d) We mention a very mild form of the analog of de Rham theorem

THEOREM 4. *Let $\mathcal{D} = \{\mathcal{D}(U), i_V^U\}$ be a cosheaf. Suppose that*

$$\dots \to \Delta_s \xrightarrow{h_{s-1}} \Delta_{s-1} \to \dots \to \Delta_1 \xrightarrow{h_0} \Delta_0 \xrightarrow{\varepsilon} \mathcal{D} \to 0$$

is a sequence of cosheaves and homomorphisms.

Let $\mathcal{U} = \{U_i\}$ be an open covering of X adapted to the family of supports Ψ with the following properties :

(i) $\forall\, s \geq 0$ *and* $\forall\, j > 0$ $H_j^\psi (\mathcal{U}, \Delta_s) = 0$

(ii) *On every open set $U = U_{i_0} \cap \dots \cap U_{i_q}$ we have an exact sequence*

$$\dots \to \Delta_s (U) \to \Delta_{s-1} (U) \to \dots \to \Delta_1 (U) \to \Delta_0 (U) \to \mathcal{D}(U) \to 0.$$

Then we get

$$H_j^\psi (\mathcal{U}, \mathcal{D}) \simeq \frac{\operatorname{Ker} \{H_0^\psi (\mathcal{U}, \Delta_j) \to H_0^\psi (\mathcal{U}, \Delta_{j-1})\}}{\operatorname{Im} \{H_0^\psi (\mathcal{U}, \Delta_{j+1}) \to H_0^\psi (\mathcal{U}, \Delta_j)\}}.$$

In particular for compact supports we get

$$H_j(\mathcal{U}, \mathcal{D}) = \frac{\operatorname{Ker}\{\Delta_j(X) \to \Delta_{j-1}(X)\}}{\operatorname{Im}\{\Delta_{j+1}(X) \to \Delta_j(X)\}}$$

which is a sort of de Rham and Leray theorem.

REMARK. We omit here the discussion of existence of an acyclic (and in particular flabby) « resolution » of a cosheaf.

7. *The Leray Theorem on Acyclic Coverings* (cf. [21]). Let $\mathcal{D} = \{D(U), \iota_V^U\}$ be a cosheaf. Let Ψ be a family of supports and let $\mathcal{U} = \{U_i\}_{i \in I}$ $\mathcal{V} = \{V_j\}_{j \in J}$ be two locally finite open coverings of X adapted to Ψ. As the case may be, it may be necessary to do everything with a privileged class $\mathcal{W}$ of open sets. We refrain from mentioning it in the sequel since the changes only amount to a more pedantic notation and lengthier statements.

We define a double complex associated with $\mathcal{D}$, $\mathcal{U}$, $\mathcal{V}$ and Ψ as follows:

$$C_{pq}^{\psi}(\mathcal{U}, \mathcal{V}; \mathcal{D}) = \{f \in \Pi\, D(U_{i_0} \cap \ldots \cap U_{i_p} \cap V_{j_0} \cap \ldots \cap V_{i_q}) \mid \operatorname{supp}(f) \in \Psi\}$$

$$= \{f \in \Pi\, D(U_{s_p} \cap V_{\sigma_q}) \mid \operatorname{supp}(f) \in \Psi\}$$

where s_p denotes a p-simplex of $\mathcal{N}(\mathcal{U})$ and σ_q a q-simplex of $\mathcal{N}(\mathcal{V})$.

We define

$$\partial_{\mathcal{U}}\colon C_{pq}^{\psi}(\mathcal{U}, \mathcal{V}; \mathcal{D}) \to C_{p-1,q}^{\psi}(\mathcal{U}, \mathcal{V}; \mathcal{D})$$

$$\partial_{\mathcal{V}}\colon C_{pq}^{\psi}(\mathcal{U}, \mathcal{V}; \mathcal{D}) \to C_{p,q-1}^{\psi}(\mathcal{U}, \mathcal{V}; \mathcal{D})$$

by

$$(\partial_{\mathcal{U}} f)_{i_0 \cdots i_{p-1} j_0 \cdots j_q} = \sum_{\gamma} \Sigma (-1)^h i : f_{i_0 \cdots i_{h-1} r i_h \cdots i_{p-1} j_0 \cdots j_q}$$

and similarly for $\partial_{\mathcal{V}}$. We have

$$\partial_{\mathcal{U}} \partial_{\mathcal{U}} = 0\,; \quad \partial_{\mathcal{V}} \partial_{\mathcal{V}} = 0\,; \quad \partial_{\mathcal{U}} \partial_{\mathcal{V}} = \partial_{\mathcal{V}} \partial_{\mathcal{U}}\,.$$

Set

$$d = \partial_{\mathcal{U}} + (-1)^p \partial_{\mathcal{V}} \quad \text{on} \quad C_{pq}^{\psi}\,.$$

Thus $C_*^{\psi} = \coprod C_{pq}^{\psi}$ is a complex with respect to $\partial_{\mathcal{U}}$, to $\partial_{\mathcal{V}}$, to d.

(*b*) We now assume that the covering $\mathcal{W} = \mathcal{U} \cap \mathcal{V}$ is also locally finite so that on each V_{σ_q} the covering $\mathcal{U}$ induces a finite covering $\mathcal{U} \cap V_{\sigma_q}$.

With this condition we get in particular:

$$H_{pq}\, C_*^{\psi}(\mathcal{U}, \mathcal{V}; \mathcal{D}) : \partial_{\mathcal{U}}) = \{ f \in \coprod_{\sigma_q} H_p(\mathcal{U} \cap V_{\sigma_q}; \mathcal{D}) \mid \operatorname{supp}(f) \in \Psi \}$$ (4)

and especially for $p = 0$

$$H_{0q}(C_*^{\psi}(\mathcal{U}, \mathcal{V}; \mathcal{D}); \partial_{\mathcal{U}}) = \{ f \in \coprod_{\sigma_q} H_0(\mathcal{U} \cap V_{\sigma_q}, \mathcal{D}) \mid \operatorname{supp}(f) \in \Psi \}$$

$$= \{ f \in \coprod_{\sigma_q} D(V_{\sigma_q}) \mid \operatorname{supp}(f) \in \Psi \}$$

$$= C_q^{\psi}(\mathcal{V}; \mathcal{D})$$

because $\mathcal{D} = \{D(U), i_V^U\}$ is a cosheaf.

We obtain therefore a natural surjection

$$C_{0q}^{\psi}(\mathcal{U}, \mathcal{V}; \mathcal{D}) \xrightarrow{\varepsilon_{\mathcal{U}}} C_q^{\psi}(\mathcal{V}, \mathcal{D})$$

given by

$$\{ f_{i_0 j_0 \cdots j_q} \} \to \{ \sum_{i_0} i_{V_{j_0} \cap \cdots \cap V_{j_q}}^{U_{i_0} \cap V_{j_0} \cap \cdots \cap V_{j_q}} f_{i_0 j_0 \cdots j_q} \}.$$

We extend this surjection on the whole complex C_*^{ψ} by setting $\varepsilon_{\mathcal{U}} = 0$ on C_{pq}^{ψ} if $p > 0$.

PROPOSITION 6. *Suppose that*

$$H_{pq}(C_*^{\psi}(\mathcal{U}, \mathcal{V}; \mathcal{D}); \partial_{\mathcal{U}}) = 0 \quad \textit{for } p > 0 \quad \textit{and} \quad q \geq 0$$

then the natural surjection

$$(\coprod_n \coprod_{p+q=n} C_{pq}^{\psi}(\mathcal{U}, \mathcal{V}; \mathcal{D}); d) \xrightarrow{\varepsilon_{\mathcal{U}}} (\coprod_n C_n^{\psi}(\mathcal{V}, \mathcal{D}), \partial_{\mathcal{V}})$$

as a homomorphism of complexes induces an isomorphism in homology.

---

(4) By this we means that f has a representative in the cycle group

$$\tilde{f} \in \prod_{\sigma_q} Z_p(\mathcal{U} \cap V_{\sigma_q}, \mathcal{D})$$

whose support is in Ψ.

PROOF. We consider the exact sequence

$$0 \to K_{pq} \to C^{\psi}_{pq}(\mathcal{U}, \mathcal{V}; \mathcal{D}) \xrightarrow{\varepsilon_{\mathcal{U}}} C^{\psi}_{pq}(\mathcal{U}, \mathcal{V}; \mathcal{D})/\mathrm{Ker}\, \varepsilon_{\mathcal{U}} \to 0$$

where $K_{pq} = \mathrm{Ker}\, \varepsilon_{\mathcal{U}} \mid C^{\psi}_{pq}$. This can be considered as an exact sequence of complexes either if we consider on C^{ψ}_{*} as boundary operator $\partial_{\mathcal{U}}$ or d. Note that $\partial_{\mathcal{U}} = 0$ on the third complex.

In the first instance the assumption tells us that $H_{pq}(K_*, \partial_{\mathcal{U}}) = 0$ $\forall\, p \geq 0,\ q \geq 0$. The conclusion amounts to proving instead that $H_i(K_*, d) = 0$ $\forall\, i \geq 0$.

Set $K_h = \coprod_{q \leq h} K_{pq}$ so that $0 = K_{-1} \subset K_0 \subset K_1 \subset \ldots$ is an increasing sequence of subcomplexes with respect to d and

$$\{K_h/K_{h-1}, d\} = \{\coprod_p K_{ph}, \partial_{\mathcal{U}}\} \quad \forall\, h \geq 0.$$

By the exact sequence

$$0 \to K_{h-1} \to K_h \to K_h/K_{h-1} \to 0 \quad \text{(diff. operator } d\text{)}$$

we get:

$$H_n(K_h/K_{h-1}, d) = 0 \quad \forall\, h \text{ and } n$$

because of the assumption; thus

$$H_n(K_{-1}, d) = 0 = H_n(K_0, d) = H_n(K_1, d) = \ldots = H_n(K_h, d) \quad \forall\, n \geq 0.$$

If $h > n$ $H_n(K_h, d) = H_n(K_*, d)$. This concludes the proof.

COROLLARY. *If for every $\sigma_q \in \mathcal{N}(\mathcal{V})$ we have*

$$H_p(\mathcal{U} \cap V_{\sigma_q}, \mathcal{D}) = 0 \quad \textit{for} \quad p > 0$$

then

$$\varepsilon^*_{\mathcal{U}} : H^{\psi}_n(\mathcal{U}, \mathcal{V}; \mathcal{D}) \to H^{\psi}_n(\mathcal{V}, \mathcal{D})$$

is an isomorphism.

LEMMA. *If $\mathcal{W} = \{W_i\}_{i \in I}$ is a covering of a space Y and if for $i = i_0$ $W_{i_0} = Y$ then for any precosheaf $\mathcal{D}$ we get*

$$H^{\psi}_p(\mathcal{W}, \mathcal{D}) = 0 \quad \forall\, p > 0.$$

We set $\mathcal{W}' = \{Y\}$ so that $\mathcal{W} < \mathcal{W}' < \mathcal{W}$. We consider the two refine-

ment functions for $\mathcal{W}' < \mathcal{W}$

$$\tau(i) = i \quad \text{and} \quad \tau(i) = i_0 .$$

They induce on $H_p^{\psi}(\mathcal{W}', \mathcal{D})$ the same homomorphism which must therefore be the identity. However the second of these homomorphisms factors as follows:

$$H_p(\mathcal{W}', \mathcal{D}) \to H_p(\mathcal{W}'', \mathcal{D}) \to H_p(\mathcal{W}', \mathcal{D}).$$

Since $H_p(\mathcal{W}'', \mathcal{D}) = 0$ for $p > 0$ we get the conclusion.

(c) In particular if $\mathcal{V} < \mathcal{U}$, it follows from the lemma that the assumptions of the corollary are satisfied and thus

$$\varepsilon^* : H_p^{\psi}(\mathcal{U}, \mathcal{V}; \mathcal{D}) \xrightarrow{\sim} H_p^{\psi}(\mathcal{V}, \mathcal{D}) \quad \forall\, p \geq 0$$

and any cosheaf $\mathcal{D}$.

We need now to prove that, for $\mathcal{V} < \mathcal{U}$, we have the following

PROPOSITION 7. *If $\mathcal{V} < \mathcal{U}$ then the homomorphism*

$$\varepsilon_{\mathcal{V}} \circ \varepsilon_{\mathcal{U}}^{-1} : H_p^{\psi}(\mathcal{V}, \mathcal{D}) \to H_p^{\psi}(\mathcal{U}, \mathcal{D})$$

coincides with the map induced by a refinement function for $\mathcal{V} < \mathcal{U}$.

PROOF. Let $\mathcal{V} = \{V_j\}_{j \in J}$, $\mathcal{U} = \{U_i\}_{i \in I}$ and let $\{f_{j_0 j_1 \dots j_q}\} \in C_q^{\psi}(\mathcal{V}, \mathcal{D})$ with $\partial_{\mathcal{V}} f = 0$.

We can write

$$f_{j_0 \dots j_q} = \sum_{i_0} k^{(0)}_{i_0 j_0 \dots j_q}$$

i. e.

$$f = \varepsilon_{\mathcal{U}} k^{(0)} \quad \text{with} \quad k^{(0)} \in C_{0q}(\mathcal{U}, \mathcal{V}; \mathcal{D}).$$

We must have

$$\varepsilon_{\mathcal{U}} \partial_{\mathcal{V}} k^{(0)} = 0$$

$$\text{thus} \quad \partial_{\mathcal{V}} k^{(0)} = \partial_{\mathcal{U}} k^{(1)} \quad \text{with} \quad k^{(1)} \in C_{1q-1}(\mathcal{U}, \mathcal{V}; \mathcal{D}).$$

Also $\partial_{\mathcal{U}} \partial_{\mathcal{V}} k^{(1)} = 0$ thus (since $\mathcal{V} < \mathcal{U}$)

$$\partial_{\mathcal{V}} k^{(1)} = \partial_{\mathcal{U}} k^{(2)} \quad \text{with} \quad k^{(2)} \in C_{2q-1}(\mathcal{U}, \mathcal{V}; \mathcal{D})$$

and so on.

We get

$$d(k^{(0)} + k^{(1)} + k^{(2)} + \dots + k^{(q)}) = 0$$

$$\varepsilon_{\mathcal{U}}(k^{(0)} + k^{(1)} + k^{(2)} + \dots + k^{(q)}) = f.$$

We set

$$g^{(0)}_{i_0 i_1 \dots i_q} = \sum_{\substack{\tau(j_0)=i_0 \\ \dots \\ \tau(j_q)=i_q}} k^{(0)}_{i_0 j_0 j_1 \dots j_q}$$

$$g^{(1)}_{i_0 i_1 \dots i_q} = \sum_{\substack{\tau(j_0)=i_2 \\ \dots \\ \tau(j_q)=i_q \\ \dots}} k^{(1)}_{i_0 i_1 j_0 \dots j_{q-1}}.$$

Note that the $k^{(i)}$'s can be so chosen that $\operatorname{supp} g^{(i)} \in \Psi$ for every i. By direct calculation we get

$$\partial_{\mathcal{U}} (g^{(0)} + g^{(1)} + \dots + g^{(q)}) = \tau_* f - \varepsilon_{\mathcal{V}} (k^{(0)} + k^{(1)} + \dots + k^{(q)})$$
$$= \tau_* f - \varepsilon_{\mathcal{V}} \varepsilon_{\mathcal{U}}^{-1} (f).$$

Since $g^{(0)} + \dots + g^{(q)} \in C_q^{\psi} (\mathcal{U}, \mathcal{D})$, our contention is proved. We conclude with the following.

THEOREM 5. *Let $\mathcal{U} = \{U_i\}_{i \in I}$, $\mathcal{V} = \{V_j\}_{j \in J}$ be locally finite coverings of X by relatively compact open subsets of X and adapted to the family of supports Ψ. Let $\mathcal{D}$ be a cosheaf on X.*

We assume that

(i) $\mathcal{V} < \mathcal{U}$

(ii) $H_q(\mathcal{V} \cap U_s, \mathcal{D}) = 0$ *for every simplex* $s \in \mathcal{N}(\mathcal{U})$ *and for every* $q > 0$.

Then the natural map

$$H_n^{\psi}(\mathcal{U}, \mathcal{D}) \leftarrow H_n^{\psi}(\mathcal{V}, \mathcal{D})$$

is an isomorphism for every $n \geq 0$.

In particular, *if there exists a family* $\mathcal{V}^{\alpha} = \{V_j^{\alpha}\}_{j \in I_{\alpha}}$ *of locally finite coverings by open relatively compact subsets of X adapted to Ψ verifying condition* (ii) *and such that*

(iii) *for every open covering $\mathcal{W}$ of X there exists an α such that $\mathcal{V}^{\alpha} < \mathcal{W}$ then the natural map*

$$H_n^{\psi}(\mathcal{U}, \mathcal{D}) \leftarrow H_n(X, \mathcal{D})$$

is an isomorphism for every $n \geq 0$.

REMARK. The fact that $\mathcal{D}$ is a cosheaf needs only to be verified on every set V_{σ}^{α} for the coverings $\mathcal{U} \cap V_{\sigma}^{\alpha}$ $(\sigma \in \mathcal{N}(\mathcal{V}^{\alpha}))$ on every set U_s for the coverings $\mathcal{V}^{\alpha} \cap U_s$ $(s \in \mathcal{N}(\mathcal{U}))$.

Since V_{σ}^{α} and U_s are relatively compact, these coverings are finite.

§ 4. Čech homology on complex spaces.

8. *The dual cosheaf of a coherent sheaf.* *a*) Let X be a (reduced) complex space for which we will assume that it has a countable basis for open sets.

We will denote by $\mathcal{W}$ the class of all open subsets of X which are relatively compact and holomorphically complete (i. e. Stein). We take $\mathcal{W}$ as the priviledged class of open sets. All notions will be referred to this priviledged class without explicit reference to $\mathcal{W}$. Given dual families of supports one can always find coverings $\mathcal{U} \subset \mathcal{W}$ which are adapted. In fact, there always exist adapted coverings $A = \{A_i\}_{i \in I}$ as easily follows from the fact that a two point compactification of X is a metrisable space. Let $\mathcal{U}'$ be the family of open Stein sets contained in some A_i. Any locally finite covering $\mathcal{U}$ extracted from $\mathcal{U}'$ has the desired property.

Let $\mathcal{O}$ denote the structure sheaf on X and let $\mathcal{F}$ be any coherent analytic sheaf of $\mathcal{O}$-modules.

It is known that for any open set U the space $\Gamma(U, \mathcal{O})$ with the topology of uniform convergence on compact sets is a complete metric space and indeed a space of Fréchet-Schwartz.

For any open set $U \in \mathcal{W}$ we can find a surjective homomorphism

$$\mathcal{O}^p \xrightarrow{\alpha} \mathcal{F} \to 0 \tag{1}$$

and thus a surjective map

$$\Gamma(U, \mathcal{O})^p \xrightarrow{\alpha^*} \Gamma(U, \mathcal{F}) \to 0.$$

It is known that there exists a unique structure of a space of Fréchet-Schwartz such that, for any presentation (1) the corresponding linear map α_* between topological vector spaces is continuous.

We will always consider the presheaves $\{\Gamma(U, \mathcal{O}), r_V^U\}$; $\{\Gamma(U, \mathcal{F}), r_V^U\}$ endowed with their natural structure of presheaves of Fréchet-Schwartz.

b) For every $U \in \mathcal{W}$ we define

$$\mathcal{F}_*(U) = \operatorname{Hom\,cont}(\Gamma(U, \mathcal{F}), \mathbb{C})$$

i. e. $\mathcal{F}_*(U)$ is the topological dual of the space $\Gamma(U, \mathcal{F})$. If $U \subset V$, to the restriction map. $r_U^V : \Gamma(V, \mathcal{F}) \to \Gamma(U, \mathcal{F})$ corresponds by transposition a linear continuous map

$$\iota_V^U : \mathcal{F}_*(U) \to \mathcal{F}_*(V).$$

We obtain in this way a precosheaf

$$\mathcal{F}_* = \{\mathcal{F}_*(U), \iota_V^U\}$$

that will be called the dual of the sheaf $\mathcal{F}$.

PROPOSITION 8. (a) *For any coherent sheaf $\mathcal{F}$ the dual precosheaf $\mathcal{F}_*$ is a cosheaf*

(b) *For any $\Omega \in \mathcal{W}$ and any locally finite covering $\mathcal{U} \subset \mathcal{W}$ of Ω we have*

$$H_q(\mathcal{U} \cap \Omega, \mathcal{F}_*) = 0 \quad \textit{for} \quad q > 0.$$

PROOF. We may assume $X = \Omega$ without loss of generality and simplification of notations.

Consider the sequence

$$(*) \qquad 0 \to \Gamma(X, \mathcal{F}) \to C^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F}) \to \dots$$
$$\dots \to C^s(\mathcal{U}, \mathcal{F}) \to C^{s+1}(\mathcal{U}, \mathcal{F}) \to \dots .$$

This is a sequence of Fréchet spaces (since $\mathcal{U}$ is countable) which is exact since $X = \Omega$ is Stein. Thus by going to the topological duals we also get an exact sequence (cf. n. 10). This exact sequence is the homology sequence

$$(**) \qquad 0 \leftarrow \mathcal{F}_*(X) \leftarrow C_0(\mathcal{U}, \mathcal{F}_*) \leftarrow C_1(\mathcal{U}, \mathcal{F}_*) \leftarrow \dots$$
$$\dots \leftarrow C_s(\mathcal{U}, \mathcal{F}_*) \leftarrow C_{s+1}(\mathcal{U}, \mathcal{F}_*) \leftarrow \dots .$$

Indeed

$$C^s(\mathcal{U}, \mathcal{F}) = \Pi\Gamma(U_{i_0} \cap \dots \cap U_{i_s}, \mathcal{F}) \quad \text{and}$$

$$\text{Hom cont}(C^s(\mathcal{U}, \mathcal{F}), \mathbb{C}) = \coprod \mathcal{F}_*(U_{i_0} \cap \dots \cap U_{i_s}) \quad ([8] \text{ page } 264)$$

and moreover, by the very exactness of (*), it follows that all maps in the sequence are topological homomorphisms.

The exactness of the sequence (**) proves the two contentions of the proposition.

c) The dual cosheaf $\mathcal{O}_*$ of the structural sheaf $\mathcal{O}$ will be called the *structural cosheaf*. For every U, $\mathcal{O}_*(U)$ will be endowed with the topology of the strong dual of the space of Fréchet-Schwartz $\Gamma(U, \mathcal{O})$.

COREMARK. $\mathcal{O}_* = \{\mathcal{O}_*(U), \iota_V^U\}$ is a cosheaf of «coalgebras» compatible with the topological structure.

Similarly for any coherent sheaf $\mathcal{F}$ the dual cosheaf $\mathcal{F}_*(U)$ for any $U \subset \mathcal{W}$ can be endowed with the structure of the strong dual of the space of Fréchet-Schwartz $\Gamma(U, \mathcal{F})$.

COREMARK. $\mathcal{F}_* = \{\mathcal{F}_*(U), \iota_V^U\}$ ($\mathcal{F}$ coherent) is a cosheaf of comodules over $\mathcal{O}_*$; for every U we get

$$\mathcal{F}_*(U) = \operatorname{Ker}\{\mathcal{O}_*(U)^{p_0} \xrightarrow{\alpha} \mathcal{O}_*(U)^{p_1}\}$$

where α is a « $\mathcal{O}_*$-cohomomorphism ».

(*c*) If Ψ is a family of supports and $\mathcal{U} \subset \mathcal{W}$ adapted to Ψ is a locally finite covering of X we get by virtue of Leray theorem

$$H_p^{\psi}(\mathcal{U}, \mathcal{F}_*) \xleftarrow{\sim} H_p^{\psi}(X, \mathcal{F}_*)$$

for any dual $\mathcal{F}_*$ of a coherent sheaf $\mathcal{F}$.

CHAPTER 2. THE DUALITY THEOREM FOR COMPLEX SPACES

§ 5. Duality between homology and cohomology.

9. *Homology and cohomology.* *a*) Let X be a complex, $\mathcal{F}$ a coherent sheaf on X and $\mathcal{F}_*$ the dual cosheaf. Let $\mathcal{U}$ be any locally finite Stein covering of X. We consider the two Čech complexes

$$\text{(I)} \qquad C^0(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta_0} C^1(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta_1} C^2(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta_2} \dots$$

$$\text{(II)} \qquad C_0(\mathcal{U}, \mathcal{F}_*) \xleftarrow{\partial_0} C_1(\mathcal{U}, \mathcal{F}_*) \xleftarrow{\partial_1} C_2(\mathcal{U}, \mathcal{F}_*) \xleftarrow{\partial_2} \dots$$

whose respective homologies are (by Leray theorem)

$$H^q(X, \mathcal{F}) \quad \text{and} \quad H_q(X, \mathcal{F}_*) \qquad \forall\, q \geq 0.$$

In complex (I) we have

$$C^q(\mathcal{U}, \mathcal{F}) = \prod_{(i_0 \dots i_q)} \Gamma(U_{i_0} \dots U_{i_q}, \mathcal{F}).$$

As a countable product of spaces of Fréchet-Schwartz, it has with the product topology again the structure of a space of Fréchet-Schwartz.

In complex (II) we have

$$C_q(\mathcal{U}, \mathcal{F}_*) = \coprod_{(i_0 \cdots i_q)} \mathcal{F}_*(U_{i_0} \dots U_{i_q}).$$

This as topological vector space with the direct sum topology has the structure of the strong dual of $C^q(\mathcal{U}, \mathcal{F})$ i. e. the structure of the strong dual of a space of Fréchet Schwartz. (cf. [18] page 137 e 138; [8] page 264).

In both cases we are dealing with Souslin spaces, inductive limits of Banach spaces ([24] page 556).

b) Let now Φ and Ψ be dual families of supports and let $\mathcal{U}$ be a locally finite Stein covering by relatively compact open sets, adapted to the dual families of supports. We have now to consider the two Čech complexes

$$(\mathrm{I})_\Phi: \qquad C^0_\Phi(\mathcal{U}, \mathcal{F}) \to C^1_\Phi(\mathcal{U}, \mathcal{F}) \to C^2_\Phi(\mathcal{U}, \mathcal{F}) \to \dots.$$

$$(\mathrm{II})^\psi: \qquad C^\psi_0(\mathcal{U}, \mathcal{F}_*) \leftarrow C^\psi_1(\mathcal{U}, \mathcal{F}_*) \leftarrow C^\psi_2(\mathcal{U}, \mathcal{F}_*) \leftarrow \dots.$$

We get in this case with the notations of section 5

$$C^q_\Phi(\mathcal{U}, \mathcal{F}) = \varinjlim_s C^q(\mathcal{U}^s, \mathcal{F})$$

the limit being a strict inductive limit of spaces of Fréchet-Schwartz. With the inductive limit topology we get a separated topology at the limit and the space is again a Souslin space inductive limit of Banach spaces.

The strong dual of this inductive limit is topologically isomorphic with the projective limit

$$\varprojlim_s C_q(\mathcal{U}^s, \mathcal{F}_*) = C^\psi_q(\mathcal{U}, \mathcal{F}_*)$$

[[18] page 140]. This again is a Souslin space as a closed subspace of a countable product of Souslin spaces.

The bilinear form

$$B: C^q_\Phi(\mathcal{U}, \mathcal{F}) \times C^\psi_q(\mathcal{U}, \mathcal{F}_*) \to \mathbb{C}$$

of this dual pair of spaces is given as follows:

for $f = \{f_{i_0 \dots i_q}\} \in C_\Phi^q(\mathcal{U}, \mathcal{F})$, $g = \{g_{i_0 \dots i_q}\} \in C_q^\psi(\mathcal{U}, \mathcal{F}_*)$

$$B(f, g) = \sum_{(i_0 \dots i_q)} \langle f_{i_0 \dots i_q}, g_{i_0 \dots i_q} \rangle$$

where $\langle \cdot, \cdot \rangle$ denote the natural pairing between $\Gamma(U_{i_0} \cap \dots \cap U_{i_q}, \mathcal{F})$ and $\mathcal{F}_*(U_{i_0} \cap \dots \cap U_{i_q})$. Note that the sum involves only finitely many terms $\neq 0$.

From the other end we also have

$$C_q^\psi(\mathcal{U}, \mathcal{F}_*) = \varinjlim_s C_q^c(\mathcal{U}_s, \mathcal{F}_*)$$

which exhibits that on $C_q^\psi(\mathcal{U}, \mathcal{F}_*)$ there is also a topology inductive limit of Banach spaces. In fact each space $C_q^c(\mathcal{U}_s, \mathcal{F}_*) = \prod_{U_i \in \mathcal{U}_s} \mathcal{F}_*(U_{i_0 \dots i_q})$, as a countable product of complete bornological spaces is a complete and bornological space ([10] p. 387). Thus it is an inductive limit of Banach spaces ([10] p. 384). The same is therefore true for the space $\varinjlim C_q^c(\mathcal{U}_s, \mathcal{F}_*)$.

LEMMA. *The identity map $C_q^\psi(\mathcal{U}, \mathcal{F}_*) \to C_q^\psi(\mathcal{U}, \mathcal{F}_*)$ where the right hand space is identified topologically with $\varinjlim C_q^c(\mathcal{U}_s, \mathcal{F}_*)$ and the left hand with $\varprojlim C_q(\mathcal{U}^s, \mathcal{F}_*)$ is a topological isomorphism.*

PROOF. In view of the open mapping theorem, it is sufficient to prove that the identity map

$$\varprojlim C_q(\mathcal{U}^s, \mathcal{F}_x) \to \varinjlim C_q^c(\mathcal{U}_s, \mathcal{F}_x)$$

is continuous.

Let C be a closed convex set in the target space such that, for every s

$$C_s = C \cap C_q^c(\mathcal{U}_s, \mathcal{F}_*)$$

is a neighborhood of the origin in $C_q^c(\mathcal{U}_s, \mathcal{F}_*)$. The sets C's form a fundamental system of neighborhoods in the target space. For every s, $C_s = K_s^0$ where K_s is a bounded set in the strong dual of $C_q^c(\mathcal{U}_s, \mathcal{F}_*)$, i. e., in $\coprod_{U_i \in \mathcal{U}_s} \Gamma(U_{i_0 \dots i_q}, \mathcal{F})$. Therefore $K_s \subset \Pi K_{i_0 \dots i_q}^{(s)}$ where $K_{i_0 \dots i_q}^{(s)}$ is bounded in $\Gamma(U_{i_0 \dots i_q}, \mathcal{F})$, where the product is finite and $U_{i_0}, \dots, U_{i_q}$ are in $\mathcal{U}_s$. Now remark that we can choose the bounded sets $K_{i_0 \dots i_q}^{(s)}$ independently of the index s,

$$K_{i_0 \dots i_q}^{(1)} = K_{i_0 \dots i_q}^{(2)} = \dots = K_{i_0 \dots i_q}.$$

Take the product of all $K_{i_0 \dots i_q}$ for $i_0, \dots, i_q \in I$, and denote it by $\Pi K_{i_0 \dots i_q}$. We have $C^0 \subset \Pi K_{i_0 \dots i_q}$. Moreover, there must exist an integer r such that $\Pi K_{i_0 \dots i_q}$ is contained (and bounded) is some $C^q(\mathcal{U}^r, \mathcal{F})$. Hence $\Pi K_{i_0 \dots i_q}$ is a bounded set in $\varinjlim C^q(\mathcal{U}^s, \mathcal{F})$. Now $C \supset (\Pi K_{i_0 \dots i_q})^0$ which is a neighborhood of the origin in $\varprojlim C_q(\mathcal{U}^s, \mathcal{F}_*)$. Thus C itself is a neighborhood of the origin in that projective limit.

COROLLARY. *The space $C_q^{\psi}(\mathcal{U}, \mathcal{F}_*)$ (with the topology of the strong dual of $C_{\Phi}^q(\mathcal{U}, \mathcal{F})$) is a Souslin space inductive limit of Banach spaces.*

10. *The duality Lemma* [cf. [20]]. Let

$$A \xrightarrow{u} B \xrightarrow{v} C$$

be a sequence of locally convex topological spaces and continuous linear maps such that $v \circ u = 0$. Let

$$A' \xleftarrow{{}^t u} B' \xleftarrow{{}^t v} C'$$

be the sequence of the dual spaces and corresponding transposed maps. Then ${}^t u \circ {}^t v = 0$. We have a natural map

$$\frac{\operatorname{Ker} {}^t u}{\operatorname{Im} {}^t v} \xrightarrow{\sigma} \operatorname{Hom\,cont}\left(\frac{\operatorname{Ker} v}{\operatorname{Im} u}, \mathbb{C}\right).$$

LEMMA. (a) *The map σ is always surjective*

(b) *if v is a topological homomorphism then σ is an isomorphism and* $\operatorname{Im} {}^t v$ *is weakly closed.*

PROOF. (*a*) Given $l: \dfrac{\operatorname{Ker} v}{\operatorname{Im} u} \to \mathbb{C}$, linear and continuous we can lift it to a continuous linear map $\lambda: \operatorname{Ker} v \to \mathbb{C}$ with the property that $\lambda \mid \operatorname{Im} u = 0$. By the Hahn-Banach theorem we can continuously extend λ to $\widehat{\lambda}: B \to \mathbb{C}$ so that $\widehat{\lambda} \in B'$. We show that $\widehat{\lambda} \in \operatorname{Ker} {}^t u$. Indeed $\forall a \in A$, since $u(A) \in \operatorname{Ker} v$ we have

$${}^t u\, \widehat{\lambda}(a) = \widehat{\lambda}(u(a)) = \lambda(u(a)) = 0.$$

(*b*) The homomorphism σ is defined by associating to every $\lambda \in \operatorname{Ker} {}^t u$ the continuous linear map $\lambda \mid \operatorname{Ker} v$ which vanishes on $\operatorname{Im} u$. We thus

have a continuous linear map

$$\widehat{\sigma}: \operatorname{Ker} {}^t u \longrightarrow \operatorname{Hom\ cont}\left(\frac{\operatorname{Ker} v}{\operatorname{Im} u}, \mathbb{C}\right).$$

We have to show that $\operatorname{Ker} \widehat{\sigma} = \operatorname{Im} {}^t v$. First note that $\operatorname{Im} {}^t v$ is weakly closed since v is a topological homomorphism. Secondly we remark that if $\lambda \in \operatorname{Ker} {}^t u \cap \operatorname{Ker} \widehat{\sigma}$, then $\lambda \mid \operatorname{Ker} v = 0$ so that λ defines a linear map

$$\widetilde{\lambda}: v(B) = B/\operatorname{Ker} v \longrightarrow \mathbb{C}$$

which is continuous not only for the quotient topology of $B/\operatorname{Ker} v$ but also for the induced topology on $v(B)$ by C, because v is a topological homomorphism. By Hahn-Banach we can extend $\widehat{\lambda}$ to a continuous linear map $\mu: C \longrightarrow \mathbb{C}$. We claim that

$${}^t v(\mu) = \lambda.$$

This is obvious by the construction of μ. This shows that

$$\operatorname{Im} {}^t v \supset \operatorname{Ker} \widehat{\sigma}.$$

But we have also $\operatorname{Im} {}^t v \subset \operatorname{Ker} \widehat{\sigma}$ because if $\lambda = {}^t v(\mu)$ then $\lambda \mid \operatorname{Ker} v = 0$ i. e. $\lambda \in \operatorname{Ker} \widehat{\sigma}$. In conclusion we get an exact sequence of continuous linear maps:

$$\operatorname{Im} {}^t v \xrightarrow{i} \operatorname{Ker} {}^t u \xrightarrow{\widehat{\sigma}} \operatorname{Hom\ cont}\left(\frac{\operatorname{Ker} v}{\mathcal{J}\mathrm{m}\, u}, \mathbb{C}\right) \longrightarrow 0.$$

This proves the lemma.

11. *Duality theorems.* *a*) Let X be a complex space. Let $\mathcal{F}$ be a coherent analytic sheaf on X and let $\mathcal{F}_*$ be the dual cosheaf. Let Φ, Ψ denote dual families of supports and let $\mathcal{U}$ be a locally finite Stein covering of X by relatively compact open sets, adapted to the dual families of supports.

Applying the duality lemma to the complex

$$\text{(I)} \qquad C_\Phi^{q-1}(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta_{q-1}} C_\Phi^q(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta_q} C_\Phi^{q+1}(\mathcal{U}, \mathcal{F})$$

we get the following

THEOREM (I). *If in the sequence* (I) δ_q *is a topological homomorphism then*

$$H_{\Phi}^{q+1}(X, \mathcal{F}) \quad \textit{is separated if } \operatorname{Im} \delta_q \textit{ is closed,}$$

$$H_q^{\psi}(X, \mathcal{F}_*) \quad \textit{is separated}$$

and

$$H_q^{\psi}(X, \mathcal{F}_*) \simeq \operatorname{Hom\,cont}(H_{\Phi}^q(X, \mathcal{F}), \mathbb{C}).$$

PROOF. First we remark that image of δ_q closed means that $H_{\Phi}^{q+1}(X, \mathcal{F}) = H_{\Phi}^{q+1}(\mathcal{U}, \mathcal{F})$ is separated. Also the dual complex of (I) is the complex

$$\text{(II)} \qquad C_{q-1}^{\psi}(\mathcal{U}, \mathcal{F}_*) \xleftarrow{\partial_{q-1}} C_q^{\psi}(\mathcal{U}, \mathcal{F}_*) \xleftarrow{\partial_q} C_{q+1}^{\psi}(\mathcal{U}, \mathcal{F}_*)$$

as we have shown in section 9. Here $\partial_q = {}^t\delta_q$ as the transpose of a homomorphism has a weakly closed image, in particular closed, and therefore $H_q^{\psi}(X, \mathcal{F}_*) \simeq H_q^{\psi}(\mathcal{U}, \mathcal{F}_*)$ is separated.

The last statement follows from the duality lemma.

REMARK. *The hypothesis that* δ_q *is a topological homomorphism is verified in the following instance:*

$$\Phi = \textit{all closed sets and thus } \Psi = \textit{all compact sets}$$

$$\textit{and if } H^{q+1}(X, \mathcal{F}) \textit{ is separated.}$$

In fact, the spaces $C^q(\mathcal{U}, \mathcal{F})$, $C^{q+1}(\mathcal{U}, \mathcal{F})$ are Fréchet spaces. Thus for δ_q to be a homomorphism is equivalent to the closure of the image of δ_q which is exactly the assumption of separation for $H^{q+1}(\mathcal{U}, \mathcal{F})$.

b) Similarly we can consider the complex

$$\text{(II)} \qquad C_{q-1}^{\psi}(\mathcal{U}, \mathcal{F}_*) \xleftarrow{\partial_{q-1}} C_q^{\psi}(\mathcal{U}, \mathcal{F}_*) \xleftarrow{\partial_q} C_{q+1}^{\psi}(\mathcal{U}, \mathcal{F}_*)$$

and we get the

THEOREM (II). *If in the sequence* (II) ∂_{q-1} *is a topological homomorphism then*

$$H_{q-1}^{\psi}(X, \mathcal{F}_*) \quad \textit{is separated if } \operatorname{Im} \partial_{q-1} \textit{ in closed,}$$

$$H_{\Phi}^q(X, \mathcal{F}) \quad \textit{is separated}$$

and

$$H_{\Phi}^{q}(X, \mathcal{F}) \simeq \operatorname{Hom} \operatorname{cont}(H_{q}^{\Psi}(X, \mathcal{F}_{*}), \mathbb{C}).$$

PROOF. First we remark that if ∂_{q-1} have closed image $H_{q-1}^{\Psi}(X, \mathcal{F}_{*}) = H_{q-1}^{\Psi}(\mathcal{U}, \mathcal{F}_{*})$ is separated. The spaces $C_{\Phi}^{q}(\mathcal{U}, \mathcal{F})$ as a strict inductive limit of spaces of Féchet-Schwartz are reflexive spaces (since they are Montel spaces). Thus the dual complex of (II) is the complex (I).

Here δ_{q-1} as the transpose of a topological homomorphism has a weakly closed image. In particular $H_{\Phi}^{q}(X, \mathcal{F}) = H_{\Phi}^{q}(\mathcal{U}, \mathcal{F})$ is separated.

The last part of the theorem follows from the duality lemma.

REMARK. *The hypothesis that ∂_{q-1} be a topological homomorphism is verified in the following instance*

$$\Phi = \textit{all closed sets and thus } \Psi = \textit{all compact sets}$$

$$\textit{and if } H_{q-1}(X, \mathcal{F}_{*}) \textit{ is separated.}$$

PROOF. We have by assumption that $\partial_{q-1}(C_{q}(\mathcal{U}, \mathcal{F}_{*}))$ is a closed subspace of $C_{q-1}(\mathcal{U}, \mathcal{F}_{*})$. But $C_{q-1}(\mathcal{U}, \mathcal{F}_{*})$ as a dual of Fréchet-Schwartz has the property that every closed subspace (in particular $\partial_{q-1}(C_{q}(\mathcal{U}, \mathcal{F}_{*}))$ is also a dual of a space of Fréchet-Schwartz. It follows then that $\partial_{q-1}: C_{q}(\mathcal{U}, \mathcal{F}_{*}) \to \partial_{q-1}(C_{q}(\mathcal{U}, \mathcal{F}_{*}))$ being a continuous surjective map is open. Thus ∂_{q-1} is a topological homomorphism.

§ 6. Čech homology and the functor Ext.

12. *The functors* $\mathcal{E}\mathrm{xt}$ *and* EXT. *a*) Let $\mathcal{F}$, $\mathcal{G}$ be sheaves of $\mathcal{O}$-modules on X, the sheaf associated to the presheaf

$$U \to \operatorname{Hom}(\mathcal{F}|U, \mathcal{G}|U) \qquad (U \text{ open in } X)$$

is denoted by $\mathcal{H}\mathrm{om}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$.

If Φ is any family of supports we set

$$\mathrm{HOM}_{\Phi}(U; \mathcal{F}, \mathcal{G}) = \Gamma_{\Phi}(U, \mathcal{H}\mathrm{om}_{\mathcal{O}}(\mathcal{F}, \mathcal{G}))$$

if Φ is the family of all closed sets the symbol Φ is omitted.

b) A sheaf of $\mathcal{O}$ modules $\mathcal{J}$ is called *injective* if for any short exact sequence of sheaves of $\mathcal{O}$-modules

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0. \tag{1}$$

The sequence

$$0 \to \mathrm{HOM}(X, \mathcal{F}'', \mathcal{J}) \to \mathrm{HOM}(X, \mathcal{F}, \mathcal{J}) \to \mathrm{HOM}(X, \mathcal{F}', \mathcal{J}) \to 0$$

is exact.

An injective sheaf is flabby; for $\mathcal{J}$ injective $\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{J})$ is also flabby. If in a short exact sequence (1) the sheaf $\mathcal{F}'$ is injective then the sequence (1) splits.

c) Every sheaf of $\mathcal{O}$-modules $\mathcal{G}$ admits a resolution by injective sheaves:

$$0 \to \mathcal{G} \to \mathcal{G}_0 \to \mathcal{G}_1 \to \dots . \tag{0}$$

We define the sheaf

$$\mathcal{E}xt^q_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) = \frac{\mathrm{Ker}\,\{\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}_q) \to \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}_{q+1})\}}{\mathrm{Im}\,\{\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}_{q-1}) \to \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}_q)\}} = H^q(\mathcal{H}om(\mathcal{F}, \mathcal{G}^*))$$

the definition being independent of the resolution of $\mathcal{G}$.

If $\mathcal{F}$ admits a resolution

$$\dots \to \mathcal{L}_1 \to \mathcal{L}_0 \to \mathcal{F} \to 0$$

by locally free sheaves then one has

$$\mathcal{E}xt^q_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) = \frac{\mathrm{Ker}\,\{\mathcal{H}om_{\mathcal{O}}(\mathcal{L}_q, \mathcal{G}) \to \mathcal{H}om_{\mathcal{O}}(\mathcal{L}_{q+1}, \mathcal{G})\}}{\mathrm{Im}\,(\mathcal{H}om_{\mathcal{O}}(\mathcal{L}_{q+1}, \mathcal{G}) \to \mathcal{H}om_{\mathcal{O}}(\mathcal{L}_q, \mathcal{G}))} = H_q(\mathcal{H}om(\mathcal{L}_*, \mathcal{G})).$$

One has the fallowing properties

i) $$\mathcal{E}xt^0_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) = \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$$

$$\mathcal{E}xt^q_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) = 0 \quad \text{if} \quad \begin{cases} \mathcal{F} & \text{is locally free or} \\ \mathcal{G} & \text{is injective} \end{cases}$$

for $q \geq 1$.

ii) If the sequence of sheaves of $\mathcal{O}$-modules

$$0 \to \mathcal{G}' \to \mathcal{G} \to \mathcal{G}'' \to 0 \tag{2}$$

is exact we have an exact sequence of sheaves

$$0 \to \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}') \to \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}'') \to \mathcal{E}xt^1_{\mathcal{O}}(\mathcal{F}, \mathcal{G}') \to \dots$$

iii) If the sequence of sheaves of $\mathcal{O}$-modules

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0 \tag{1}$$

is exact we have an exact sequence of sheaves

$$0 \to \mathcal{H}om_{\mathcal{O}}(\mathcal{F}'', \mathcal{G}) \to \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) \to \mathcal{E}xt^1_{\mathcal{O}}(\mathcal{F}'', \mathcal{G}) \to \dots .$$

d) Analogously one defines, using a resolution (0), for any paracompactifying family of supports Φ,

$$\mathrm{EXT}^q_{\Phi}(X, \mathcal{F}, \mathcal{G}) = \frac{\mathrm{Ker}\,\{\mathrm{Hom}_{\Phi}(X, \mathcal{F}, \mathcal{G}_q) \to \mathrm{Hom}_{\Phi}(X, \mathcal{F}, \mathcal{G}_{q+1})\}}{\mathrm{Im}\,\{\mathrm{Hom}_{\Phi}(X, \mathcal{F}, \mathcal{G}_{q-1}) \to \mathrm{Hom}_{\Phi}(X, \mathcal{F}, \mathcal{G}_q)\}}$$

one has the following properties

i) $\mathrm{EXT}^0_{\Phi}(X; \mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{\Phi}(X; \mathcal{F}, \mathcal{G})$

$\mathrm{EXT}^q_{\Phi}(X, \mathcal{F}, \mathcal{G}) = 0$ if $\mathcal{G}$ is injective

for $q \geq 1$.

ii) For an exact sequence of sheaves (2) we get an exact sequence

$$0 \to \mathrm{HOM}_{\Phi}(X; \mathcal{F}, \mathcal{G}') \to \mathrm{HOM}_{\Phi}(X; \mathcal{F}, \mathcal{G}) \to \mathrm{HOM}_{\Phi}(X; \mathcal{F}, \mathcal{G}'') \to$$
$$\to \mathrm{EXT}^1_{\Phi}(X; \mathcal{F}, \mathcal{G}') \to \dots .$$

iii) For an exact sequence (1) we get an exact sequence

$$0 \to \mathrm{HOM}_{\Phi}(X; \mathcal{F}'', \mathcal{G}) \to \mathrm{HOM}_{\Phi}(X; \mathcal{F}, \mathcal{G}) \to \mathrm{HOM}_{\Phi}(X; \mathcal{F}', \mathcal{G}) \to$$
$$\to \mathrm{EXT}^1_{\Phi}(X; \mathcal{F}'', \mathcal{G}) \to \dots .$$

iv) There exists a spectral sequence $E_2^{pq} \Longrightarrow \mathrm{EXT}^n_{\Phi}(X; \mathcal{F}, \mathcal{G})$ with

$$E_2^{pq} = H^p_{\Phi}(X; \mathcal{E}xt^q(\mathcal{F}, \mathcal{G}))$$

[cf. [6] page 264].

e) PROPOSITION 9. *Let*

$$0 \to \mathcal{G} \to \mathcal{G}_0 \to \mathcal{G}_1 \to \mathcal{G}_2 \to \dots$$

be an exact sequence of sheaves of $\mathcal{O}$*-modules and let* $\mathcal{F}$ *be any sheaf of* $\mathcal{O}$*-modules;*

(a) *If* $\mathcal{E}\mathrm{xt}^q_{\mathcal{O}}(\mathcal{F}, \mathcal{G}_i) = 0$ *for* $i \geq 0$ *and any* $q \geq 1$ *then*

$$\mathcal{E}\mathrm{xt}^q_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) = \frac{\mathrm{Ker}\,\{\mathcal{H}\mathrm{om}_{\mathcal{O}}(\mathcal{F}, \mathcal{G}_q) \to \mathcal{H}\mathrm{om}_{\mathcal{O}}(\mathcal{F}, \mathcal{G}_{q+1})\}}{\mathrm{Im}\,\{\mathcal{H}\mathrm{om}_{\mathcal{O}}(\mathcal{F}, \mathcal{G}_{q-1}) \to \mathcal{H}\mathrm{om}_{\mathcal{O}}(\mathcal{F}, \mathcal{G}_q)\}}\,.$$

(b) *If* $\mathrm{EXT}^q_{\Phi}(X; \mathcal{F}, \mathcal{G}_i) = 0$ *for* $i \geq 0$ *and* $q \geq 1$ *then*

$$\mathrm{EXT}^q_{\Phi}(X; \mathcal{F}, \mathcal{G}) = \frac{\mathrm{Ker}\,\{\mathrm{HOM}_{\Phi}(X; \mathcal{F}, \mathcal{G}_q) \to \mathrm{HOM}_{\Phi}(X; \mathcal{F}, \mathcal{G}_{q+1})\}}{\mathrm{Im}\,\{\mathrm{HOM}_{\Phi}(X; \mathcal{F}, \mathcal{G}_{q-1}) \to \mathrm{HOM}_{\Phi}(X; \mathcal{F}, \mathcal{G}_q)\}}\,.$$

PROOF. Split the resolution of $\mathcal{G}$ in short exact sequences:

$$0 \to \mathcal{G} \to \mathcal{G}_0 \to \mathcal{W}_1 \to 0$$

$$0 \to \mathcal{W}_1 \to \mathcal{G}_1 \to \mathcal{W}_2 \to 0$$

where $\mathcal{W}_i = \mathrm{Ker}\,\{\mathcal{G}_{i-1} \to \mathcal{G}_i\}$. Then apply the property ii) and the assumption.

REMARK. (*a*) If we call $\mathcal{F}$*-coflat* a sheaf $\mathcal{T}$ such that $\mathcal{E}\mathrm{xt}^q_{\mathcal{O}}(\mathcal{F}, \mathcal{T}) = 0$ for $q \geq 1$, then part (*a*) of the proposition says that instead of an injective resolution of $\mathcal{G}$ we may as well use a $\mathcal{F}$-coflat resolution to compute $\mathcal{E}\mathrm{xt}^q_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$.

(*b*) Analogously if we call $\mathcal{T}$ globally $\mathcal{F}$ *coflat* (with respect to Φ) if $\mathrm{EXT}^q_{\Phi}(X, \mathcal{F}, \mathcal{T}) = 0$ for $q \geq 1$, a resolution of $\mathcal{G}$ by globally $\mathcal{F}$-coflat sheaves can be used to compute $\mathrm{EXT}^q_{\Phi}(X, \mathcal{F}, \mathcal{G})$.

13. *The functors* $\mathcal{E}\mathrm{xt}$ *and* EXT *on complex spaces.* *a*) Let X be a complex space. Let $\mathcal{F}$, $\mathcal{G}$ be sheaves $\mathcal{O}$ modules on X then (cf. Serre [21])

i) if $\mathcal{F}$ is a coherent sheaf then

$$\mathcal{H}\mathrm{om}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})_x \xrightarrow{\sim} \mathcal{H}\mathrm{om}_{\mathcal{O}_x}(\mathcal{F}_x, \mathcal{G}_x)$$

and consequently

$$\mathcal{E}\mathrm{xt}^q_{\mathcal{O}}(\mathcal{F}, \mathcal{G})_x \xrightarrow{\sim} \mathcal{E}\mathrm{xt}^q_{\mathcal{O}_x}(\mathcal{F}_x, \mathcal{G}_x) \qquad \forall\, q \geq 0$$

ii) if $\mathcal{F}$ and $\mathcal{G}$ are coherent then

$$\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) \text{ is a coherent sheaf}$$

and consequently

$$\mathcal{E}xt^q_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) \text{ is a coherent sheaf } \quad \forall\, q \geq 0$$

(to prove this second property one uses a local free resolution of $\mathcal{F}$ and applies the second definition of the functor $\mathcal{E}xt^q_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$).

b) Let $\mathcal{F}$ and $\mathcal{G}$ be coherent sheaves on an open set $U \subset \mathbb{C}^n$. We assume that U is a domain of holomorphy (i. e. Stein) sufficiently small to have on U free resolutions

$$0 \to \mathcal{L}_r \to \mathcal{L}_{r-1} \to \dots \to \mathcal{L}_1 \to \mathcal{L}_0 \to \mathcal{F} \to 0$$

$$0 \to \mathcal{E}_s \to \mathcal{E}_{s-1} \to \dots \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{G} \to 0.$$

LEMMA 1. *If $\mathcal{F}$ and $\mathcal{G}$ are locally free sheaves then*

$$\mathrm{EXT}^q_k(U\,;\, \mathcal{F}, \mathcal{G}) = 0 \quad \textit{if} \quad q \neq n$$

and

$$\mathrm{EXT}^n_k(U\,;\, \mathcal{F}, \mathcal{G}) = H^n_k(U, \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})).$$

PROOF. We have a spectral sequence coverging to $\mathrm{EXT}^*_k(U\,;\, \mathcal{F}, \mathcal{G})$ whose term $E_2^{pq} = H^p_k(U\,;\, \mathcal{E}xt^q_{\mathcal{O}}(\mathcal{F}, \mathcal{G}))$.

Since $\mathcal{F}$ is locally free $\mathcal{E}xt^q_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) = 0$ if $q \geq 1$ and $\mathcal{E}xt^0_{\mathcal{O}}(\mathcal{F}, \mathcal{G}) = \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$ is a locally free sheaf. Since U is Stein $H^p_k(U; \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})) = 0$ if $p \neq n$ by Serre duality. Hence $\mathrm{EXT}^n_k(U; \mathcal{F}, \mathcal{G}) = E_k^{n0} = H^n_k(U, \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G}))$.

LEMMA 2. *For any $\mathcal{F}$ and $\mathcal{G}$ we have*

$$\mathrm{EXT}^p_k(U\,;\, \mathcal{F}, \mathcal{G}) = 0 \quad \text{if} \quad p > n.$$

PROOF. Splitting the resolution of $\mathcal{G}$ in short exact sequences

$$0 \to \mathcal{M}_1 \to \mathcal{E}_0 \to \mathcal{G} \to 0$$

$$0 \to \mathcal{M}_2 \to \mathcal{E}_1 \to \mathcal{M}_1 \to 0$$

$$\cdot \qquad \cdot \qquad \cdot \qquad \cdot$$

applying ii) of the previous section and lemma 1 we get for $l \geq 1$

$$\mathrm{EXT}^{n+l}_k(U\,;\, \mathcal{F}, \mathcal{G}) \simeq \mathrm{EXT}^{n+l+1}_k(U\,;\, \mathcal{F}, \mathcal{M}_1) \simeq \dots$$

$$\dots \simeq \mathrm{EXT}^{n+l+s}_k(U\,;\, \mathcal{F}, \mathcal{E}_s) \simeq \mathrm{EXT}^{n+l+s+1}_k(U, \mathcal{F}, 0) = 0.$$

PROPOSITION 10. *If $\mathcal{G}$ is a locally free sheaf we have*

$$\mathrm{EXT}_k^p(U;\mathcal{F},\mathcal{G})=0 \quad \textit{if} \quad p\neq n$$

and an exact sequence for $p=n$

$$0\to \mathrm{EXT}_k^n(U;\mathcal{F},\mathcal{G})\to H_k^n(U;\mathcal{H}\mathrm{om}(\mathcal{L}_0,\mathcal{G}))\to H_k^n(U;\mathcal{H}\mathrm{om}(\mathcal{L}_1,\mathcal{G}))\to\dots$$

PROOF. If $p>n$ the first statement follows from lemma 2. If $p<n$ we get, by splitting the resolution of $\mathcal{F}$ in short exact sequences

$$0\to\mathcal{W}_1\to\mathcal{L}_0\to\mathcal{F}\to 0$$

$$0\to\mathcal{W}_2\to\mathcal{L}_1\to\mathcal{W}_1\to 0$$

$$\cdot\quad\cdot\quad\cdot$$

$$\mathrm{EXT}_k^p(U;\mathcal{F},\mathcal{G})\simeq\mathrm{EXT}_k^{p-1}(U,\mathcal{W}_1,\mathcal{G})\simeq\dots\simeq\mathrm{HOM}_k(U,\mathcal{W}_p,\mathcal{G}).$$

But

$$\mathrm{HOM}_k(U;\mathcal{W}_p,\mathcal{G})\subset\mathrm{HOM}_k(U;\mathcal{L}_p,\mathcal{G})\simeq\Gamma_k(U,\mathcal{G})^{r_p}=0,\ \text{if}\ \mathcal{L}_p\simeq\mathcal{O}^{r_p}.$$

Finally for $p=n$ using the same exact sequences we get the short exact sequences (by virtue of lemma 2),

$$0\to\mathrm{EXT}_k^n(U;\mathcal{F},\mathcal{G})\to\mathrm{EXT}_k^n(U,\mathcal{L}_0,\mathcal{G})\to\mathrm{EXT}_k^n(U,\mathcal{W}_1,\mathcal{G})\to 0$$

$$0\to\mathrm{EXT}_k^n(U;\mathcal{W}_1,\mathcal{G})\to\mathrm{EXT}_k^n(U,\mathcal{L}_1,\mathcal{G})\to\mathrm{EXT}_k^n(U,\mathcal{W}_2,\mathcal{G})\to 0.$$

From these, the last part of the proposition follows.

COROLLARY. *Let X be a complex manifold of dimension n. Let Ω^n denote the sheaf of germs of holomorphic n-forms on X. Let $\mathcal{F}$ be any coherent sheaf on X. For any sufficiently small Stein open set $U\subset X$ we have*

$$\mathrm{EXT}_k^n(U;\mathcal{F},\Omega^n)\simeq\mathrm{Hom\ cont}(\Gamma(U,\mathcal{F}),\mathbb{C})=\mathcal{F}_*(U).$$

PROOF. We have an exact sequence of Fréchet spaces:

$$(*)\qquad 0\leftarrow\Gamma(U,\mathcal{F})\leftarrow\Gamma(U,\mathcal{L}_0)\xleftarrow{\alpha}\Gamma(U,\mathcal{L}_1)\leftarrow\dots$$

whose duals (by Serre duality) are given also topologically by

$$\mathrm{Hom\ cont}(\Gamma(U,\mathcal{L}_i),\mathbb{C})\simeq H_k^n(U,\mathcal{H}\mathrm{om}(\mathcal{L}_i,\Omega^n)).$$

Thus by dualising (*) we get an exact sequence of topological vector spaces:

$$0 \to \operatorname{Hom\,cont}(\Gamma(U, \mathcal{F}), \mathbb{C}) \to$$
$$\to H_k^n(U, \mathcal{H}om(\mathcal{L}_0, \Omega^n) \xrightarrow{{}^t\alpha} H_k^n(U, \mathcal{H}om(\mathcal{L}_1, \Omega^n)) \to \dots .$$

Therefore

$$\operatorname{Hom\,cont}(\Gamma(U, \mathcal{F}), \mathbb{C}) = \operatorname{Ker} {}^t\alpha = \mathrm{EXT}_k^n(U; \mathcal{F}, \Omega^n).$$

c) *The prechoseaves* $\mathcal{E}\mathrm{xt}_k^q(\mathcal{F}, \mathcal{G})$. Let

$$0 \to \mathcal{G} \to \mathcal{G}_0 \to \mathcal{G}_1 \to \mathcal{G}_2 \to \dots \tag{1}$$

be an injective resolution of $\mathcal{G}$. Let $U \subset V$ be open sets in X. We can consider when Φ is the family of compact supports the two complexes

$$\begin{array}{ccccccc}
\mathrm{HOM}_k(U; \mathcal{F}, \mathcal{G}_0) & \to & \mathrm{HOM}_k(U; \mathcal{F}, \mathcal{G}_1) & \to & \mathrm{HOM}_k(U; \mathcal{F}, \mathcal{G}_2) & \to & \dots \\
\downarrow i_V^U & & \downarrow i_V^U & & \downarrow i_V^U & & \\
\mathrm{HOM}_k(V; \mathcal{F}, \mathcal{G}_0) & \to & \mathrm{HOM}_k(V; \mathcal{F}, \mathcal{G}_1) & \to & \mathrm{HOM}_k(V; \mathcal{F}, \mathcal{G}_2) & \to & \dots
\end{array}$$

and the natural map i_V^U of the first complex into the second (which is an injection). This diagram commutes thus we get a map in cohomology

$$\iota_V^U : \mathrm{EXT}_k^q(U, \mathcal{F}, \mathcal{G}) \to \mathrm{EXT}_k^q(V, \mathcal{F}, \mathcal{G}).$$

It is clear then that $\{\mathrm{EXT}_k^q(U; \mathcal{F}, \mathcal{G}), \iota_V^U\}$ is a precosheaf that we will denote by $\mathcal{E}\mathrm{xt}_k^q(\mathcal{F}, \mathcal{G})$:

$$\mathcal{E}\mathrm{xt}_k^q(\mathcal{F}, \mathcal{G}) = \{\mathrm{EXT}_k^q(U; \mathcal{F}, \mathcal{G}), \iota_V^U\}.$$

PROPOSITION 11. *Let Ψ be a family of supports and $\mathcal{U}$ an adapted covering for Ψ. Then we have a spectral sequence*

$$E_2^{-pq} = H_p^\psi(\mathcal{U}, \mathcal{E}\mathrm{xt}_k^q(\mathcal{F}, \mathcal{G})) \Longrightarrow \mathrm{EXT}_\psi^{-p+q}(X; \mathcal{F}, \mathcal{G}).$$

PROOF. We consider the double complex

$$E_0^{-pq} = C_p^\psi(\mathcal{U}, \mathcal{H}om_k(\mathcal{F}, \mathcal{G}_q)).$$

We note that since $\mathcal{G}_q$ is injective $\mathcal{H}om(\mathcal{F}, \mathcal{G}_q)$ is a soft sheaf. Therefore the precosheaf

$$\mathcal{H}om_k(\mathcal{F}, \mathcal{G}_q) = \{\mathrm{HOM}_k(U, \mathcal{F}, \mathcal{G}_q), i_V^U\}$$

$$= \{\Gamma_k(U, \mathcal{H}om(\mathcal{F}, \mathcal{G}_q), \iota_V^U\}$$

is a *flabby cosheaf* (no. 6 Proposition 5). Hence taking first the homology with respect to the boundary operator $\partial_{\mathcal{U}}$ coming from the Čech complex we get

$$E_1^{-pq} = \begin{cases} 0 & \text{if} \quad p \neq 0 \\ \Gamma_\psi(X, \mathcal{H}om(\mathcal{F}, \mathcal{G}_q)) = \mathrm{HOM}_\psi(X; \mathcal{F}, \mathcal{G}_q) & \text{if} \quad p = 0. \end{cases}$$

Taking then the homology with respect to the boundary operator coming from the resolution (1) we get that

$$E_2^{-pq} = \begin{cases} 0 & \text{if} \quad p \neq 0 \\ \mathrm{EXT}_\psi^q(X, \mathcal{F}, \mathcal{G}) & \text{if} \quad p = 0. \end{cases}$$

Taking the boundary operators in the reverse order we get instead a second spectral sequence, converging to the same limit whose term

$$E_2^{-pq} = H_p^\psi(\mathcal{U}; \mathcal{E}xt_k^q(\mathcal{F}, \mathcal{G})).$$

14. *Homology and* EXT *on complex manifolds*

THEOREM 6. *Let X be a complex manifold of pure dimension n and let Ω^n denote the sheaf of germs of holomorphic n-forms. For any family Ψ of supports* (5) *and for any coherent sheaf $\mathcal{F}$ on X we have:*

$$H_p^\psi(X, \mathcal{F}_*) \simeq \mathrm{EXT}_\psi^{n-p}(X; \mathcal{F}, \Omega^n).$$

PROOF. We use the spectral sequence of proposition 11 taking $\mathcal{G} = \Omega^n$. By proposition 10 $\mathcal{E}xt_k^q(\mathcal{F}, \Omega^n) = 0$ if $q \neq n$ and for $q = n$ we get on any small Stein set U

$$\mathrm{EXT}_k^n(U, \mathcal{F}, \Omega^n) = \mathcal{F}_*(U)$$

as explicitely stated in the corollary to proposition 10. Thus

$$E_2^{-pq} = \begin{cases} 0 & \text{if} \quad q \neq n \\ H_p^\psi(\mathcal{U}, \mathcal{F}_*(U)) & \text{if} \quad q = n. \end{cases}$$

(5) As considered here cf. § 1 note 1.

If $\mathcal{U}$ is a sufficiently small Stein covering adapted to Ψ we thus get, since $\mathcal{F}_*(U)$ is a cosheaf for which the covering is acyclic,

$$E_2^{-pn} = H_p^{\Psi}(X, \mathcal{F}_*).$$

We must therefore have

$$\mathrm{EXT}_{\Psi}^{n-p}(X, \mathcal{F}, \Omega^n) = H_p^{\Psi}(X, \mathcal{F}_*).$$

§ 7. Comparison of Topologies on Cohomology and Homology Groups.

15. *De Rham topologies.* *a*) On the cohomology (and homology) groups we have considered the topology inherited from the corresponding Čech complexes. However, via de Rham theorem the same groups can be obtained as the cohomology of other complexes of topological vector spaces. These topologies may be very different at the level of the cochain-complex but have the tendency to give the same topology on the corresponding cohomology groups. We give here some criteria which enable us to establish this fact.

By a *complex of topological vector spaces* we mean a graded complex $A = \{\coprod A_n, d\}$ having the following properties:

(i) each space A_n is a topological vector space over $\mathbb{C}$ with a Hausdorff topology:

(ii) for each n the differential map $d: A^n \longrightarrow A^{n+1}$ is continuous.

For each n we consider the spaces $\mathcal{Z}^n(A)$ of n-cocycles and $\mathcal{B}^n(A)$ of n-coboundaries. Both spaces, as subspaces of A^n have natural structures of topological vector spaces; moreover, by the assumption (ii) $\mathcal{Z}^n(A)$ is a closed subspace of A^n. On the cohomology group

$$H^n(A) = \mathcal{Z}^n(A)/\mathcal{B}^n(A)$$

we consider the quotient topology. If $\theta: \mathcal{Z}^n(A) \longrightarrow H^n(A)$ is the natural map, a set S in $H^n(A)$ is open if and only if $\theta^{-1}(S)$ is open in $\mathcal{Z}^n(A)$. With this structure $H^n(A)$ is a topological vector space which however may not be Hausdorff, the closure of zero being $\theta(\overline{\mathcal{B}^n(A)})$ which may be larger than $0 = \theta(\mathcal{B}^n(A))$. If A, B are complexes of topological vector spaces and if

$$\alpha: A \longrightarrow B$$

is a homomorphism of complexes which is continuous then the induced

map

$$\alpha^* : H^n(A) \longrightarrow H^n(B)$$

is also continuous.

LEMMA. *Let A, B, C, be complexes of topological vector spaces and let*

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

be a short exact sequence of homomorphisms of complexes. If α and β are topological homomorphisms then the connecting homomorphisms

$$\delta_q : H^q(C) \longrightarrow H^{q+1}(A)$$

are continuous.

PROOF. We want to prove that

$$\tilde{\delta}_q : \mathcal{Z}^q(C) \longrightarrow H^{q+1}(A)$$

where $\tilde{\delta}_q = \delta_q \circ \theta$, is a continuous linear map.

Consider the subspace $\beta^{-1}(\mathcal{Z}^q(C)) \subset B^q$ and the continuous linear map

$$d_\beta : \beta^{-1}(\mathcal{Z}^q(C)) \longrightarrow B^{q+1}$$

induced by the differential operator $d : B^q \longrightarrow B^{q+1}$. The image of d_β is contained in $\alpha(\mathcal{Z}^{q+1}(A))$ because it is contained in the kernel of β ($\beta \circ d_\beta = d \circ \beta = 0$ on $\beta^{-1}(\mathcal{Z}^q(C))$) and it consists of cocycles. Since α is a topological isomorphism between A^{q+1} and $\alpha(A^{q+1})$, we obtain a continuous linear map $d'_\beta = \alpha^{-1} \circ d_\beta$:

$$d'_\beta : \beta^{-1}(\mathcal{Z}^q(C)) \longrightarrow \mathcal{Z}^{q+1}(A).$$

Consider the composed map $\theta \circ d'_\beta = \varrho$

$$\varrho : \beta^{-1}(\mathcal{Z}^q(C)) \longrightarrow H^{q+1}(A).$$

Since θ is continuous ϱ is also continuous. Now $\alpha(A^q) = \beta^{-1}(0) \subset \beta^{-1}(Z^q(C))$ and, moreover,

$$\alpha(A^q) \subset \operatorname{Ker} \varrho.$$

In fact, $d'_\beta \alpha(a) = \alpha^{-1} d\alpha(a) = da \in \mathcal{B}^{q+1}(A)$ for every $a \in A^q$. Therefore ϱ can be factored as follows :

$$\begin{tikzcd}
\beta^{-1}(\mathcal{Z}^q(C)) \arrow[rr, "\varrho"] \arrow[dr, "\sigma"'] & & H^{q+1}(A) \\
& \beta^{-1}(\mathcal{Z}^q(C))/\alpha(A^q) \arrow[ur, "\mu"'] &
\end{tikzcd}$$

where σ is the natural projection map and where μ is continuous if $\beta^{-1}(Z^q(C))/\alpha(A^q)$ is endowed with its quotient topology.

Since β is a topological homomorphism with $\operatorname{Ker}\beta = \operatorname{Im}\alpha$, that factor space, with its quotient topology can be topologically identified with $\mathcal{Z}^q(C)$. The map μ remains continuous, but, μ as a map from $\mathcal{Z}^q(C)$ to $H^{q+1}(A)$, by its very construction, coincides with $\tilde{\delta}_q$.

b) Čech and Dolbeault cohomology. Let X be a complex manifold and $\mathcal{U} = \{U_i\}_{i \in I}$ be a countable locally finite covering of X by open, relatively compact Stein sets. Let $\mathcal{F} = \mathcal{O}(E)$ be a locally free sheaf on X, i. e., the sheaf of germs of holomorphic sections of a holomorphic vector-bundle E on X.

Let $\mathcal{E}^{p,q}(E)$ denote the sheaf of germs of C^∞ forms of type (p, q) with values in E. We have on X the Dolbeault resolution

$$(1) \qquad 0 \to \mathcal{O}(E) \to \mathcal{E}^{00}(E) \xrightarrow{\bar{\partial}} \mathcal{E}^{01}(E) \xrightarrow{\bar{\partial}} \dots .$$

Because the resolution is fine we have algebraically

$$H^r(X, \mathcal{O}(E)) = H^r(\{\Gamma(X, \mathcal{E}^{0*}(E)), \bar{\partial}\}).$$

Now $H^r(X, \mathcal{O}(E)) = H^r(\mathcal{U}, \mathcal{O}(E))$ because the covering $\mathcal{U}$ is acyclic. Thus from the Čech complex for $\mathcal{U}$, $H^r(\mathcal{U}, \mathcal{O}(E))$ inherits a topological vector space structure that we denote by $\check{H}^r(\mathcal{U}, \mathcal{O}(E))$. Also, the complex $\{\Gamma(X, \mathcal{E}^{0,*}(E)), \bar{\partial}\}$ is a complex of topological vector spaces and we can endow the corresponding cohomology groups with a topological vector space structure. Note that each space $\Gamma(X, \mathcal{E}^{0,r}(E))$ has the structure of a Fréchet space.

PROPOSITION 12. *The isomorphism*

$$\check{H}^r(\mathcal{U}, \mathcal{O}(E)) \simeq H^r(\{\Gamma(X, \mathcal{E}^{0,*}(E)), \bar{\partial}\})$$

is a topological isomorphism.

PROOF. Let $U = U_{i_0 \dots i_p}$ for $p = 0, 1, \dots$. Since U Stein, the sequence (1) gives an exact sequence

$$(2) \qquad 0 \to \Gamma(U, \mathcal{O}(E)) \to \Gamma(U, \mathcal{E}^{0,0}(E)) \xrightarrow{\bar\partial} \Gamma(U, \mathcal{E}^{0,1}(E)) \to \dots$$

In it each map is a topological homomorphism. Splitting this sequence into short exact sequences we get:

$$0 \to \Gamma(U, \mathcal{O}(E)) \to \Gamma(U, \mathcal{E}^{0,0}(E)) \to \Gamma(U, \bar\partial\, \mathcal{E}^{0,0}(E)) \to 0$$

$$0 \to \Gamma(U, \bar\partial\, \mathcal{E}^{0,0}(E)) \to \Gamma(U, \mathcal{E}^{0,1}(E)) \to \Gamma(U, \bar\partial\, \mathcal{E}^{0,1}(E)) \to 0$$

$$\cdot \qquad \cdot \qquad \cdot$$

Taking the direct product over all $U's$ we get the exact sequences of Čech complexes

$$0 \to \Pi\Gamma(U_{i_0 \dots i_p}, \mathcal{O}(E)) \to \Pi\Gamma(U_{i_0 \dots i_p}, \mathcal{E}^{0,0}(E)) \to \Pi\Gamma(U_{i_0 \dots i_p}, \bar\partial\, \mathcal{E}^{0,0}(E)) \to 0$$

$$0 \to \Pi\Gamma(U_{i_0 \dots i_p}, \bar\partial \mathcal{E}^{0,0}(E)) \to \Pi\Gamma(U_{i_0 \dots i_p}, \mathcal{E}^{0,1}(E)) \to \Pi\Gamma(U_{i_0 \dots i_p}, \bar\partial \mathcal{E}^{0,1}(E)) \to 0$$

$$\cdot \qquad \cdot \qquad \cdot$$

In these complexes each space is a Fréchet space and the maps are topological homomorphisms. In the corresponding cohomology sequences, all maps are therefore continuous. In particular we get a sequence of continuous bijections:

$$\check{H}^q(\mathcal{U}, \mathcal{O}(E)) \leftarrow \check{H}^{q-1}(\mathcal{U}, \bar\partial\, \mathcal{E}^{0,0}(E)) \leftarrow \check{H}^{q-2}(\mathcal{U}, \bar\partial\, \mathcal{E}^{0,1}(E)) \leftarrow \dots$$

$$\dots \leftarrow \check{H}^1(\mathcal{U}, \bar\partial\, \mathcal{E}^{0,q-2}(E))$$

and the exact sequences of topological vector spaces and continuous maps,

$$\check{H}^0(\mathcal{U}, \mathcal{E}^{0\,q-1}(E)) \xrightarrow{\bar\partial} \check{H}^0(\mathcal{U}, \bar\partial\, \mathcal{E}^{0,q-1}(E)) \to \check{H}^1(\mathcal{U}, \bar\partial\, \mathcal{E}^{0,q-2}(E)) \to 0$$

$$0 \to \check{H}^0(\mathcal{U}, \bar\partial\, \mathcal{E}^{0,q-1}(E)) \to \check{H}^0(\mathcal{U}, \mathcal{E}^{0,q}(E)) \xrightarrow{\bar\partial} \check{H}^0(\mathcal{U}, \mathcal{E}^{0,q+1}(E)).$$

Note that $\check{H}^0(\mathcal{U}, \mathcal{E}^{0,s}(E)) \simeq \Gamma(U, \mathcal{E}^{0,s}(E))$ also topologically. Therefore a con-

tinuous bijection

$$H^q(\{\Gamma(X, \mathcal{E}^{0,*}(E), \bar{\partial}\}) \to H^1(\mathcal{U}, \bar{\partial}\mathcal{E}^{0,q-2}(E)).$$

In conclusion we have a continuous bijection

$$\lambda : H^q(\{\Gamma(X, \mathcal{E}^{0,*}(E), \bar{\partial}\}) \to H^q(\mathcal{U}, \mathcal{O}(E)).$$

Let $\{\varrho_i\}_{i \in I}$ be a C^∞ partition of unity subordinate to the covering $\mathcal{U}$. We define continuous mappings

$$\check{H}^{r+1}(\mathcal{U}, \bar{\partial}\mathcal{E}^{0,s}(E)) \to \check{H}^r(\mathcal{U}, \bar{\partial}\mathcal{E}^{0,s+1}(E))$$

$$\check{H}^{r+1}(\mathcal{U}, \mathcal{O}(E)) \to \check{H}^r(\mathcal{U}, \bar{\partial}\mathcal{E}^{0,0}(E))$$

by the mapping of cochains

$$\{\gamma_{i_0 \dots i_{r+1}}\} \to \{\sigma_{j_0 \dots j_r}\}$$

where

$$\sigma_{j_0 \dots j_r} = \bar{\partial}(\sum_j \varrho_j \gamma_{jj_0 \dots j_r}).$$

These induce on cohomology the sequence of inverse mappings of the continuous bijections considered above. Therefore λ^{-1} is also continuous.

c) The previous proof can be repeated substituting direct sums for direct products. Indeed one has the following useful

LEMMA. *Let A_n, B_n, $n = 1, 2, 3, \dots$ be two sequences of topological vector spaces over $\mathbb{C}$ and let $\Phi_n : A_n \to B_n$ be a sequence of topological homomorphisms. Then*

$$\Phi \equiv \coprod \Phi_n : \coprod A_n \to \coprod B_n$$

is a topological homomorphism.

PROOF. Let C_n, $n = 1, 2, 3, \dots$ be a sequence of locally convex spaces. Let W_n be a neighborhood of the origin in C_n which is convex and balanced. Let $W = \coprod W_n$; it is convex balanced and a neighborhood of the origin in $C = \coprod C_n$ as $W \cap C_n = W_n$.

We claim that if the sets W_n describe a fundamental system of neighborhoods of the origin in C_n then W describes a fundamental system of neighborhoods of the origin in C. Indeed, by definition, the convex balanced envelope $\Gamma(\cup W_n)$ of the set $\cup W_n$ in C, describe a fundamental system of

neighborhoods of the origin in C. Now

$$U = \coprod_{n=1}^{\infty} \frac{1}{2^n} W_n$$

is a neighborhood of the origin of the type of W and moreover $U \subset \Gamma(\cup W_n)$.

Let now U_n be a neighborhood of the origin in A_n, convex and balanced. By assumption there exists a convex balanced neighborhood of the origin $V_n \subset B_n$ such that $\Phi_n(U_n) = \Phi(A_n) \cap V_n$. Therefore if $U = \coprod U_n$, $V = \coprod V_n$ we have

$$\Phi(U) = \Phi(A) \cap V$$

where $A = \coprod A_n$. This shows that $\Phi : A \to \Phi(A)$ is open.

COROLLARY. *The isomorphism*

$$\check{H}_k^r(\mathcal{U}, \mathcal{O}(E)) \simeq H^r(\Gamma_k(X, \mathcal{E}^{0,*}(E)), \bar{\partial})$$

is topological.

PROOF. Let $\{K_i\}_{i \in N}$ be a sequence of compact subsets of X such that $\mathring{K}_i \subset K_{i+1}$, $\cup K_i = X$. If we set

$$\Gamma_{K_i}(X, \mathcal{E}) = \{s \in \Gamma(X, \mathcal{E}) \mid \operatorname{supp} s \subset K_i\}$$

with the induced topology from $\Gamma(X, \mathcal{E})$ we obtain a Fréchet space. The topology on $\Gamma_k(X, \mathcal{E})$ is defined as the inductive limit topology

$$\Gamma_k(X, \mathcal{E}) = \varinjlim \Gamma_{K_i}(X, \mathcal{E}).$$

The only thing that needs proof is that the bijection

$$(*) \qquad \check{H}_k^0(\mathcal{U}, \mathcal{E}) \to \Gamma_k^0(X, \mathcal{E})$$

is a topological isomorphism. Now the target space is Souslin inductive limit of Banach spaces, the source space is also a Souslin space as a closed subspace of $\coprod \Gamma(U_i, \mathcal{E})$. We need only to verify that the map (*) is continuous. This verification is left to the reader.

d) Čech and Dolbeault homology. Let E^* be the dual bundle of E and let $\mathcal{D}^{r,s}(E^*)(U)$ denote the space of currents of type (r, s) with values in E^* and compact support in U (U open in X). We have

$$\mathcal{D}^{r,s}(E^*)(U) = \text{strong dual of } \Gamma(U, \mathcal{E}^{n-r,n-s}(E));$$

as each such space $\mathcal{D}^{r,s}(E^*)(U)$ has a structure of a dual of a space of Fréchet-Schwartz.

Let $\mathcal{O}_*(E)$ denote the dual cosheaf of the sheaf $\mathcal{O}(E)$. By dualising the sequence (2) we get on U Stein an exact sequence

$$0 \leftarrow \mathcal{O}_*(E)(U) \leftarrow \mathcal{D}^{n,n}(E^*)(U) \xleftarrow{\bar{\partial}} \mathcal{D}^{n,n-1}(E^*)(U) \leftarrow \dots$$

in which each map is a topological homomorphism. Replacing direct products with direct sums in the proof of the previous proposition one obtains the following

PROPOSITION 13. *The isomorphism*

$$\check{H}_r(\mathcal{U}, \mathcal{O}_*(E)) \simeq H_r(\{\mathcal{D}^{n,*}(X), \bar{\partial}\})$$

is a topological isomorphism.

REMARK: Using the fact that a product of topological homomorphisms is a topological homomorphism, one can prove a statement corresponding to the corollary to proposition 12 for homology with closed supports.

e) (6) The previous propositions can be extended to the case of any coherent sheaf making use of the theorem of division of distributions of Lojasiewicz and Malgrange [12]. For the sake of completeness we recall the basic facts.

Let $\mathcal{T}^p(\Omega)$ denote the space of distributions with values in $\mathbb{C}^p$ on the open set $\Omega \in \mathbb{R}^m$. If $\mathcal{E}_0^p(\Omega)$ denotes the space of C^∞ functions with values in $\mathbb{C}^p$ and compact support in Ω, topologized with the usual inductive limit topology, then $\mathcal{T}^p(\Omega)$ denotes the strong dual of $\mathcal{E}_0^p(\Omega)$. Let us also denote by Σ_a, $a \in \mathbb{R}^m$, the ring of formal power series with center in a and complex coefficients. Let $\mathcal{A}(\Omega)$ denote the ring of complex valued real analytic functions in Ω.

The theory of division of distributions is based on the following two theorems (cf. [12] for references).

THEOREM OF WHITNEY. *Let $\mathcal{M}$ be a closed submodule of the Fréchet space $\mathcal{E}^p(\Omega)$ of C^∞ functions in Ω with values in $\mathbb{C}^p$, and let $f \in \mathcal{E}^p(\Omega)$. A necessary and sufficient condition for $f \in \mathcal{M}$ is that*

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(6) This section and the following one are never used in the sequel.

for every $a \in \Omega$ the Taylor series $t_a(f)$ of f with center at a is the Taylor series of some element in $\mathcal{M}$.

THEOREM OF LOJASIEWICZ-MALGRANGE. *Let A be a $p \times q$-matrix with elements in $\mathcal{A}(\Omega)$ and let $\mathcal{S} \in \mathcal{T}^p(\Omega)$. The equation*

$$AX = \mathcal{S}$$

admits a solution $X \in \mathcal{T}^q(\Omega)$ if and only if for any vector $G = (g_1, \dots, g_r)$ with components $g_i \in \mathcal{A}(\Omega)$ and such that $GA = 0$ we also have $G\mathcal{S} = 0$.

By transposition and using the theorem of Whitney one deduces the following corollaries

COROLLARY 1. *The linear map*

$$A : \mathcal{E}^q(\Omega) \longrightarrow \mathcal{E}^p(\Omega)$$

given by $f \longrightarrow Af$, has a closed image.

COROLLARY 2. *Given $f \in \mathcal{E}^p(\Omega)$ the equation*

$$AX = f$$

admits a solution $X \in \mathcal{E}^q(\Omega)$ if and only if for every $a \in \Omega$ the equation

$$t_a(A)\, X = t_a(f)$$

admits a solution $X \in \Sigma_a^q$.

Let us now suppose that $\mathbb{R}^m = \mathbb{C}^n$ and let $\mathcal{O}$ be the structure sheaf of $\mathbb{C}^n$, using the fact that for every $x \in \mathbb{C}^n$, Σ_x is a flat ring over $\mathcal{O}_x$, from the previous corollary one deduces the following

FLATNESS THEOREM. *The sheaf $\mathcal{E}$ of germs of C^∞ complex valued functions is $\mathcal{O}$-flat* (i. e., $\forall\, x \in \mathbb{C}^n$, *$\mathcal{E}_x$ is a flat $\mathcal{O}_x$-module*).

Let now X be a complex manifold, we consider on X the Dolbeault resolution (1) (with $E =$ trivial bundle)

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{E}^{00} \xrightarrow{\bar\partial} \mathcal{E}^{01} \xrightarrow{\bar\partial} \dots$$

Let $\mathcal{F}$ be any analytic sheaf on X. From the flatness theorem one deduces that the following sequence of sheaves is also exact:

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E}^{0,0} \otimes_{\mathcal{O}} \mathcal{F} \xrightarrow{\bar\partial} \mathcal{E}^{0,1} \otimes_{\mathcal{O}} \mathcal{F} \xrightarrow{\bar\partial} \dots$$

Since the sheaves $\mathcal{E}^{0,q} \otimes_{\mathcal{O}} \mathcal{F}$ are fine sheaves we thus have

$$H^r(X, \mathcal{F}) \simeq H^r(\{\Gamma(X, \mathcal{E}^{0,*} \otimes_{\mathcal{O}} \mathcal{F}), \bar{\partial}\}).$$

Let us now suppose that $\mathcal{F}$ is coherent. Let U be an open set of X on which we have an exact sequence of the form

$$\mathcal{O}^{p_1} \xrightarrow{A} \mathcal{O}^{p_0} \to \mathcal{F} \to 0. \tag{1}$$

LEMMA. *Applying to the sequence* (2) *the functor* $\Gamma(U, \cdot \otimes_{\mathcal{O}} \mathcal{E}^{0,j})$ *we get an exact sequence*

$$\Gamma(U, \mathcal{E}^{0,j})^{p_1} \xrightarrow{A} \Gamma(U, \mathcal{E}^{0,j})^{p_0} \to \Gamma(U, \mathcal{F} \otimes \mathcal{E}^{0,j}) \to 0.$$

PROOF. Split the sequence (2) in two short exact sequences

$$0 \to \mathcal{Z}_0 \to \mathcal{O}^{p_0} \to \mathcal{Z} \to 0$$

$$0 \to \mathcal{N} \to \mathcal{O}^{p_1} \to \mathcal{Z}_0 \to 0.$$

By tensoring these sequences with $\mathcal{E}^{0,j}$, they remain exact by the flatness theorem. Writing the cohomology sequences of the exact sequences thus obtained we get

$$0 \to \Gamma(U, \mathcal{Z}_0 \otimes \mathcal{E}^{0,j}) \to \Gamma(U, \mathcal{E}^{0,j})^{p_0} \to \Gamma(U, \mathcal{F} \otimes \mathcal{E}^{0,j}) \to$$
$$\to H^1(U, \mathcal{Z}_0 \otimes \mathcal{E}^{0j}) \to \dots$$

and

$$0 \to \Gamma(U, \mathcal{N} \otimes \mathcal{E}^{0,j}) \to \Gamma(U, \mathcal{E}^{0,j})^{p_1} \to \Gamma(U, \mathcal{Z}_0 \otimes \mathcal{E}^{0,j}) \to$$
$$\to H^1(U, \mathcal{N} \otimes \mathcal{E}^{0,j}) \to \dots$$

But $\mathcal{Z}_0 \otimes \mathcal{E}^{0,j}$ and $\mathcal{N} \otimes \mathcal{E}^{0,j}$ are fine sheaves thus their first cohomology groups vanish. From the short exact sequences that we obtain, follows the contention of the lemma.

Now $\Gamma(U, \mathcal{E}^{0,j})$ has a natural structure of Fréchet space and by Corollary 1 the image of the map

$$\Gamma(U, \mathcal{E}^{0,j})^{p_1} \xrightarrow{A} \Gamma(U, \mathcal{E}^{0,j})^{p_0} \tag{*}$$

is closed. Therefore on $\Gamma(U, \mathcal{F} \otimes \mathcal{E}^{0,j})$ we can introduce the quotient topology derived from (*) which makes that space a Fréchet space.

If we have on U another resolution

$$\mathcal{O}^{q_1} \longrightarrow \mathcal{O}^{q_0} \longrightarrow \mathcal{F} \longrightarrow 0$$

there exists a linear map $\beta : \Gamma(U, \mathcal{E}^{0,j})^{p_0} \longrightarrow \Gamma(U, \mathcal{E}^{0,j})^{q_0}$ given by a matrix with C^∞ entries such that the diagram

$$\begin{tikzcd}
\Gamma(U, \mathcal{E}^{0,j})^{p_0} \arrow[d, "\beta"] \arrow[dr] & \\
\Gamma(U, \mathcal{E}^{0,j})^{q_0} \arrow[r] & \Gamma(U, \mathcal{F} \otimes \mathcal{E}^{0,j})
\end{tikzcd}$$

is commutative. By the open mapping theorem it follows that the topology introduced on $\Gamma(U, \mathcal{F} \otimes \mathcal{E}^{0,j})$ is independent of the choice of the resolution (2).

If $W \subset U$ then the restriction map

$$r_W^U : \Gamma(U, \mathcal{F} \otimes \mathcal{E}^{0,j}) \longrightarrow \Gamma(W, \mathcal{F} \otimes \mathcal{E}^{0,j})$$

is a continuous map.

Moreover, from the commutative diagram

$$\begin{tikzcd}
\Gamma(U, \mathcal{E}^{0,j})^{p_0} \arrow[r, "\bar\partial"] \arrow[d] & \Gamma(U, \mathcal{E}^{0,j+1})^{p_0} \arrow[d] \\
\Gamma(U, \mathcal{F} \otimes \mathcal{E}^{0,j}) \arrow[r, "\bar\partial"] & \Gamma(U, \mathcal{F} \otimes \mathcal{E}^{0,j+1})
\end{tikzcd}$$

it follows that the operator

$$\bar\partial : \Gamma(U, \mathcal{F} \otimes \mathcal{E}^{0,j}) \longrightarrow \Gamma(U, \mathcal{F} \otimes \mathcal{E}^{0,j+1})$$

is continuous.

Let $\mathcal{U} = \{U_i\}_{i \in \mathfrak{N}}$ be a countable covering of X by relatively compact Stein open sets. From the exact sequence

$$0 \longrightarrow \Gamma(X, \mathcal{F} \otimes \mathcal{E}^{0,j}) \longrightarrow \prod_i \Gamma(U_i, \mathcal{F} \otimes \mathcal{E}^{0,j}) \xrightarrow{\delta} \prod_{ik} \Gamma(U_i \cap U_k, \mathcal{F} \otimes \mathcal{E}^{0,j})$$

we see that $\Gamma(X, \mathcal{F} \otimes \mathcal{E}^{0,j})$ as the kernel of δ has the structure of a Fréchet

space. Since the restriction homomorphisms are continuous that topology is independent of the choice of the covering.

In conclusion, the spaces

$$\Gamma(X, \mathcal{F} \otimes \mathcal{E}^{0,j})$$

have natural structures of Fréchet spaces and the operators $\bar{\partial}$ are therefore continuous.

Applying the argument of proposition 12 we then obtain

PROPOSITION 12′. *Let $\mathcal{F}$ be a coherent sheaf on X. The identity maps on cohomology groups*

$$\check{H}^q(\mathcal{U}, \mathcal{F}) = H^q(\{\Gamma(X, \mathcal{F} \otimes \mathcal{E}^{0,*}), \bar{\partial}\})$$

are topological isomorphisms.

f) Similar considerations can be developed for homology groups. We limit ourself to some brief indications.

We consider the exact sequence of sheaves

$$0 \to \mathcal{F} \to \Lambda^0 \xrightarrow{\bar{\partial}} \Lambda^1 \xrightarrow{\bar{\partial}} \dots$$

where $\Lambda^i = \mathcal{F} \otimes \mathcal{E}^{0,i}$.

Let U be a relatively compact Stein open set on which $\mathcal{F}$ admits a resolution (2). Taking sections we get an exact sequence of Fréchet spaces and continuous maps :

$$(3) \qquad 0 \to \Gamma(U, \mathcal{F}) \xrightarrow{i} (\Gamma(U, \Lambda^0) \xrightarrow{\bar{\partial}} \Gamma(U, \Lambda^1) \xrightarrow{\bar{\partial}} \dots$$

where $\Gamma(U, \mathcal{F})$ is endowed with the topology of the coker $\{\Gamma(U, \mathcal{O}^{p_1}) \to \to \Gamma(U, \mathcal{O}^{p_0})\}$. The map i is continuous as the composed map

$$\alpha : \Gamma(U, \mathcal{O}^{p_0}) \to \Gamma(U, \mathcal{F}) \to \Gamma(U, \Lambda^0)$$

is continuous.

By dualising (3) we get an exact sequence of topological vector spaces (duals of Fréchet Schwartz) and continuous maps,

$$(4) \qquad 0 \leftarrow \mathcal{F}_*(U) \leftarrow \Lambda^0_*(U) \leftarrow \Lambda^1_*(U) \leftarrow \dots$$

where $\Lambda^i_* = \text{Hom cont}(\Gamma(U, \Lambda^i), \mathbb{C})$.

Let $\mathcal{K}^{n,n-j}$ be the sheaf of germs of currents of type $(n, n-j)$; one can verify that

$$\Lambda^i_*(U) = \Gamma_k(U, \mathcal{H}om(\mathcal{F}, \mathcal{K}^{n,n-j})\}.$$

It follows that (4) gives a resolution of the cosheaf $\mathcal{F}_*$ by flabby cosheaves. From here the situation is completely analogous to the one developed before.

It is worth noticing that the sheaf $\mathcal{K}$ is a coflat sheaf. One obtains thus the following

PROPOSITION 13′. *Let $\mathcal{F}$ be a coherent sheaf on X. Then the isomorphism*

$$\check{H}_r(\mathcal{U}, \mathcal{F}_*) \simeq H_r(\{\Lambda_*(X), \bar{\partial}\})$$

is a topological isomorphism.

16. *Cohomology with compact supports. a)* This type of cohomology, and dually homology with closed supports presents some unpleasant features that we are able to overcome with a different type of argument.

Let K be a compact set in $\mathbb{C}^n$ and let $\{W_s\}_{s \in \mathfrak{D}}$ be a fundamental sequence of neighborhoods of K. We set

$$\mathcal{O}(K) = \varinjlim \Gamma(W_s, \mathcal{O})$$

where each $\Gamma(W_s, \mathcal{O})$ is endowed with its natural Fréchet topology and where $\mathcal{O}(K)$ has the topology of the considered inductive limit. The topology of $\mathcal{O}(K)$ is independent of the choice of the sequence $\{W_s\}$ and is the topology of a strong dual of a space of Fréchet-Schwartz ([8] p. 315 and 337).

If $V \subset U$ are open bounded in $\mathbb{C}^n$ then the natural restriction maps

$$\mathcal{O}(\bar{U}) \to \Gamma(V, \mathcal{O})$$

$$\Gamma(U, \mathcal{O}) \to \mathcal{O}(\bar{V})$$

are continuous.

By a *Stein compact set* K we mean a compact set having a fundamental sequence of neighborhoods each of which is Stein.

Let $\mathcal{F}$ be a coherent sheaf defined in a neighborhood of K, $\mathcal{F}$ admits a presentation of the form

$$\mathcal{O}^p \xrightarrow{\alpha} \mathcal{O}^q \xrightarrow{\varepsilon} \mathcal{F} \to 0. \tag{1}$$

From (1) we deduce an exact sequence

$$\mathcal{O}^p(K) \xrightarrow{\alpha_*} \mathcal{O}^q(K) \xrightarrow{\varepsilon_*} \mathcal{F}(K) \to 0 \tag{2}$$

where $\mathcal{F}(K) = \Gamma(K, \mathcal{F}) = \varinjlim \Gamma(W_s, \mathcal{F})$.

In the sequence (2) the spaces $\mathcal{O}^p(X)$ and $\mathcal{O}^q(K)$ are strong duals of Fréchet-Schwartz and α_* is continuous.

LEMMA 1. *In the sequence* (2) $\operatorname{Im} \alpha_* = \operatorname{Ker} \varepsilon_*$ *is closed.*

PROOF. For $x \in K$ let $\mathfrak{m}_x$ denote the maximal ideal of the ring $\mathcal{O}_x$. Consider the composite linear mapping, for each positive integer k,

$$\mathcal{O}^q(K) \xrightarrow{\lambda_x^k} \frac{\mathcal{O}_x^q}{\mathfrak{m}_x^{k+1}\mathcal{O}_x^q} \xrightarrow{\mu_x^k} \frac{\mathcal{F}_x}{\mathfrak{m}_x^{k+1}\mathcal{F}_x}$$

where λ_x^k is the natural map and where μ_x^k is induced by ε_*.

The second and third spaces are finite dimensional over $\mathbb{C}$ and μ_x^k is $\mathbb{C}$ linear and therefore continuous. Also λ_x^k is continuous since for each W_s the composite map

$$\Gamma(W_s, \mathcal{O})^q \to \mathcal{O}^q(K) \xrightarrow{\lambda_x^k} \frac{\mathcal{O}_x^q}{\mathfrak{m}_x^{k+1}\mathcal{O}_x^q}$$

is continuous being the evaluation at x of a function of $\Gamma(W_s, \mathcal{O})^q$ with all its partial derivatives up to order k included.

Therefore $\varepsilon_x^k = \mu_x^k \circ \lambda_x^k$ is continuous and consequently $\operatorname{Ker} \varepsilon_x^k$ ls closed. By Krull's theorem we have

$$\operatorname{Ker} \varepsilon_* = \bigcap_{x \in K} \bigcap_{k \in \mathbb{N}} \operatorname{Ker} \varepsilon_x^k .$$

Therefore $\operatorname{Ker} \varepsilon_*$ is closed.

We can therefore consider on $\mathcal{F}(K)$ the topological structure of coker ε_* which makes $\mathcal{F}(K)$ into a strong dual of a space of Fréchet-Schwartz. One verifies as usual that this structure is independent of the choice of the resolution (1). Note that the topological structure on $\mathcal{F}(K)$ must coincide with the inductive limit topology $\varinjlim \Gamma(W_s, \mathcal{F})$ as the natural map

$$\varinjlim \Gamma(W_s, \mathcal{F}) \to \mathcal{F}(K)$$

is continuous, the source space being Souslin and the target space inductive limit of Banach spaces.

These considerations can be extended to compact Stein subsets K of a complex space X either directly or by making use of an imbedding of K into some compact Stein subset of some numerical space. We thus have:

PROPOSITION 14. *Let K be a compact Stein subset of a complex space X having a fundamental sequence of Stein neighborhoods $\{W_s\}_{s \in \mathfrak{N}}$. Let $\mathcal{F}$ be a coherent sheaf on X. Then the topological inductive limit*

$$\mathcal{F}(K) = \varinjlim \Gamma(W_s, \mathcal{F})$$

has a natural structure of a strong dual of a space of Fréchet-Schwartz.

(*b*) Let X be a complex space and let $\mathcal{F}$ be a coherent sheaf on X: Let K be a compact Stein subset of X. We define

$$\mathcal{F}_*(K) = \text{strong dual of } \mathcal{F}(K).$$

If $V \subset U$ are open Stein subsets of X whose closure $\bar{V} \subset \bar{U}$ are compact Stein, we get continuous maps:

$$\mathcal{F}(\bar{U}) \to \Gamma(V, \mathcal{F})$$

$$\Gamma(U, \mathcal{F}) \to \mathcal{F}(\bar{V}).$$

And therefore, by transposition, continuous maps

$$\mathcal{F}_*(V) \to \mathcal{F}_*(\bar{U})$$

$$\mathcal{F}_*(\bar{V}) \to \mathcal{F}_*(U).$$

We can thus consider the presheaf $\bar{\mathcal{F}}$:

$$U \to \mathcal{F}(\bar{U})$$

defined on the class $\mathcal{W}'$ of all relatively compact Stein subsets U of X with Stein closure. Dually we can consider on $\mathcal{W}'$ the precosheaf $\bar{\mathcal{F}}_*$

$$U \to \mathcal{F}_*(\bar{U}).$$

Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a countable locally finite covering of X with $\mathcal{U} \subset \mathcal{W}'$. We can consider the Čech complexes

$$(\mathrm{I}') = \{C_k^*(\mathcal{U}, \bar{\mathcal{F}}), \delta\} \quad \text{and} \quad (\mathrm{II}') = \{C_*^c(\mathcal{U}, \bar{\mathcal{F}}_*), \partial\}$$

which are dual of each other. Both are topological complexes; in the first

the cochain groups

$$C_k^q(\mathcal{U}, \overline{\mathcal{F}}) = \coprod_{U_{i_0} \cap \dots \cap U_{i_q} \neq \emptyset} \mathcal{F}(\overline{U_{i_0} \cap \dots \cap U_{i_q}})$$

have the structure of strong duals of Fréchet-Schwartz; in the second the chain groups

$$C_q^c(\mathcal{U}, \overline{\mathcal{F}}_*) = \prod_{U_{i_0} \cap \dots \cap U_{i_q} \neq \emptyset} \mathcal{F}_*(\overline{U_{i_0} \cap \dots \cap U_{i_q}})$$

have the structure of spaces of Fréchet-Schwartz.

LEMMA 2. *Let Ω be an open Stein subset of X with Stein compact closure. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a finite covering of Ω by Stein open subsets with compact Stein closures. Then*

$$H^q(\mathcal{U}, \overline{\mathcal{F}}) = 0 \quad \textit{for} \quad q > 0 \quad \textit{and} \quad H_q(\mathcal{U}, \overline{\mathcal{F}}_*) = 0 \quad \textit{for} \quad q > 0.$$

PROOF. Let $\{W_i^{(n)}\}$ be a fundamental sequence of Stein neighborhoods of $\overline{U}_i$ and $\{\Omega^{(n)}\}$ a fundamental sequence of Stein neighborhoods of Ω. It is no restriction to assume that $\Omega^{(n)} = \cup W_i^{(n)}$.

The sequences

$$0 \to \Gamma(\Omega^{(n)}, \mathcal{F}) \to \Pi\Gamma(W_i^{(n)}, \mathcal{F}) \xrightarrow{\delta} \Pi\Gamma(W_i^{(n)} \cap W_j^{(n)}, \mathcal{F}) \xrightarrow{\delta} \dots$$

are exact since $\Omega^{(n)}$ is Stein. Taking the inductive limits for $n \to \infty$ we get an exact sequence

$$0 \to \mathcal{F}(\overline{\Omega}) \to \Pi\, \mathcal{F}(\overline{U}_i) \to \Pi\, \mathcal{F}(\overline{U_i \cap U_j}) \to \dots$$

This proves the first part of the lemma.

Now we remark that in the above sequence, since the covering $\mathcal{U}$ is finite, each space is a strong dual of a space of Fréchet-Schwartz. Therefore all maps in the sequence, being continuous are topological homomorphisms.

We can therefore apply the duality lemma and we obtain an exact sequence

$$0 \leftarrow \mathcal{F}_*(\overline{\Omega}) \leftarrow \coprod \mathcal{F}_*(\overline{U}_i) \xleftarrow{\partial} \coprod \mathcal{F}_*(\overline{U_i \cap U_j}) \xleftarrow{\partial} \dots$$

(each space being a Fréchet space and each map being a topological homomorphism).

This proves the second part of the lemma.

COROLLARY 1. *If $\mathcal{U} < \mathcal{V}$ are locally finite coverings of X extracted from $\mathcal{W}'$ then we have continuous bijections for any $q > 0$,*

$$H_k^q(\mathcal{U}, \overline{\mathcal{F}}) \to H_k^q(\mathcal{V}, \mathcal{F}), \quad H_k^q(\mathcal{U}, \mathcal{F}) \to H_k^q(\mathcal{V}, \overline{\mathcal{F}})$$

$$H_q^c(\mathcal{U}, \overline{\mathcal{F}}_*) \leftarrow H_q^c(\mathcal{V}, \mathcal{F}_*), \quad H_q^c(\mathcal{U}, \mathcal{F}_*) \leftarrow H_q^c(\mathcal{V}, \overline{\mathcal{F}}_*).$$

PROOF. Let us prove the first of these statements. We can select locally finite coverings extracted from $\mathcal{W}'$, $\tilde{\mathcal{U}}$ and $\tilde{\mathcal{V}}$ such that

$$\tilde{\mathcal{U}} > \mathcal{U} > \mathcal{V} > \tilde{\mathcal{V}}.$$

We have then a sequence of continuous maps

$$H_k^q(\tilde{\mathcal{U}}, \mathcal{F}) \to H_k^q(\mathcal{U}, \overline{\mathcal{F}}) \to H_k^q(\mathcal{V}, \mathcal{F}) \to H_k^q(\tilde{\mathcal{V}}, \overline{\mathcal{F}}).$$

By the Leray theorem $H_k^q(\tilde{\mathcal{U}}, \mathcal{F}) \xrightarrow{\sim} H_k^q(\mathcal{V}, \mathcal{F})$, hence

$$H_k^q(\mathcal{U}, \overline{\mathcal{F}}) \to H_k^q(\mathcal{V}, \mathcal{F}) \quad \text{is surjective.}$$

By lemma 2, we can apply the Leray theorem (see the remark after theorem 5) to the presheaf $\overline{\mathcal{F}}$, thus $H_k^q(\mathcal{U}, \overline{\mathcal{F}}) \to H_k^q(\mathcal{U}, \mathcal{F})$ is injective.

The other statements are proved in the same way.

COROLLARY 2. (a) *If for every locally finite countable covering $\mathcal{U} \subset \mathcal{W}'$ the spaces $H_k^q(\mathcal{U}, \overline{\mathcal{F}})$ (or the spaces $H_k^q(\mathcal{U}, \mathcal{F})$) are Hausdorff, then the spaces $H_k^q(\mathcal{U}, \overline{\mathcal{F}}) = H_k^q(\mathcal{U}, \mathcal{F}) = H_k^q(X, \mathcal{F})$ are topologically isomorphic and have the structure of a strong dual of a space of Fréchet-Schwartz.*

(b) *If for every locally finite countable covering $\mathcal{U} \subset \mathcal{W}'$ the spaces $H_q^c(\mathcal{U}, \overline{\mathcal{F}}_*)$ (or the spaces $H_q^c(\mathcal{U}, \mathcal{F}_*)$) are Hausdorff, then the spaces $H_q^c(\mathcal{U}, \overline{\mathcal{F}}_*) = H_q^c(\mathcal{U}, \mathcal{F}_*) = H_q^c(X, \mathcal{F}_*)$ are topologically isomorphic and have the structure of a space of Fréchet-Schwartz.*

PROOF. (*a*) If $\mathcal{V}$ is locally finite countable, $\mathcal{V} \subset \mathcal{W}'$ and if $\mathcal{V} < \mathcal{U}$ we have continuous bijections

$$H_k^q(\mathcal{U}, \mathcal{F}) \to H_k^q(\mathcal{V}, \overline{\mathcal{F}})$$

$$(\text{or} \quad H_k^q(\mathcal{U}, \overline{\mathcal{F}}) \to H_k^q(\mathcal{V}, \mathcal{F})).$$

Therefore the spaces $H_k^q(\mathcal{U}, \mathcal{F})$ (or $H_k^q(\mathcal{U}, \overline{\mathcal{F}})$) are also Hausdorff. But then $H_k^q(\mathcal{V}, \overline{\mathcal{F}})$ has a structure of strong dual of a space of Fréchet-Schwartz. Therefore, since $H_k^q(\mathcal{U}, \mathcal{F})$ is a Souslin space, the continuous bijection $H_k^q(\mathcal{U}, \mathcal{F}) \to H_k^q(\mathcal{V}, \overline{\mathcal{F}})$ is a topological isomorphism. Hence for every $\mathcal{U}$ the spaces $H_k^q(\mathcal{U}, \mathcal{F})$ and $H_k^q(\mathcal{U}, \overline{\mathcal{F}})$ are strong duals of Fréchet Schwartz. Since continuous bijections among spaces of this type are topological isomorphisms, the assertion follows.

(*b*) The argument is the same with only formal changes.

ADDITION TO THEOREM I. *If $H_k^{q+1}(X, \mathcal{F})$ is separated, then $H_q^c(X, \mathcal{F}_*)$ is also separated and*

$$H_q^c(X, \mathcal{F}_*) = \operatorname{Hom\,cont}(H_k^q(X, \mathcal{F}), \mathbb{C}).$$

PROOF. Replace in the argument of Theorem I the complexes (I) and (II) by the complexes (I′) and (II″).

Similarly, we get the following

ADDITION TO THEOREM II. *If for every locally finite covering $\mathcal{U} \subset \mathcal{W}'$ the spaces $H_{q-1}^c(\mathcal{U}, \mathcal{F}_*)$ are separated. Then $H_{q-1}^c(X, \mathcal{F}_*)$ is separated and $H_k^q(X, \mathcal{F})$ is separated. Moreover*

$$H_k^q(X, \mathcal{F}) = \operatorname{Hom\,cont}(H_q^c(X, \mathcal{F}_*), \mathbb{C}).$$

§ 8. Some applications.

18. *Duality on q-pseudoconvex manifolds.* Let X be a q-pseudoconvex manifold of finite dimension $\dim_{\mathbb{C}} X = n$ and let $\mathcal{F}$ be any coherent shear on X.

The behavior of cohomology groups on X is given by the following theorem which we borrow from [1].

THEOREM 7. (a) *If X is a q-pseudoconvex space then for any coherent sheaf $\mathcal{F}$*

$$\dim_{\mathbb{C}} H^j(X, \mathcal{F}) < \infty \quad \textit{for} \quad j > q.$$

(b) *If X is q complete then for any coherent sheaf $\mathcal{F}$*

$$H^j(X, \mathcal{F}) = 0 \quad \textit{for} \quad j > q.$$

Applying Theorem (I) we obtain the following

COROLLARY 1. (a) *If X is q-pseudoconvex, then for any coherent sheaf $\mathcal{F}$*

$$H_q(X, \mathcal{F}_*) \quad \textit{is separated}$$

and

$$\dim_{\mathbb{C}} H_j(X, \mathcal{F}_*) < \infty \quad \textit{for} \quad j > q.$$

(b) *If X is in particular q-complete then for any coherent sheaf $\mathcal{F}$*

$$H_j(X, \mathcal{F}_*) = 0 \quad \textit{for} \quad j > q.$$

COROLLARY 2. (a) *If X is q-pseudoconvex, then for any coherent sheaf $\mathcal{F}$*

$$\dim_{\mathbb{C}} H_j^c(X, \mathcal{F}_*) < \infty \quad \textit{if} \quad j < \operatorname{prof} \mathcal{F} - q.$$

(b) *If X is q-complete then for any coherent sheaf $\mathcal{F}$*

$$H_j^c(X, \mathcal{F}_*) = 0 \quad \textit{if} \quad j < \operatorname{prof} \mathcal{F} - q.$$

PROOF OF COROLLARY 2. From Theorem 6 we get

$$H_j^c(X, \mathcal{F}_*) = \operatorname{EXT}^{n-j}(X; \mathcal{F}, \Omega^n)$$

we have a spectral sequence

$$E_2^{r,s} = H^r(X, \mathcal{E}\mathrm{xt}^s(\mathcal{F}, \Omega^n)) \Longrightarrow \operatorname{EXT}^*(X; \mathcal{F}, \Omega^n).$$

The sheaf $\mathcal{E}\mathrm{xt}^s(\mathcal{F}, \Omega^n)$ being coherent we get $\dim_{\mathbb{C}} E_2^{r,s} < \infty$ (0 for X q-complete) if $r > q$ from the theorem mentioned above. Also by definition of depth we have

$$\mathcal{E}\mathrm{xt}^s(\mathcal{F}, \Omega^n) = 0 \quad \text{if} \quad s > n - \operatorname{prof} \mathcal{F}.$$

Therefore if $r + s = n - j > n - \operatorname{prof} \mathcal{F} + q$ i. e. $j < \operatorname{prof} \mathcal{F} - q$ we have $\dim E_2^{r,s} < \infty$ (0 for X q-complete).

By an application of the addition to Theorem (II) we then obtain

COROLLARY 3. (a) *If X is q-pseudoconvex, for any coherent sheaf $\mathcal{F}$ we have*

$$H_k^{\operatorname{prof}\mathcal{F}-q}(X, \mathcal{F}) \textit{ is separated and } \dim H_k^j(X, \mathcal{F}) < \infty \textit{ for } j < \operatorname{prof} \mathcal{F} - q.$$

(b) *If X is in particular q complete, then for any coherent sheaf $\mathcal{F}$,*

$$H_k^j(X, \mathcal{F}) = 0 \quad \textit{for} \quad j < \operatorname{prof} \mathcal{F} - q.$$

18. *Duality on q-pseudoconcave manifolds.* For a q-pseudoconcave manifold X of finite dimension n we may argue analogously. Now the basic fact is the following.

THEOREM 8. *If X is a q-pseudoconcave space and $\mathcal{F}$ is any coherent sheaf on X then*

$$\dim H_k^j(X, \mathcal{F}) < \infty \quad \textit{if} \quad j > q + 1.$$

The proof of this theorem can be obtained following the argument given in [2] section 21 using the following finiteness criterion.

LEMMA OF FINITENESS. *Let X be a complex space. Let $\mathcal{F}$ be a coherent sheaf on X.*

Suppose there exists an open relatively compact subset A of X such that the natural map

$$H_k^s(A, \mathcal{F}) \to H_k^s(X, \mathcal{F})$$

is surjective.

Then

$$\dim_{\mathbb{C}} H_k^s(X, \mathcal{F}) < \infty.$$

PROOF. Let B be open relatively compact subset in X such that $A \subset\subset B \subset\subset X$. Let $\mathcal{U} = \{U_i\}_{i \in \mathbb{Z}}$ be a countable locally finite covering of X by relatively compact Stein open sets and such that if

$$U_i \cap A \neq \emptyset \quad \text{then} \quad U_i \subset B.$$

Let

$$0 \to \mathcal{F} \to \mathcal{C}^0 \to \mathcal{C}^1 \to \dots$$

be a flabby resolution of $\mathcal{F}$ and $\xi \in H_k^s(X, \mathcal{F})$ be represented by

$$\mathfrak{z} \in \operatorname{Ker}\{\Gamma_k(X, \mathcal{C}^s) \to \Gamma_k(X, \mathcal{C}^{s+1})\}$$

with support of $\mathfrak{z} \subset A$. By transforming the representative $\mathfrak{z}$ into the Čech representative one realizes that this Čech representative $\eta = \{\eta_{i_0 \dots i_s}\}$ can be so choosen that

$$\eta_{i_0 \dots i_s} = 0 \quad \text{if} \quad U_{i_0} \cap A = \emptyset, \dots, U_{i_s} \cap A = \emptyset$$

i. e.

$$\eta_{i_0 \dots i_s} \neq 0 \quad \text{imply} \quad U_{i_0} \cap \dots \cap U_{i_s} \cap B \neq \emptyset.$$

Consider the following spaces

$$Z_N^s(\mathcal{U}, \mathcal{F}) = \{\eta \in C^s(\mathcal{U}, \mathcal{F}) \mid \delta\eta = 0, \quad \eta_{i_0 \dots i_s} = 0 \quad \text{if} \quad \inf |i_j| > N\}.$$

These are closed subspaces of $C^s(\mathcal{U}, \mathcal{F})$ and thus have the structure of a space of Fréchet-Schwartz.

We have an algebraic isomorphism

$$Z_k^s(\mathcal{U}, \mathcal{F}) = \varinjlim Z_N^s(\mathcal{U}, \mathcal{F}).$$

On the space $Z_k^s(\mathcal{U}, \mathcal{F})$ we consider the structure of inductive limit of the spaces of Fréchet-Schwartz $Z_N^s(\mathcal{U}, \mathcal{F})$.

Analogously we consider the spaces

$$C_N^{s-1}(\mathcal{U}, \mathcal{F}) = \{\eta \in C^{s-1}(\mathcal{U}, \mathcal{F}) \mid \eta_{i_0 \dots i_{s-1}} = 0 \quad \text{if} \quad \inf |i_j| > N\}$$

as spaees of Fréchet-Schwartz and on

$$C_k^{s-1}(\mathcal{U}, \mathcal{F}) = \varinjlim_N C_N^{s-1}(\mathcal{U}, \mathcal{F})$$

we consider the topology of the inductive limit.

Note that the coboundary map

$$C_k^{s-1}(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} Z_k^s(\mathcal{U}, \mathcal{F})$$

is continuous, as the composite map

$$\delta_N : C_N^{s-1}(\mathcal{U}, \mathcal{F}) \to C_k^{s-1}(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} Z_k^s(\mathcal{U}, \mathcal{F})$$

can be factored as follows

$$\begin{tikzcd}
C_N^{s-1}(\mathcal{U}, \mathcal{F}) \arrow[rr, "\delta_N"] \arrow[dr, "\delta'_N"'] & & Z_k^s(\mathcal{U}, \mathcal{F}) \arrow[dl] \\
& Z_{\lambda(N)}^s(\mathcal{U}, \mathcal{F}) &
\end{tikzcd}$$

with δ'_N continuous, $\lambda(N) \geq N$ being a sufficiently large integer. From the remark made at the beginning it follows that there exists a positive integer N_0 such that every $\xi \in H_k^s(X, \mathcal{F})$ has a representative in $Z_{N_0}^s(\mathcal{U}, \mathcal{F})$.

Let $\mathcal{U}^* = \{U_i^*\}_{i \in \mathbb{Z}}$ be a new Stein covering of X with $U_i^* \subset\subset U_i$ for all $i \in \mathbb{Z}$.

We consider the map

$$\sigma : Z^s_{N_0}(\mathcal{U}, \mathcal{F}) \oplus C^{s-1}_k(\mathcal{U}^*, \mathcal{F}) \longrightarrow Z^s_k(\mathcal{U}^*, \mathcal{F})$$

defined by

$$\sigma\{\mathfrak{z} \oplus \eta\} = r(\mathfrak{z}) \longrightarrow \delta(\eta)$$

where r is the restriction maps

$$r : Z^s_{N_0}(\mathcal{U}, \mathcal{F}) \longrightarrow Z^s_k(\mathcal{U}^*, \mathcal{F}).$$

Note that r is a compact map. Now σ is surjective and both source and target spaces for σ are Souslin spaces inductive limit of Banach spaces. Therefore σ is a topological homomorphism. Moreover, every compact disk in the target space of σ is the image by σ of a compact disk in the source space. This follows from the fact that both source and target spaces of σ are spaces $\mathcal{LF}$. (Cf. [8] p. 270 and [2] p. 119). By a standard argument of L. Schwartz if follows that

$$\delta(C^{s-1}_k(\mathcal{U}^*, \mathcal{F})) \quad \text{is closed in } Z^s_k(\mathcal{U}^*, \mathcal{F})$$

and

$$\dim_{\mathbb{C}} H^s_k(\mathcal{U}^*, \mathcal{F}) = \dim_{\mathbb{C}} H^s_k(X, \mathcal{F}) < \infty.$$

Applying the addition to Theorem (I) we get

COROLLARY 1. *If X is q-pseudoconcave then for any coherent sheaf $\mathcal{F}$*

$$H^c_{q+1}(X, \mathcal{F}_*) \quad \textit{is separated}$$

and

$$\dim H^c_j(X, \mathcal{F}_*) < \infty \quad \textit{for} \quad j > q+1.$$

COROLLARY 2. *If X is q-pseudoconcave then for any coherent sheaf $\mathcal{F}$*

$$\dim_{\mathbb{C}} H_i(X, \mathcal{F}_*) < \infty \quad \textit{for} \quad i < \operatorname{prof} \mathcal{F} - q - 1.$$

PROOF. We have $H_i(X, \mathcal{F}_*) = \mathrm{EXT}^{n-i}_k(X, \mathcal{F}, \Omega^n)$ and we have a spectral sequence

$$E_2^{r,s} = H^r_k(X, \mathcal{E}\mathrm{xt}^s(\mathcal{F}, \Omega^n)) \Longrightarrow \mathrm{EXT}^*_k(X; \mathcal{F}, \Omega^n).$$

$$\dim E_2^{r,s} < \infty \text{ if } r > q+1 \quad \text{and} \quad \mathcal{E}\mathrm{xt}^s(\mathcal{F}, \Omega^n) = 0 \text{ if } s > n - \operatorname{prof} \mathcal{F}.$$

Thus if $r + s = n - i > q + 1 + n - \operatorname{prof} \mathcal{F}$ we get finite dimensionality.

COROLLARY 3. *If X is q-pseudoconcave we get for any coherent sheaf $\mathcal{F}$ that*

$$H^{\operatorname{prof}\mathcal{F}-q-1}(X, \mathcal{F}) \text{ is separated and } \dim_{\mathbb{C}} H^i(X, \mathcal{F}) < \infty \quad \text{if } i < \operatorname{prof}\mathcal{F} - q - 1.$$

Note the proof of the separation of $H^{\operatorname{prof}\mathcal{F}-q-1}(X, \mathcal{F})$ is given here for the first time.

§ 9. Duality on complex spaces.

19. *Cohomology with compact support of domain of holomorphy.* *a*) Let X be a Stein manifold of pure dimension n. Let $\mathcal{F}$ be a coherent analytic sheaf on X.

The situation with cohomology and homology groups on X is the following

$$H^j(X, \mathcal{F}) = 0 \text{ if } j > 0, \quad H^0(X, \mathcal{F}) \text{ is a Fréchet-Schwartz space}$$

$$H_j(X, \mathcal{F}_*) = 0 \text{ if } j > 0, \quad H_0(X, \mathcal{F}_*) = \text{strong dual of } H^0(X, \mathcal{F}).$$

The situation for cohomology with compact supports and homology with closed support is more complicated. From Theorem 6 and the spectral sequence for the functor EXT we deduce an algebraic isomorphism

$$H_j^c(X, \mathcal{F}_*) = \Gamma(X, \operatorname{Ext}^{n-j}(\mathcal{F}, \Omega^n)).$$

One is lead to ask if this is a topological isomorphism or, equivalently if the following is true

$$H_k^j(X, \mathcal{F}) = \text{strong dual of } \Gamma(X, \operatorname{Ext}^{n-j}(\mathcal{F}, \Omega^n)).$$

The aim of the following considerations is to prove this fact.

b) We assume first that $X = D \subset \mathbb{C}^n$ is a domain of holomorphy. By Serre's duality we have an algebraic isomorphism $H_k^n(D, \mathcal{O}) = \operatorname{Hom}\,\text{cont}\cdot\{\Gamma(D, \Omega^n), \mathbb{C}\}$ (7).

---

(7) Serre's duality says that the isomorphism is topological when $H_k^n(D, \mathcal{O})$ is endowed with the topology coming from the Dolbeault resolution of $\mathcal{O}$ by currents. Here, however, $H_k^n(D, \mathcal{O})$ is considered with the topology coming from the corresponding Čech complex.

LEMMA 1. *Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a locally finite countable Stein covering of D, by relatively compact subsets of D. Each element $\omega \in \Gamma(D, \Omega^n)$ defines a continuous linear map*

$$\langle \omega, \cdot \rangle : H_k^n(\mathcal{U}, \mathcal{O}) \longrightarrow \mathbb{C}.$$

In particular $H_k^n(\mathcal{U}, \mathcal{O})$ is Hausdorff.

PROOF. Let $\xi \in H_k^n(\mathcal{U}, \mathcal{O})$ and let $\Phi_\xi^{0,n}$ be a $(0, n)$ C^∞ form representing by the Dolbeault isomorphism the cohomology class of ξ. The form Φ has compact support and the pairing given by Serre duality is given by

$$\langle \omega, \xi \rangle = \int_D \omega \wedge \Phi.$$

Let $\{\varrho_i\}_{i \in I}$ be a C^∞ partition of unity subordinate to the covering $\mathcal{U}$. For

$$\eta = \{\eta_{i_0 \dots i_n}\} \in C_k^n(\mathcal{U}, \mathcal{O})$$

we construct the sequence of C^∞ forms

$$\Phi^0_{i_0 \dots i_{n-1}} = \Sigma\, \varrho_k\, \eta_{k i_0 \dots i_{n-1}} \qquad \text{on} \quad U_{i_0 \dots i_{n-1}}$$

$$\Phi^1_{i_0 \dots i_{n-2}} = \Sigma\, \varrho_k\, \bar\partial \Phi^0_{k i_0 \dots i_{n-2}} \qquad \text{on} \quad U_{i_0 \dots i_{n-2}}$$

$$\cdot \qquad \cdot \qquad \cdot$$

$$\Phi^{n-1}_{i_0} = \Sigma\, \varrho_k\, \bar\partial \Phi^{n-2}_{k i_0} \qquad \text{on} \quad U_{i_0}$$

$$\Phi^n = \Sigma\, \varrho_k\, \bar\partial \Phi^{n-1}_k \qquad \text{on} \quad D.$$

Note that Φ^n has compact support. Moreover if η is a cocycle, then Φ^n is a Dolbeault epresentative of $\{\eta\}$. It follows that the linear map $\langle \omega, \cdot \rangle$ is induced by the linear map

$$\lambda_\omega : C_k^n(\mathcal{U}, \mathcal{O}) \longrightarrow \mathbb{C}$$

given by

$$\lambda_\omega(\eta) = \int_D \omega \wedge \Phi^n.$$

For each $U_{i_0 \dots i_n}$ the composed map

$$\Gamma(U_{i_0 \dots i_n} \mathcal{O}) \longrightarrow C_k^n(\mathcal{U}, \mathcal{O}) \xrightarrow{\lambda_\omega} \mathbb{C}$$

is continuous. Therefore λ_ω is continuous. The same is true for its restriction $\lambda_\omega \mid \mathcal{Z}_k^n(\mathcal{U}, \mathcal{O})$ and therefore $\langle \omega, \cdot \rangle$ is continuous.

COROLLARY. *The space $H_k^n(D, \mathcal{O})$ with its Cech topology is topologically isomorphic to the strong dual of $\Gamma(D, \Omega^n)$ (with its Fréchet-Schwartz topology).*

PROOF. By corollary 2 to lemma 2 of n. 16 $H_k^n(D, \mathcal{O})$ has the topology of a strong dual of a space of Fréchet-Schwartz. We have also a natural bijection

$$\lambda : H_k^n(D, \mathcal{O}) \longrightarrow \text{strong dual of } \Gamma(D, \Omega^n).$$

Both spaces are reflexive and λ, by lemma 1, is weakly continuous. Therefore λ is also strongly continuous. Consequently λ is a topological isomorphism.

LEMMA 2. *Let D be a bounded domain of holomorphy in $\mathbb{C}^n$ and let $\mathcal{F}$ be a coherent analytic sheaf on $\mathbb{C}^n$. Then $H_k^n(D, \mathcal{F})$ is algebraically isomorphic to the dual of the Fréchet space $\Gamma(D, \mathcal{H}om(\mathcal{F}, \Omega^n))$.*

PROOF. (α) We consider on D a presentation of $\mathcal{F}$ of the form

$$\mathcal{O}^{\varrho_1} \xrightarrow{\alpha} \mathcal{O}^{\varrho_0} \xrightarrow{\varepsilon} \mathcal{F} \longrightarrow 0.$$

From this applying the functor $\mathcal{H}om(\cdot, \Omega^n)$ and identifying $(\Omega^n)^{\varrho_i}$ with $\mathcal{H}om(\mathcal{O}^{\varrho_i}, \Omega^n)$ we get an exact sequence

$$(\Omega^n)^{\varrho_1} \xleftarrow{{}^t\alpha} (\Omega^n)^{\varrho_0} \xleftarrow{{}^t\varepsilon} \mathcal{H}om(\mathcal{F}, \Omega^n) \longleftarrow 0$$

and thus an exact sequence of Fréchet spaces,

$$(0) \qquad \Gamma(D, (\Omega^n)^{\varrho_1}) \xleftarrow{{}^t\alpha_*} \Gamma(D, (\Omega^n)^{\varrho_0}) \xleftarrow{{}^t\varepsilon_*} \Gamma(D, \mathcal{H}om(\mathcal{F}, \Omega^n) \longleftarrow 0.$$

In it the maps ${}^t\alpha_*$ and ${}^t\varepsilon_*$ are topological homomorphisms.

(β) Now we remark that for any coherent sheaf $\mathcal{F}$ on D admitting on D a finite free resolution

$$(3) \qquad 0 \longrightarrow \mathcal{O}^{\varrho_d} \longrightarrow \dots \longrightarrow \mathcal{O}^{\varrho_0} \longrightarrow \mathcal{O}^{\varrho_1} \longrightarrow \mathcal{F} \longrightarrow 0$$

we must have

$$H_k^{n+i}(D, \mathcal{F}) = 0 \quad \text{if} \quad i > 0.$$

This can be proved by induction on the length d of the resolution as for $d = 0$ the statement follows by Serre duality. Indeed, setting $\mathcal{L} = \operatorname{Ker}\{\mathcal{O}^{\varrho_1} \to \mathcal{O}^{\varrho_0}\}$ we get a short exact sequence

$$(*) \qquad 0 \to \mathcal{L} \to \mathcal{O}^{\varrho_0} \to \mathcal{F} \to 0$$

and $\mathcal{L}$ admits on D a free resolution of length $\leq d - 1$.

From the exact cohomology sequence with compact support derived from (*) we get the exact sequences

$$H_k^{n+i}(D, \mathcal{O}^{\varrho_0}) \to H_k^{n+i}(D, \mathcal{F}) \to H_k^{n+i+1}(D, \mathcal{L})$$

and if $i > 0$ we have $H_k^{n+i}(D, \mathcal{O}^{\varrho_0}) = 0 = H^{n+i+1}(D, \mathcal{L})$ by the inductive hypothesis. Thus the conclusion (cf. also [15]).

(γ) Since $\mathcal{F}$ is given on $\mathbb{C}^n$, it admits on D a finite free resolution (3). Splitting that into short exact sequences and writing the corresponding cohomology sequences with compact support we derive in particular the exact sequence (taking (β) into account)

$$H_k^n(D, \mathcal{O}^{\varrho_1}) \xrightarrow{\alpha_*} H_k^n(D, \mathcal{O}^{\varrho}) \xrightarrow{\varepsilon_*} H_k^n(D, \mathcal{F}) \to 0.$$

By the corollary to lemma 1 $H_k^n(D, \mathcal{O}^{\varrho_i})$ are duals of Fréchet-Schwartz spaces and α_* is continuous and the transpose of the topological homomorphism ${}^t\alpha_*$. Thus α_* is a topological homomorphism. Therefore $H_k^n(D, \mathcal{F})$ can be algebraically identified with the topological vector space $\operatorname{Coker} \alpha_*$.

The assertion then follows from the duality lemma.

LEMMA 3. *Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a countable covering of D by Stein open sets U_i such that each $\overline{U}_i$ is a compact Stein subset of D. Then each element $\omega \in \Gamma(D, \mathcal{H}om(\mathcal{F}, \Omega^n))$ defines a continuous linear map of $H_k^n(\mathcal{U}, \overline{\mathcal{F}})$ into $\mathbb{C}$. In particular $H_k^n(\mathcal{U}, \overline{\mathcal{F}})$ is Hausdorff.*

PROOF. With the same notations as in the proof of the previous lemma, for each $i \in I$ we have an exact sequence

$$\mathcal{O}(\overline{U}_{i_0 \dots i_n})^{\varrho_1} \xrightarrow{\overline{\alpha}} \mathcal{O}(\overline{U}_{i_0 \dots i_n})^{\varrho_0} \xrightarrow{\overline{\varepsilon}} \mathcal{F}(\overline{U}_{i_0 \dots i_n}) \to 0$$

in which each space has the structure of a strong dual of a space of Fréchet-Schwartz and where $\overline{\alpha}$ and $\overline{\varepsilon}$ are topological homomorphisms. By taking

direct sums we get an exact sequence

$$\coprod_{U_{i_0 \dots i_n} \neq \emptyset} \mathcal{O}(\overline{U}_{i_0 \dots i_n})^{\varrho_1} \xrightarrow{\overline{\alpha}} \coprod_{U_{i_0 \dots i_n} \neq \emptyset} \mathcal{O}(\overline{U}_{i_0 \dots i_n})^{\varrho_0} \xrightarrow{\overline{\varepsilon}} \coprod_{U_{i_0 \dots i_n} \neq \emptyset} \mathcal{F}(\overline{U}_{i_0 \dots i_n}) \to 0$$

in which each space is again the strong dual of a space of Fréchet-Schwartz and where $\overline{\alpha}$ and $\overline{\varepsilon}$ are topological homomorphisms.

Given $\omega \in \Gamma(D, \mathcal{H}om(\mathcal{F}, \Omega^n))$, we can consider ${}^t\varepsilon_*(\omega) \in \Gamma(D, (\Omega^n)^{\varrho_0})$. Note that ${}^t\alpha_* {}^t\varepsilon_*(\omega) = 0$. Now given a C^∞ partition of unity subordinate to the covering $\mathcal{U}$, we can define as in lemma 1 a continuous linear map

$$\lambda_{t_{\varepsilon*(\omega)}} : \coprod \mathcal{O}(\overline{U}_{i_0 \dots i_n})^{\varrho_0} \to \mathbb{C}.$$

Note that $\lambda_{t_{\varepsilon*(\omega)}}$ vanishes on $\operatorname{Im} \overline{\alpha}$ because $\lambda_{t_{\varepsilon(\omega)}}(\alpha(x)) = \lambda_{t_\alpha t_{\varepsilon(\omega)}}(x) = 0$. Therefore $\lambda_{t_{\varepsilon*(\omega)}}$ defines a continuous linear functional θ_ω on

$$\coprod_{U_{i_0 \dots i_n} \neq \emptyset} \mathcal{F}(\overline{U}_{i_0 \dots i_n}).$$

If $\eta = \{\eta_{i_0 \dots i_n}\}$ is a cocycle representing a cohomology class $\xi \in H_k^n(\mathcal{U}, \overline{\mathcal{F}})$ then

$$\theta_\omega(\eta) = \langle \omega, \xi \rangle.$$

This proves that $\langle \omega, \cdot \rangle$ is continuous on $H_k^n(D, \overline{\mathcal{F}})$.

COROLLARY. *The topological vector space $H_k^n(D, \mathcal{F})$ is topologically isomorphic to the strong dual of the Fréchet space $\Gamma(D, \mathcal{H}om(\mathcal{F}, \Omega^n))$.*

c) Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{C}^n$. Then on any bounded domain of holomorphy $D \subset \mathbb{C}^n$, $\mathcal{F}$ admits a finite free resolution

$$(3) \qquad 0 \to \mathcal{O}^{\varrho_d} \xrightarrow{\alpha_d} \dots \xrightarrow{\alpha_2} \mathcal{O}^{\varrho_1} \xrightarrow{\alpha_1} \mathcal{O}^{\varrho_0} \xrightarrow{\varepsilon} \mathcal{F} \to 0.$$

Applying the functor $\mathcal{H}om(\cdot, \Omega^n)$ and identifying $\mathcal{H}om(\mathcal{O}^{\varrho_i}, \Omega^n)$ with $(\Omega^n)^{\varrho_i}$ we get a complex ef coherent sheaves and homomorphisms

$$(4) \qquad 0 \to (\Omega^n)^{\varrho_0} \xrightarrow{{}^t\alpha_1} (\Omega^n)^{\varrho_1} \xrightarrow{{}^t\alpha_2} \dots \xrightarrow{{}^t\alpha_d} (\Omega^n)^{\varrho_d} \to 0.$$

By definition the p-th cohomology sheaf of this complex is the sheaf $\mathcal{E}xt_{\mathcal{O}}^p(\mathcal{F}, \Omega^n)$.

Taking global sections we get a complex

$$(5)\qquad 0 \to \Gamma(D, (\Omega^n)^{\varrho_0}) \xrightarrow{{}^t\alpha_1^*} \Gamma(D, \Omega^n)^{\varrho_1}) \xrightarrow{{}^t\alpha_2^*} \dots \xrightarrow{{}^t\alpha_d^*} \Gamma(D, (\Omega^n)^{\varrho_d}) \to 0$$

of Fréchet spaces in which each map is a topological homomorphism. By definition the p-th cohomology group of the complex (5) is the group $\mathrm{EXT}^p(D; \mathcal{F}, \Omega^n)$. As it is constructed it has a natural structure of Fréchet space. Also since $\mathcal{E}\mathrm{xt}^p(\mathcal{F}, \Omega^n)$ is coherent the space $\Gamma(D, \mathcal{E}\mathrm{xt}^p(\mathcal{F}, \Omega^n))$ has a natural structure of a Fréchet space. Moreover, since D is Stein we have a canonical algebraic isomorphism

$$(6)\qquad \mathrm{EXT}^p(D; \mathcal{F}, \Omega^n) \simeq \Gamma(D, \mathcal{E}\mathrm{xt}^p(\mathcal{F}, \Omega^n)).$$

LEMMA 4. *The isomorphism* (6) *is a topological isomorphism. In particular the Fréchet structure on* $\mathrm{EXT}^p(D; \mathcal{F}, \Omega^n)$ *is independent of the choice of the resolution* (3).

PROOF. By definition we have the exact sequence of sheaves

$$\mathrm{Im}\, {}^t\alpha_p \to \mathrm{Ker}\, {}^t\alpha_{p+1} \to \mathcal{E}\mathrm{xt}^p(\mathcal{F}, \Omega^n) \to 0.$$

Since D is Stein we get the exact sequence

$$\Gamma(D, \mathrm{Im}\, {}^t\alpha_p) \to \Gamma(D, \mathrm{Ker}\, {}^t\alpha_{p+1}) \to \Gamma(D, \mathcal{E}\mathrm{xt}^p(\mathcal{F}, \Omega^n)) \to 0.$$

The spaces are Fréchet spaces and the maps are topological homomorphisms. From the sequence

$$0 \to \mathrm{Ker}\, {}^t\alpha_{p+1} \to (\Omega^n)^{\varrho_p} \xrightarrow{{}^t\alpha_{p+1}} (\Omega^n)^{\varrho_{p+1}}$$

we get the exact sequence of Fréchet spaces and continuous maps

$$0 \to \Gamma(D, \mathrm{Ker}\, {}^t\alpha_{p+1}) \to \Gamma(D, (\Omega^n)^{\varrho_p}) \xrightarrow{{}^t\alpha_{p+1}^*} \Gamma(D, (\Omega^n)^{\varrho_{p+1}}).$$

Thus a topological isomorphism

$$\Gamma(D, \mathrm{Ker}\, {}^t\alpha_{p+1}) \simeq \mathrm{Ker}\, \{\Gamma(D, (\Omega^n)^{\varrho_p}) \xrightarrow{{}^t\alpha_{p+1}^*} \Gamma(D, (\Omega^n)^{\varrho_{p+1}})\}.$$

Analogously from the exact sequences

$$0 \to \mathrm{Im}\, {}^t\alpha_p \to (\Omega^n)^{\varrho_p}; \quad (\Omega^n)^{\varrho_{p-1}} \to \mathrm{Im}\, {}^t\alpha_p \to 0$$

we get topological homomorphisms of Fréchet spaces in the exact sequences

$$0 \longrightarrow \Gamma(D, \operatorname{Im} {}^t\alpha_p) \longrightarrow \Gamma(D, (\Omega^n)^{\varrho_p})$$

$$\Gamma(D, (\Omega^n)^{\varrho_{p-1}}) \longrightarrow \Gamma(D, \operatorname{Im} {}^t\alpha_p) \longrightarrow 0.$$

This shows that the image of $\Gamma(D, \operatorname{Im} {}^t\alpha_p)$ into $\Gamma(D, (\Omega^n)^{\varrho_p})$ is the closed subspace $\operatorname{Im}\{\Gamma(D, (\Omega^n)^{\varrho_{p-1}}) \xrightarrow{{}^t\alpha_p^*} \Gamma(D, (\Omega^n)^{\varrho_p}\}$. From this the contention of the lemma follows.

We consider now the complex obtained from the resolution (3) applying the functor $H_k^n(D, \cdot)$,

$$(6) \qquad 0 \longrightarrow H_k^n(D, \mathcal{O}^{\varrho_p}) \xrightarrow{\alpha_p^*} \dots \xrightarrow{\alpha_2^*} H_k^n(D, \mathcal{O}^{\varrho_1}) \xrightarrow{\alpha_1^*} H_k^n(D, \mathcal{O}^{\varrho_0}) \longrightarrow 0.$$

Each space has a structure of a dual of a space of Fréchet-Schwartz and each map α_i^* is continuous. But the complex (6) is the dual of the complex (5) therefore each map α_i^* is a topological homomorphism. Considering this as a complex of chains we get that the homology groups

$$H_q(\{H_k^n(D, \mathcal{O}^{\varrho_\cdot}), \alpha_\cdot^*\})$$

are all Hausdorff and have a natural structure of a strong dual of a space of Fréchet-Schwartz.

By splitting the sequence (3) into short exact sequences and writing the corresponding cohomology sequences with compact supports we get an algebraic isomorphism

$$(7) \qquad H_k^p(D, \mathcal{F}) \simeq H_{n-p}(\{H_k^n(D, \mathcal{O}^{\varrho_\cdot}), \alpha_\cdot^*\}).$$

LEMMA 5. *The isomorphism* (7) *is a topological isomorphism.*

PROOF. For $p = n$ the statement follows from the exact sequence

$$H_k^n(D, \mathcal{O}^{\varrho_1}) \longrightarrow H_k^n(D, \mathcal{O}^{\varrho_0}) \longrightarrow H_k^n(D, \mathcal{F}) \longrightarrow 0$$

in which each space is a dual of a space of Fréchet-Schwartz and where all maps are continuous. Note that here we make use of lemma 3.

Suppose $p < n$. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a countable covering of D by Stein open sets such that for each $i \in I$ $\overline{U}_i$ is a compact Stein set in D.

Let $\overline{U} = \overline{U}_{i_0 \dots i_p}$ and split the resolution (3) in short exact sequences

$$0 \to \mathcal{Z}_0 \to \mathcal{O}^{e_0} \to \mathcal{F} \to 0$$

$$0 \to \mathcal{Z}_1 \to \mathcal{O}^{e_1} \to z_0 \to 0$$

$$\cdot \qquad \cdot \qquad \cdot$$

where $\mathcal{Z}_i = \operatorname{Ker} \alpha_i$. From these we deduce short exact sequences

$$0 \to \mathcal{Z}_0(\overline{U}) \to \mathcal{O}^{e_0}(\overline{U}) \to \mathcal{F}(\overline{U}) \to 0$$

$$0 \to \mathcal{Z}_1(\overline{U}) \to \mathcal{O}^{e_1}(\overline{U}) \to \mathcal{Z}_0(\overline{U}) \to 0$$

$$\cdot \qquad \cdot \qquad \cdot$$

In these each space is a strong dual of a space of Fréchet-Schwartz and all maps, being continuous, are topological homomorphisms. Taking direct sume we get exact sequences of complexes

$$0 \to \coprod_{U_{i_0 \dots i_p} \neq \emptyset} \mathcal{Z}_0(\overline{U}_{i_0 \dots i_p}) \to \coprod_{U_{i_0 \dots i_p} \neq \emptyset} \mathcal{O}^{e_0}(\overline{U}_{i_0 \dots i_p}) \to$$
$$\to \coprod_{U_{i_0 \dots i_p} \neq \emptyset} \mathcal{F}(\overline{U}_{i_0 \dots i_p}) \to 0$$

$$0 \to \coprod_{U_{i_0 \dots i_p} \neq \emptyset} \mathcal{Z}_1(\overline{U}_{i_0 \dots i_p}) \to \coprod_{U_{i_0 \dots i_p} \neq \emptyset} \mathcal{O}^{e_1}(\overline{U}_{i_0 \dots i_p}) \to$$
$$\to \coprod_{U_{i_0 \dots i_p} \neq \emptyset} \mathcal{Z}_0(\overline{U}_{i_0 \dots i_p}) \to 0.$$

In these complexes the spaces are all duals of spaces of Fréchet-Schwartz and the maps are continuous, therefore topological homomorphisms. We can thus apply the lemma of n. 15 (*a*) and we deduce a sequence of continuous bijections

$$H_k^p(\mathcal{U}, \overline{\mathcal{F}}) \xrightarrow{\sim} H_k^p(\mathcal{U}, \overline{\mathcal{Z}}_0) \xrightarrow{\sim} \dots \xrightarrow{\sim} H_k^{n-1}(\mathcal{U}, \overline{\mathcal{Z}}_{n-p-2}).$$

Moreover we have the exact sequences

$$0 \to H_k^{n-1}(\mathcal{U}, \overline{\mathcal{Z}}_{n-p-2}) \to H_k^n(\mathcal{U}, \overline{\mathcal{Z}}_{n-p-1}) \xrightarrow{\alpha} H_k^n(\mathcal{U}, \overline{\mathcal{O}}^{e_{n-p-1}})$$

$$H_k^n(\mathcal{U}, \overline{\mathcal{Z}}_{n-p}) \xrightarrow{\beta} H_k^n(\mathcal{U}, \overline{\mathcal{O}}^{e_{n-p}}) \xrightarrow{\gamma} H_k^n(\mathcal{U}, \overline{\mathcal{Z}}_{n-p-1}) \to 0$$

$$H_k^n(\mathcal{U}, \overline{\mathcal{O}}^{e_{n-p+1}}) \xrightarrow{\sigma} H_k^n(\mathcal{U}, \overline{\mathcal{Z}}_{n-p}) \to 0.$$

Note that the maps α, β, γ, σ must be topological homomorphisms. Therefore

$$H_k^p(\mathcal{U}, \overline{\mathcal{F}}) \xrightarrow{\sim} \operatorname{Ker} \alpha \simeq \frac{\operatorname{Ker}\{H_k^n(\mathcal{U}, \overline{O}^{\varrho_{n-p}}) \to H_k^n(\mathcal{U}, \overline{O}^{\varrho_{n-p-1}})\}}{\operatorname{Im}\{H_k^n(\mathcal{U}, \overline{O}^{\varrho_{n-p-1}}) \to H_k^n(\mathcal{U}, \overline{O}^{\varrho_{n-p}})\}}.$$

Hence we have a continuous bijection

$$H_k^p(\mathcal{U}, \overline{\mathcal{F}}) \xrightarrow{\sim} H_{n-p}(\{H_k^n(\mathcal{U}, \overline{O}^{\varrho}\cdot), \alpha_\cdot^*\}).$$

It follows that $H_k^p(\mathcal{U}, \overline{\mathcal{F}})$ must be Hausdorff and consequently that bijection must be a topological isomorphism.

For $p > n$ we know that $H_k^p(D, \mathcal{F}) = 0$.

PROPOSITION 15. *For every integer $p \geq 0$ $H_k^p(D, \mathcal{F})$ is topologically isomorphic to the strong dual of the space Fréchet-Schwartz $\Gamma(D, \mathcal{E}\mathrm{xt}^{n-p}(\mathcal{F}, \Omega^n))$.*

PROOF. Apply the duality lemma to the dual sequences (5) and (6) and use lemma 4 and lemma 5.

20. *Cohomology with compact support of a Stein space.* Let $\mathcal{F}$ be a coherent analytic sheaf on $\mathbb{C}^n$. We consider the countable covering of $\mathbb{C}^n$ given by open balls of rational radius centered at points with coordinates having rational real and imaginary parts.

Let $\mathcal{U} = \{U_i\}_{i \in I}$ be this covering. For each $i \in I$ $\overline{U}_i$ is a Stein compact set.

Let $\{D_m\}_{m \in \mathbb{N}^*}$ be the sequence of open balls centered at the origin, with radius $m = 1, 2, 3, \ldots$. Let

$$\mathcal{U}_m = \{U_i \in \mathcal{U} \mid \overline{U}_i \subset D_m\}.$$

Then each $\mathcal{U}_m$ is a covering of D_m.

Consider the cohomology groups

$$H_k^p(D_m, \mathcal{F}) = H_k^p(\mathcal{U}_m, \overline{\mathcal{F}}).$$

We have a sequence of natural maps

$$H_k^p(D_m, \mathcal{F}) \xrightarrow{{}^t r_m^{m+1}} H_k^p(D_{m+1}, \mathcal{F})$$

and

$$H_k^p(D_m, \mathcal{F}) \xrightarrow{{}^t r_m^{\infty}} H_k^p(\mathbb{C}^n, \mathcal{F}).$$

These maps are continuous as they are induced at the cochain level by the continuous maps

$$C_k^p(\mathcal{U}_m, \overline{\mathcal{F}}) \longrightarrow C_k^p(\mathcal{U}_{m+1}, \overline{\mathcal{F}})$$

$$C_k^p(\mathcal{U}_m, \overline{\mathcal{F}}) \longrightarrow C_k^p(\mathcal{U}, \overline{\mathcal{F}}).$$

Note that ${}^t r_m^{m+1}$ is the transpose of the restriction map

$$r_m^{m+1} : \Gamma(D_{m+1}, \mathcal{E}xt^{n-p}(\mathcal{F}, \Omega^n)) \longrightarrow \Gamma(D_m, \mathcal{E}xt^{n-p}(\mathcal{F}, \Omega^n)).$$

Since D_m is Runge in D_{m+1} the maps r_m^{m+1} have a dense image. Therefore the maps ${}^t r_m^{m+1}$ are injective. We can consider the inductive limit of the family of strong duals of spaces of Fréchet-Schwartz and continuous maps $\{H_k^p(D_m, \mathcal{F}), {}^t r_m^{m+1}\}$. Let $\varinjlim H_k^p(D_m, \mathcal{F})$ be this limit space.

Note that each element $\omega \in \Gamma(\mathbb{C}^n, \mathcal{E}xt^{n-p}(\mathcal{F}, \Omega^n))$ defines a continuous linear map

$$\lambda_\omega : \lim H_k^p(D_m, \mathcal{F}) \longrightarrow \mathbb{C}$$

by setting

$$\lambda_\omega \mid H_k^p(D_m, \mathcal{F}) = \langle r_m^\infty \omega, \cdot \rangle .$$

Since the image of r_m^∞ is dense in $\Gamma(D_m, \mathcal{E}xt^{n-p}(\mathcal{F}, \Omega^n))$ the maps λ_ω for $\omega \in \Gamma(\mathbb{C}^n, \mathcal{E}xt^{n-p}(\mathcal{F}, \Omega^n))$ separate points. Therefore the space $\varinjlim H_k^p(D_m, \mathcal{F})$ is Hausdorff and therefore has the structure of a strong dual of a space of Fréchet-Schwartz (this space being a quotient of the direct sum $\coprod H_k^p(D_m, \mathcal{F})$).

Also we have a natural continuous map

$$\varinjlim H_k^p(D_m, \mathcal{F}) \to H_k^p(\mathbb{C}^n, \mathcal{F}) \tag{1}$$

which is bijective.

LEMMA. *The linear functions λ_ω on $H_k^p(\mathbb{C}^n, \mathcal{F})$ are also continuous.*

PROOF. Let $\theta : Z_k^p(\mathcal{U}, \overline{\mathcal{F}}) \longrightarrow H_k^p(\mathcal{U}, \overline{\mathcal{F}})$ be the natural map. We have to show that $\widetilde{\lambda}_\omega = \lambda_\omega \circ \theta$ is continuous. Since $Z_k^p(\mathcal{U}, \overline{\mathcal{F}})$ is a dual of a space of Fréchet-Schwartz, it is enough to show that the restriction of $\widetilde{\lambda}_\omega$ to bounded sets is continuous (cf. [8] p. 318). Now each bounded set of

$Z_k^p(\mathcal{U}, \overline{\mathcal{F}})$ is contained in a set of the form

$$Z_k^p(\mathcal{U}, \overline{\mathcal{F}}) \cap \coprod_J \mathcal{F}(\overline{U}_{i_0 \dots i_p})$$

where the sum is over a finite set J of p tuples $i_0 \dots i_p$. Therefore there exists an integer m_0 such that

$$Z_k^p(\mathcal{U}, \overline{\mathcal{F}}) \cap \coprod_J \mathcal{F}(\overline{U}_{i_0 \dots i_p}) \subset Z_k^p(\mathcal{U}_m, \overline{\mathcal{F}}).$$

The inclusion is a topological homomorphism since it is continuous and the image is the set

$$\mathcal{L} = \{\{\mathfrak{z}_{i_0 \dots i_p}\} \in Z_k^p(\overline{\mathcal{U}}_m, \mathcal{F}) \mid \mathfrak{z}_{i_0 \dots i_p} = 0 \quad \text{if} \quad (i_0, \dots, i_p) \notin J\}$$

which is closed.

It is enough to show therefore that $\widetilde{\lambda}_\omega \mid Z_k^p(\overline{\mathcal{U}}_m, \mathcal{F})$ is continuous. But this follows from proposition 15.

COROLLARY. *The space $H_k^p(\mathbb{C}^n, \mathcal{F})$ is Hausdorff, therefore the continuous bjiection* (1) *is a topological isomorphism.*

PROPOSITION 16. *The space $H_k^p(\mathbb{C}^n, \mathcal{F})$ is isomorphic to the strong dual of the space of Fréchet-Schwartz $\Gamma(\mathbb{C}^n, \mathcal{E}\mathrm{xt}^{n-p}(\mathcal{F}, \Omega^n))$.*

PROOF. Denoting by a dash the strong dual of a space, we get

$$\begin{aligned} H_k^p(\mathbb{C}^n, \mathcal{F}) &\simeq \varinjlim H_k^p(D_m, \mathcal{F}) \\ &\simeq \varinjlim \Gamma(D_m, \mathcal{E}\mathrm{xt}^{n-p}(\mathcal{F}, \Omega^n))' \\ &\simeq \{\varprojlim \Gamma(D_m, \mathcal{E}\mathrm{xt}^{n-p}(\mathcal{F}, \Omega^n)\}' \\ &\simeq \Gamma(\mathbb{C}^n, \mathcal{E}\mathrm{xt}^{n-p}(\mathcal{F}, \Omega^n))'. \end{aligned}$$

This result completes an algebraic statement of C. Banica and O. Stanasila [3].

PROPOSITION 17. *Let X be a finite, dimensional Stein space and let $\mathcal{F}$ be any coherent analytic sheaf on X. Then, for each $p \geq 0$ the spaces $H_k^p(X, \mathcal{F})$ are Hausdorff strong duals of spaces of Fréchet Schwartz.*

PROOF. We have a holomorphic homeomorphism $\pi: X \to \mathbb{C}^n$ which is proper. For every $U_{i_0} \cap \ldots \cap U_{i_p} = U$, $U_i \in \mathcal{U}$ we have topological isomorphisms

$$\Gamma(\pi^{-1}(U), \mathcal{F}) \simeq \Gamma(U, \pi_* \mathcal{F}).$$

Thus, the corresponding Čech complexes on X with respect to $\pi^{-1}\mathcal{U}$ with values in $\mathcal{F}$ and on $\mathbb{C}^n$ with respect to $\mathcal{U}$ with values in $\pi_* \mathcal{F}$ are topologically isomorphic.

Since $\pi_* \mathcal{F}$ is coherent the assertion follows from proposition 16.

21. *Dualizing sheaves and cosheaves.* *a*) Let X be a complex analytic space and let $\mathcal{F}$ be a coherent analytic sheaf on X. For each integer $q \geq 0$ we consider the presheaf

$$U \to \mathcal{D}^q \mathcal{F}(U) = \text{strong dual of } H_k^q(U, \mathcal{F})$$

for all $U \subset \mathcal{W}$, where $\mathcal{W}$ is the collection of all relatively compact Stein open sets of X.

By proposition 15 each space $\mathcal{D}^q\mathcal{F}(U)$ has a natural structure of a space of Fréchet-Schwartz. The restrictions mappings, for $V \subset U$,

$$r_V^U: \mathcal{D}^q \mathcal{F}(U) \to \mathcal{D}^q \mathcal{F}(V)$$

are the transposed of the extension maps

$$i_U^V: H_k^q(V, \mathcal{F}) \to H_k^q(U, \mathcal{F}).$$

We denote by $\mathcal{D}^q \mathcal{F}$ the sheaf associated to the presheaf $\{\mathcal{D}^q \mathcal{F}(U), r_V^U\}$.

PROPOSITION 18. (a) *The sheaf $\mathcal{D}^q \mathcal{F}$ is a coherent analytic sheaf on X for every $q = 0, 1, \ldots$.*

(b) *If $U \subset X$ is a Stein open set, finite dimensional, then $\Gamma(U, \mathcal{D}^q\mathcal{F}) =$ strong dual of $H_k^q(U, \mathcal{F})$.*

PROOF. Let $\pi: U \to \mathbb{C}^n$ be a proper holomorphic bijection of U into some numerical space $\mathbb{C}^n$. Let $V \subset U$. Then

$$\begin{aligned}
\mathcal{D}^q \mathcal{F}(V) &= \text{strong dual of } H_k^q(V, \mathcal{F}) \\
&= \text{strong dual of } H_k^q(\pi V, \pi_* \mathcal{F}) \text{ (cf. proposition 17)} \\
&= \Gamma(\pi V, \mathcal{E}xt_{\mathcal{O}_{\mathbb{C}^n}}^{n-q}(\pi_* \mathcal{F}, \Omega^n)) \\
&= \Gamma(V, \mathcal{E}xt_{\mathcal{O}_V}^{n-q}(\mathcal{F}, \pi^* \Omega^n)).
\end{aligned}$$

It follows that

$$\mathcal{D}^q \mathcal{F} = \mathcal{E}\mathrm{xt}_{\mathcal{O}}^{n-q}(\mathcal{F}, \pi^* \Omega^n)$$

and therefore it is a coherent sheaf.

Moreover

$$\Gamma(U, \mathcal{D}^q \mathcal{F}) = \Gamma(U, \mathcal{E}\mathrm{xt}^{n-q}(\mathcal{F}, \pi^* \Omega^n))$$

$$= \text{strong dual of } H_k^q(U, \mathcal{F}).$$

REMARKS. (*a*) We have

$$\mathcal{D}^q\mathcal{F} = 0 \quad \text{if} \quad q < \operatorname{prof} \mathcal{F} \quad \text{and} \quad q > \dim_{\mathbb{C}} (\text{support of } \mathcal{F}).$$

The first fact follows from (1), the second from a result of Reiffen [15].

(*b*) If X is non-singular of pure complex dimension n then

$$\mathcal{D}^q \mathcal{F} = \mathcal{E}\mathrm{xt}_{\mathcal{O}}^{n-q}(\mathcal{F}, \Omega^n).$$

(*c*) If X is non-singular of pure dimension n and if $\mathcal{F}$ is locally free then

$$\mathcal{D}^q\mathcal{F} = 0 \quad \text{if} \quad q \neq n$$

and

$$\mathcal{D}^n \mathcal{F} = \mathcal{H}\mathrm{om}_{\mathcal{O}}(\mathcal{F}, \Omega^n).$$

b) Dually we can consider on X the precosheaf $\mathcal{H}_k^q(\mathcal{F})$ given by

$$U \longrightarrow H_k^q(U, \mathcal{F})$$

for $U \in \mathcal{W}$, with the natural inclusion maps if $U \subset V$

$$\iota_V^U : H_k^q(U, \mathcal{F}) \longrightarrow H_k^q(V, \mathcal{F}).$$

As a consequence of the previous proposition, since spaces of Fréchet-Schwartz are reflexive, we obtain the following:

PROPOSITION 19. *The precosheaf $\mathcal{H}_k^q(\mathcal{F})$ is the dual cosheaf to the coherent sheaf $\mathcal{D}^q\mathcal{F}$. In particular*

(a) *$\mathcal{H}_k^q(\mathcal{F})$ is a $\mathcal{W}$ cosheaf*

(b) *for every $U \in \mathcal{W}$ we have*

$$\mathcal{H}_k^q(\mathcal{F})(U) = H_k^q(U, \mathcal{F}).$$

REMARKS. (*a*) We have

$$\mathcal{H}_k^q(\mathcal{F}) = 0 \quad \text{if} \quad q < \operatorname{prof} \mathcal{F} \quad \text{or} \quad q > \dim_{\mathbb{C}} (\text{support of } \mathcal{F}).$$

(*b*) If X is non singular of complex dimension n then

$$\mathcal{H}_k^q(\mathcal{F}) = \mathcal{E}\mathrm{xt}^{n-q}(\mathcal{F}, \Omega^n)_*.$$

(*c*) If moreover $\mathcal{F}$ is locally free, then

$$\mathcal{H}_k^q(\mathcal{F}) = 0 \quad \text{if} \quad q \neq n$$

$$\mathcal{H}_k^n(\mathcal{F}) = \mathcal{H}\mathrm{om}(\mathcal{F}, \Omega^n)_*.$$

21. *The spectral sequence of homology and cohomology.* (*a*) Let X be a complex space, let $\mathcal{F}$ be an analytic coherent sheaf on X, and let Φ, Ψ be dual families of supports as in n. 5 (*b*).

THEOREM 9. *There exists a spectral sequence $E_r^{p,q}$ converging to $H_\Phi^{q-p}(X, \mathcal{F})$ with E_2 term:*

$$E_2^{-p,q} = H_p^\Phi(X, \mathcal{H}_k^q(\mathcal{F})).$$

PROOF. Let

$$0 \to \mathcal{F} \to \mathcal{J}_0 \to \mathcal{J}_1 \to \dots$$

be an injective (or flabby) resolution of $\mathcal{F}$. Let $D(\mathcal{J}_p)$ be the associated cosheaf (n. 6 (*c*)) where

$$D(\mathcal{J}_p)(U) = \Gamma_k(U, \mathcal{J}_p).$$

Consider the double complex $K = \coprod K^{p,q}$, where

$$K^{-p,q} = C_p^\Phi(\mathcal{U}, D(\mathcal{J}_q))$$

where $\mathcal{U}$ is a countable locally finite covering by relatively compact open Stein sets, adapted to Φ.

Taking first the homology with respect to the differential of the Čech complex we get (proposition 5)

$$'E_1^{-p,q} = H_p^\Phi(\mathcal{U}, D(\mathcal{J}_q)) = \begin{cases} 0 & \text{if} \quad p > 0 \\ \Gamma_\Phi(X, \mathcal{J}_q) & \text{if} \quad p = q. \end{cases}$$

Therefore

$$'E_2^{-p,q} = \begin{cases} 0 & \text{if} \quad p > 0 \\ H_\Phi^q(X, \mathcal{F}) & \text{if} \quad p = 0. \end{cases}$$

Now we consider the second spectral sequence.

We first take the cohomology of the complex

$$\dots \to C_p^\Phi(\mathcal{U}, D(\mathcal{J}_{q-1})) \to C_p^\Phi(\mathcal{U}, D(\mathcal{J}_q)) \to C_p^\Phi(\mathcal{U}, D(\mathcal{J}_{q+1})) \to \dots$$

i. e.,

$$\dots \to \varinjlim C_p^c(\mathcal{U}_s, D(\mathcal{J}_{q-1})) \to \varinjlim C_p^c(\mathcal{U}_s, D(\mathcal{J}_q)) \to \varinjlim C_p^c(\mathcal{U}_s, D(\mathcal{J}_{q+1})) \to \dots$$

i. e.,

$$\dots \to \varinjlim \prod_s \Gamma_k(U_{i_0 \dots i_p}, \mathcal{J}_{q-1}) \to$$

$$\to \varinjlim \prod_s \Gamma_k(U_{i_0 \dots i_p}, \mathcal{J}_q) \varinjlim \prod_s \Gamma_k(U_{i_0 \dots i_p}, \mathcal{J}_{q+1}) \to \dots$$

Therefore, this cohomology is given by

$$''E_1^{-p,q} = \varinjlim \prod_s H_k^q(U_{i_0 \dots i_p}, \mathcal{F}) = C_p^\Phi(\mathcal{U}, \mathcal{H}_k^q(\mathcal{F})).$$

Taking then homology with respect to the differential of the Čech complex we get

$$''E_2^{-p,q} = H_p(\mathcal{U}, \mathcal{H}_k^q(\mathcal{F}))$$

$$= H_p(X, \mathcal{H}_k^q(\mathcal{F}))$$

by the Leray theorem.

EXERCISE. Deduce the same result from proposition 11 taking $\mathcal{J} = \mathcal{O}$, $\mathcal{G} = \mathcal{F}$.

REMARKS. (*a*) If $H_k^{p-1}(X, D^q(\mathcal{F}))$ is separated or if $H_{p-1}^c(X, \mathcal{H}_k^q(\mathcal{F}))$ is separated then we have

$$H_p^c(X, \mathcal{H}_k^q(\mathcal{F})) = (H_k^p(X, \mathcal{D}^q(\mathcal{F})))'$$

(*b*) If $H_{p-1}(X, \mathcal{H}_k^q(\mathcal{J}))$ is separated or if $H^{p+1}(X, \mathcal{D}^q(\mathcal{F}))$ is separated, then we have

$$H_p(X, \mathcal{H}_k^q(\mathcal{F})) = (H^p(X, \mathcal{D}^q(\mathcal{F})))'$$

COROLLARY 1. *If X is a compact complex space then there exists a spectral sequence*

$$E_2^{-p,q} = H^p(X, \mathcal{D}^q(\mathcal{F})) \Longrightarrow H_{q-p}(X, \mathcal{F}_*)$$

or dually

$$E_2^{-p,q} = (H^p(X, \mathcal{D}^q(\mathcal{F}))' \Longrightarrow H^{q-p}(X, \mathcal{F}).$$

COROLLVRY 2. (*a*) *For a Stein space (finite dimensional) we have*

$$H_p^c(X, \mathcal{F}_*) = H^0(X, \mathcal{D}^p(\mathcal{F}))$$

or dually

$$H_k^p(X, \mathcal{F}) = (H^0(X, \mathcal{D}^p(\mathcal{F}))'.$$

(*b*) *Also the spectral sequence*

$$E_2^{-p,q} = H_k^p(X, \mathcal{D}^q(\mathcal{F}))$$

converges to 0 *if* $q - p \neq 0$ *and to* $\mathcal{F}_*(X)$ *if* $q - p = 0$.

COROLLARY 3. *Let X be a complex space imbedded as a closed analytic subspace of some complex manifold. Let $\mathcal{F}$ be a coherent analytic sheaf on X, then there exists a spectral sequence*

$$E_2^{-p,q} = H_\psi^p(X, \mathcal{D}^q(\mathcal{F})) \Longrightarrow H_{-p+q}^\psi(X, \mathcal{F}_*).$$

PROOF. We may assume X to be an analytic subset of a connected complex manifold M of $\dim_{\mathbb{C}} M = n$.

We consider on M a family of supports $\overline{\Psi}$ such that

$$\overline{\Psi} \cap X = \Psi$$

We have

$$H_j^\psi(X, \mathcal{F}_*) = \mathrm{EXT}_{\overline{\Psi}}^{n-j}(M, \widehat{\mathcal{F}}, \Omega^n)$$

where $\widehat{\mathcal{F}}$ is the trivial extension of $\mathcal{F}$ to M.

Now we have a spectral sequence

$$E_2^{r,s} = H_{\overline{\Psi}}^r(M, \mathcal{E}\mathrm{xt}^s(\widehat{\mathcal{F}}, \Omega^n)) \Longrightarrow \mathrm{Ext}^{n-j}(M, \widehat{\mathcal{F}}, \Omega^n)$$

Locally, for a small Stein set $U \subset M$, we have

$$\begin{aligned} \Gamma(U, \mathcal{E}\mathrm{xt}^s(\widehat{\mathcal{F}}, \Omega^n)) &= \mathrm{EXT}^s(U, \widehat{\mathcal{F}}, \Omega^n) \\ &\simeq (H_k^{n-s}(U, \mathcal{F}))' \qquad \text{(proposition 15)} \\ &= \Gamma(U, \mathcal{D}^{n-s}(\mathcal{F})). \end{aligned}$$

Thus a spectral sequence

$$E_2^{r,s} = H_\psi^r(X, \mathcal{D}^{n-s}(\mathcal{F})) \Longrightarrow \mathrm{EXT}_\psi^{n-j}(M, \widehat{\mathcal{F}}, \Omega^n),$$

i. e.,

$$E_2^{-r,s} = H_\psi^r(X, \mathcal{D}^s(\mathcal{F})) \Longrightarrow H_{s-r}^\psi(X, \mathcal{F}_*).$$

22. *The connections with the dualizing complex.* (*a*) Let $(X, \mathcal{O})$ be a complex space, let $\mathcal{K}^\cdot$ be a graded complex of $\mathcal{O}$-modules and let $0 \to \mathcal{K}^\cdot \to \mathcal{J}^\cdot$ be an injective resolution of $\mathcal{K}^\cdot$. Let F be an additive functor defined in the category of $\mathcal{O}$-modules. One defines for every integer p-th right derived functor or F

$$R^p F(X^\cdot) = H^p(F(\mathcal{J}^\cdot))$$

where $F(\mathcal{J}^\cdot)$ is the simple complex associated to the double complex obtained from $\mathcal{J}^\cdot$ by application of the functor F. The definition is meaningful as it is independent of the choice of the injective resolution.

In [17] the following theorems are proved.

THEOREM. *For any complex space $(X, \mathcal{O})$ there exists a complex $\mathcal{K}_X^\cdot$ of $\mathcal{O}$-modules having the following properties*

(i) *If X is reduced and finite dimensional then the complex $\mathcal{K}_X^\cdot$ is bounded.*

(ii) *If $f: X \to Y$ is a closed or open imbedding of complex spaces there is defined an isomorphism of complexes*

$$\widetilde{f}: \mathcal{K}_X^\cdot \simeq \mathcal{H}\mathrm{om}_{\mathcal{O}_Y}(f_*(\mathcal{O}_X), \mathcal{K}_Y^\cdot)|_X$$

with natural compotibility conditions for compositions of imbeddings.

(iii) *If X is a manifold of dimension n aud $\Omega[n]$ is the complex with all components zero except that in degree $-n$ which equals Ω^n, then $\mathcal{K}_X^\cdot$ is a resolution of $\Omega[n]$ and the stalks $\mathcal{K}_{X,x}^p$ are injective $\mathcal{O}_{X,x}$-modules for every $x \in X$ and every integer p.*

(iv) *Let V be an analytic subset of a Stein manifold Y of dimension n and let $\mathcal{F}$ be a coherent sheaf on X such that its trivial extension $\widehat{\mathcal{F}}$ admits a finite free resolution. Then*

$$\mathbb{E}\mathrm{XT}_k^p(X; \mathcal{F}, \mathcal{K}_X^\cdot) = \begin{cases} 0 \text{ if } p \neq 0 \\ (H^0(X, \mathcal{F}))' \text{ if } p = 0 \end{cases}$$

where $\mathbb{E}\mathrm{XT}_k$ denotes the right derived functors of the functor $\mathrm{HOM}_k(X, \mathcal{F}, \cdot)$.

(v) *We have, under the assumptions of* (iv),

$$\mathbb{E}xt^{p}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K}^{\cdot}_X) \simeq \mathcal{E}xt^{n+p}_{\mathcal{O}_Y}(\widehat{\mathcal{F}}, \Omega^n)|_X, \qquad \text{for every integer } p,$$

where $\mathbb{E}\mathrm{xt}$ *denotes the right derived functor of* $\mathcal{H}om(\mathcal{F},\cdot)$.

PROPOSITION 20. *Let X be a reduced finite dimensional complex space then*

$$\mathbb{E}\mathrm{XT}^{-p}_{\psi}(X; \mathcal{F}, \mathcal{K}^{\cdot}_X) \simeq H^{\psi}_p(X, \mathcal{F}_*)$$

for any coherent sheaf $\mathcal{F}$ on X.

PROOF. Consider an injective resolution $\mathcal{K}^{\cdot}_X \to \mathcal{J}^{\cdot}$ of $\mathcal{K}^{\cdot}_X$ and the double complex

$$C^{p,q} = C^{\psi}_p(\mathcal{U}, \mathcal{H}\mathrm{om}_k(\mathcal{F}, \mathcal{J}^q))$$

where $\mathcal{U}$ is a locally finite Stein covering by relatively compact open subsets adapted to the family of supports Ψ. Here

$$\mathcal{H}om_k(\mathcal{F}, \mathcal{J}^q) = \coprod_{r+s=q} \mathcal{H}om_k(\mathcal{F}, \mathcal{J}^{r,s}).$$

Since $\coprod \mathcal{J}^{r,s} = \mathcal{J}^q$ is injective by proposition 5 the precosheaf $\mathcal{H}om_k(\mathcal{F}, \mathcal{J}^q) = \{\mathrm{Hom}_k(U, \mathcal{F}, \mathcal{J}^q), \iota^U_V\}$ is flabby. Differentiation with respect to the Čech operator gives therefore

$$'E_1^{p,q} = \begin{cases} 0 \text{ if } p > 0 \\ \Gamma_{\psi}(X, \mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{J}^q)) = \mathrm{HOM}_{\psi}(X; \mathcal{F}, \mathcal{J}^q) \text{ if } p = 0. \end{cases}$$

Taking the homology with respect to the operator induced by the simple complex associated to the double complex $\mathcal{J}^{\cdot}$ we get

$$'E_2^{p,q} = \begin{cases} 0 \text{ if } p > 0 \\ \mathbb{E}\mathrm{XT}_{\psi}(X; \mathcal{F}, \mathcal{K}^{\cdot}_X) \text{ if } p = 0 \end{cases}$$

We now calculate the second spectral sequence. The term $''E_1^{p,q}$ is the homology group of the complex

$$C^{\psi}_p(\mathcal{U}, \mathcal{H}\mathrm{om}_k(\mathcal{F}, \mathcal{J}^{q-1})) \to C^{\psi}_p(\mathcal{U}, \mathcal{H}\mathrm{om}_k(\mathcal{F}, \mathcal{J}^q) \to C^{\psi}_p(\mathcal{U}, \mathcal{H}\mathrm{om}_k(\mathcal{F}, \mathcal{J}^{q+1})$$

Now, with the notations of n. 5, we have for any cosheaf $\mathcal{D}$:

$$C_p^{\psi}(\mathcal{U}, \mathcal{D}) = \varinjlim_s C_p^c(\mathcal{U}_s, \mathcal{D}).$$

Since homology commutes with direct limits and products we get

$$''E_1^{p,q} = \varinjlim_s C_p^c(\mathcal{U}_s, \mathbb{E}xt_k^q(\mathcal{F}, \mathcal{K}_X^{\cdot}))$$

$$= C_p^{\psi}(\mathcal{U}, \mathbb{E}xt_k^q(\mathcal{F}, \mathcal{K}_X^{\cdot}))$$

where $\mathbb{E}xt_k^q(\mathcal{F}, \mathcal{K}_X^{\cdot})$ is the precosheaf

$$U \to \mathbb{E}\mathrm{XT}_k^q(U; \mathcal{F}, \mathcal{K}_X^{\cdot}).$$

By the previous theorem (iv) we have therefore

$$\mathbb{E}xt_k^q(\mathcal{F}, \mathcal{K}_X^{\cdot}) = \begin{cases} 0 \text{ if } q \neq 0 \\ \mathcal{F}_* \text{ if } q = 0 \end{cases}$$

Therefore

$$''E_2^{p,q} = \begin{cases} 0 \text{ if } q \neq 0 \\ H_p^{\psi}(\mathcal{U}, \mathcal{F}_*) \text{ if } q = 0 \end{cases}$$

By the Leray theorem $H_p^{\psi}(\mathcal{U}, \mathcal{F}_*) = H_p^{\psi}(X, \mathcal{F}_*)$ and this concludes the proof

COROLLARY. *We have an isomorphism*

$$\mathcal{D}^p(\mathcal{F}) \simeq \mathbb{E}xt^{-p}(\mathcal{F}, \mathcal{K}_X^{\cdot}).$$

In particular for $\mathcal{F} = \mathcal{O}$, $\mathcal{D}^p(\mathcal{O})$ *is the* p-th *cohomology object of the dualizing complex* $\mathcal{K}_X^{\cdot}$.

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BIBLIOGRAPHY

[1] A. ANDREOTTI and H. GRAUERT, *Théorèmes definitude pour la cohomologie des espaces complexes*, Bull. Soc. Math. France v. 90 (1962) p. 193-259.

[2] A. ANDREOTTI and E. VESENTINI, *Carleman estimates for the Laplace-Beltrami equation on complex manifolds*, Publ. I. H. E. S. (1965) n. 25.

[3] C. BANICA and O. STANASILA, *Sur la profondeur d'un faisceau analytique cohérent sur un espace de Stein*, C. R. Ac. Sci. Paris v. 269 (1969) p. 636-639.

[4] G. E. BREDON, *Sheaf theory*, McGraw Hill, New York, 1967.

[5] A. CASSA, *Coomologia separate sulle varietà analitiche complesse*, Ann. Sc. Nor. Pisa, v. 25 (1971) p. 291-323.

[6] R. GODEMENT, *Théorie des faisceaux*, Hermann, Paris, 1958.

[7] R. C. GUNNING and H. ROSSI, *Analytic functions of several complex variables*, Prentice-Hall, Inc. Englewood Cliffs, N. J. 1965.

[8] A. GROTHENDIECK, *Espaces vectoriels topologiques*, Societad Mat. Sao Paolo, 1958.

[9] J. L. KELLEY and T. NAMIOKA, *Linear topological vectors spaces*, D. Van Nostrand, Princeton, 1963.

[10] G. KÖTHE, *Topologische lineare Räume*, Springer, Berlin, 1960.

[11] R. KULTZE, *Dualität von Homologie und Cohomologiegruppen*, Arch. Math. v. 10 (1959)

[12] B. MALGRANGE. *Systémes differentiels a coefficients constants*, Sem. Bourbaki 265-01 (1962).

[13] A. MARTINEAU, *Sur le théorème du graphe fermé*, C. R. Ac. Sci Paris, n. 263 (1966) p. 870-871.

[14] R. NARASIMHAN, *Introduction to the theory of analytic spaces*, Springer Lecture Notes 25, 1966.

[15] H. J. REIFFEN. *Riemannische Hebbarkeitssätze für Cohomologieklassen mit kompaktem Träger*, Math. Ann. v. 164 (1966) p 272-273.

[16] D. A. RAIKOV. *Double closed-graph theorem for topological linear spaces*, Sibirski Mat. Zhurnal v. 7 (1966) p. 353-372.

[17] J. P. RAMIS and G. RUGET, *Complexes dualisants et théorème de dualité en géometrie analytique complexe*, Publ. I. H. E. S. (1970) n. 38 p. 77-91.

[18] H. H. SCHAEFER, *Topological vector spaces*, MacMillan, New York, 1966.

[19] L. SCHWARTZ, *Sur le théorème du graphe fermé*, C. R. Ac. Sci. Paris v. 263 (1966) p. 602-605.

[20] J. P. SERRE, *Un théorème de dualité*, Comm. Math. Helvetici, v. 29 (1955) pp. 9-26.

[21] J. P. SERRE, *Faisceaux algébriques cohérents*, Annals of Math. v. 61 (1955) pp. 197-278.

[22] S e SILVA, *Su certe classi di spazi convessi importanti per le applicazioni*, Rend. Mat. v. 5 (1955) p. 388-410.

[23] K. SOUMINEN, *Duality for coherent sheaves on analytic manifolds*, An. Ac. Fennicae (1968).

[24] F. TREVES, *Topological vector spaces distributions and kernels*, Academic press, New York, 1967.