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E. Vesentini

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ON BANACH ALGEBRAS SATISFYING A SPECTRAL MAXIMUM PRINCIPLE

by E. VESENTINI (*)

Let $\mathfrak{A}$ be a complex Banach algebra with unit. For any $x \in \mathfrak{A}$, $Sp\, x$ will denote the spectrum of x.

Let $F: D \longrightarrow \mathfrak{A}$ be a holomorphic map of a domain D in $\mathbb{C}$ into $\mathfrak{A}$. A point $z_0 \in D$ such that

$$Sp\, F(z) \subset Sp\, F(z_0) \text{ for all } z \in D \tag{1}$$

will be called a *spectral maximum point.* The Banach algebra $\mathfrak{A}$ will be said to satisfy the *spectral maximum principle* if, for any holomorphic map $F: D \longrightarrow \mathfrak{A}$ having a spectral maximum point, $Sp\, F(z)$ is independent of z. If the existence of a spectral maximum point for F always implies that F is constant, $\mathfrak{A}$ will be said to satisfy the *strong spectral maximum principle.*

Let $F: D \longrightarrow \mathfrak{A}$ be a holomorphic map for which there is a point $z_0 \in D$ satisfying (1). It has been shown in [7, propositions 5 and 7, p. 116] that, if either $Sp\, F(z_0)$ has no interior points or $Sp\, F(z)$ does not divide the complex plane (i. e., $\mathbb{C}$-$Sp\, F(z)$ is connected) for all $z \in D$, then $Sp\, F(z)$ is independent of z. This result implies that if for all $x \in \mathfrak{A}$ either $Sp\, x$ has no interior points or does not divide the plane, then $\mathfrak{A}$ satisfies the spectral maximum principle. It has been proved in [8] that, if $\mathfrak{A}$ is commutative and semi-simple, the validity of the spectral maximum principle entails that of the strong spectral maximum principle provided that $Sp\, x$ has no interior points for all $x \in \mathfrak{A}$.

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The purpose of this note is to find a necessary and sufficient condition for the spectral maximum principle to hold for a certain class of Banach algebras. A condition of this kind (Theorem 1) depends on the topological structure of the spectra of the elements of the algebra. Theorem 2 shows that the spectral maximum principle and the strong spectral maximun principle are equivalent conditions for these algebras.

The class of Banach algebras to which Theorems 1 and 2 apply contains any commutative Banach algebra for which the Gelfand transform is a continuous isomorphism onto a closed subalgebra of all continuous functions on the space of maximal ideals (with the Gelfand topology). Theorem 3 characterizes the uniform algebras $C(X)$ of all continuous functions on a compact Hausdorff *scattered* space X as the only function algebras satisfying the spectral maximum principle.

According to Theorem 4, the Banach algebra of all bounded linear operators in a complex Hilbert space satisfies the spectral maximum principle if, and only if, the latter space has finite dimension.

In the final part of this paper the validity of the spectral maximum principle in regular self-adjoint Banach algebras is investigated. Theorem 5 characterizes the compact abelian groups as the only locally compact abelian groups whose group algebras satisfy the spectral maximum principle.

1. Let X be a compact Hausdorff space. Let $C(X)$ be the complex Banach algebra of all continuous complex-valued functions on X with the uniform norm.

In [4] A. Pelczynski has proved the following

PROPOSITION 1. *Let $\mathfrak{A}$ be a (uniform) function algebra on the compact metric space X. Then X is uncountable (if, and) only if there exists a subset S of X homeomorphic to the Cantor set and a linear operator $T: C(S) \to \mathfrak{A}$ such that for every $f \in C(S)$, $\| Tf \| = \| f \|$ and $(Tf)(s) = f(s)$ for every $s \in S$.*

Taking as $f \in C(S)$ a Peano curve, i. e., a continuous mapping of T onto the unit disc

$$\Delta = \{z : z \in \mathbb{C},\ |z| \leq 1\},$$

the above theorem implies the following

COROLLARY 2. *If X is an uncountable compact metric space, and if S is a compact subset of X homeomorphic to the Cantor set, there is a function $f \in \mathfrak{A}$ such that $\| f \| = 1$ and*

$$f(S) = f(X) = \Delta.$$

REMARK. Actually Proposition 1 of [4] is slightly more precise than Corollary 2, but the latter will be sufficient for the purposes of this note.

Let K be a compact subset of $\mathbb{C}$. We denote by $A(K)$ the uniform function algebra of all continuous functions on K which are holomorphic on $\mathrm{Int}(K)$ (endowed with the uniform norm).

Let K_1 be the complement of the unbounded component of $\mathbb{C} - K$. The boundary $\mathfrak{F} K_1$ of K_1, i. e., the outer boundary of K, is the Šilov boundary of $A(K_1)$. Let $B(K_1)$ be the uniform function algebra on $\mathfrak{F} K_1$ consisting of the restrictions to $\mathfrak{F} K_1$ of the elements of $A(K_1)$. Applying Corollary 2 to $B(K_1)$ we obtain

LEMMA 3. *If K is an uncountable compact subset of $\mathbb{C}$, there exists a function $f \in A(K_1)$ such that*

$$f(K) = f(K_1) = \Delta.$$

LEMMA 4. *The spectral maximum principle does not hold for the disc algebra $A(\Delta)$.*

PROOF. Let K_1 and K_2 be two disjoint compact subsets of $T = \{z : |z| = 1\}$ with Lebesgue measure zero, both homeomorphic to the Cantor set. Let Φ_1 and Φ_2 be two continuous mappings of $K_1 \cup K_2$ onto the unit interval $I = [0,1]$ such that $\Phi_1(K_2) = \Phi_2(K_1) = \{0\}$. Let ψ be a continuous mapping of I onto the rectangle R_a of $\mathbb{C}$ having vertices $1 + \pi i$, $-a + \pi i$, $-a - \pi i$, $1 - \pi i$ $(a > 0)$, such that $\psi(0) = 1 + \pi i$. By Rudin-Carleson's theorem [6] there is an element $\Phi \in A(\Delta)$ mapping Δ onto R_a and whose restriction to $K_1 \cup K_2$ is $\psi \circ \Phi_1$. Thus $\Phi(K_1 \cup K_2) = \Phi(\Delta) = R_a$. The entire function $\frac{1}{e}\exp : z \longmapsto e^{z-1}$ maps R_a onto the annulus with center 0 and radii 1 and e^{-a-1}. Let $F = \frac{1}{e}\exp \circ \Phi$. Then $F \in A(\Delta)$, and $F^{-1}(1) \subset T$.

Consider now the conformal mapping

$$z \longmapsto \frac{z+1}{z-1}$$

of $\mathrm{Int}(\Delta)$ onto the half-plane $\Pi = \{z : \mathrm{Re}\, z < 0\}$. The image of the disc $\{z : |z| \leq e^{-a-1}\}$ is, for a sufficiently large, a neighborhood, V_a, of the point -1. The diameter of V_a tends to zero as a tends to $+\infty$.

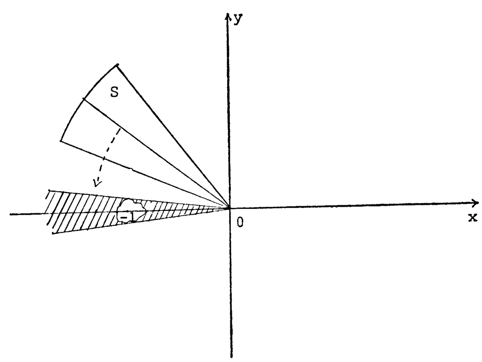
Let f be the holomorphic function defined on $\Delta' = \Delta - F^{-1}(1)$ by

$$f(z) = \frac{F(z)+1}{F(z)-1}.$$

Then $f(\Delta') = f K_1' = \bar{\Pi} - \mathrm{Int}(V_a)$, where $K_1' = K_1 - F_1^{-1}(1)$.

Choose now a so large that V_a is seen from 0 under an angle less than $\frac{\pi}{12}$.

Let S be the closed circular sector with center 0 and radius 2 determined by the points $2e^{i\frac{7\pi}{8}}$ and $2e^{i\frac{17\pi}{24}}$. Its amplitude is $\frac{\pi}{6}$. Let λ be a continuous mapping of I onto S such that $\lambda(0) = 0$. The function $\lambda \circ \Phi_2$ maps $K_1 \cup K_2$ onto S, while $\lambda \circ \Phi_2(K_1) = \{0\}$. By Rudin-Carleson's theorem, there is an element $g \in A(\Delta)$ such that $g|_{K_1 \cup K_2} = \lambda \circ \Phi_2$, and $g(\Delta) = S$. Since $K_2 \cap F^{-1}(1) = \emptyset$, then $g(\Delta') = S$.



Let ζ be any complex number, and consider the function $f + \zeta g$. For all $\zeta \in \mathbb{C}$

$$(f + \zeta g)(K_1') = f(K_1') = \overline{\Pi} - \mathrm{Int}(V_a), \quad (f + \zeta g)(K_2) = \zeta S.$$

Thus

$$(f + \zeta g)(\Delta') \supset (\overline{\Pi} - \mathrm{Int}(V_a)) \cup \zeta S$$

for all $\zeta \in \mathbb{C}$.

Since for $\zeta = 0$, $(f + \zeta g)(\Delta') = \overline{\Pi} - \mathrm{Int}.(V_a)$, when $|\zeta|$ is sufficiently small $(f+\zeta g)(\Delta')$ does not contain V_a. Since when $\zeta = e^{i\frac{\pi}{6}}$, $V_a \subset \mathrm{Int}(S)$, then V_a is contained in the interior of $(f + \zeta g)(\Delta')$, provided that $|\zeta - e^{i\frac{\pi}{6}}|$ is sufficiently small.

Let B be the open circular sector with center 0, and vertices 1 and $e^{i\frac{\pi}{6}}$. For every $\zeta \in B$ the function F_ζ defined by

$$F_\zeta(z) = \frac{f(z) + \zeta g(z) + 1}{f(z) + \zeta g(z) - 1} = \frac{2F(z) + \zeta g(z)(F(z) - 1)}{2 + \zeta g(z)(F(z) - 1)}$$

belongs to $A(\Delta)$.

For all $\zeta \in B$, $F_\zeta(\Delta) \subset \Delta$; for some $\zeta \in A$, $F_\zeta(\Delta) \neq \Delta$, while for some other $\zeta \in B$, $F_\zeta(\Delta) = \Delta$. Thus the mapping $B \ni \zeta \longmapsto F_\zeta$ is a holomorphic function with values in $A(\Delta)$ for which the spectral maximum principle does not hold. Q. E. D.

2. Let $\mathfrak{A}$ be a complex Banach algebra with unit. For any $x \in \mathfrak{A}$, $\| x \|$ and $\varrho(x)$ will denote the norm and the spectral radius of x.

Let x_0 be an element of $\mathfrak{A}$ and let $\mathfrak{B}$ be the closed subalgebra of $\mathfrak{A}$ generated by x and by the unit.

LEMMA 5. *If $Sp\, x_0$ is uncountable, and if there exists a constant $k > 0$ such that*

$$\| x \| \leq k\, \varrho(x) \quad (*) \tag{2}$$

for all $x \in \mathfrak{B}$, then there exists an element $y \in \mathfrak{A}$ such that

$$Sp\, y = \Delta.$$

PROOF. The spectrum of x_0 in $\mathfrak{B}$, $Sp_{\mathfrak{B}}\, x_0$, is the union of $Sp\, x_0$ and of all bounded connected components of $\mathbb{C} - Sp\, x_0$. Thus $\mathbb{C} - Sp_{\mathfrak{B}}\, x_0$ is connected. Since $Sp\, x_0$ is uncountable, then $Sp\, x_0$ contains a Cantor set. Thus also $Sp_{\mathfrak{B}}\, x_0$ contains a Cantor set. By Lemma 3 there is a function $f \in A\,(Sp_{\mathfrak{B}}\, x_0)$ such that $f\,(Sp\, x_0) = f\,(Sp_{\mathfrak{B}}\, x_0) = \Delta$. In view of Mergelyan's theorem, $\mathfrak{B}$ is topologically isomorphic with the commutative Banach algebra $A\,(Sp_{\mathfrak{B}}\, x_0)$, the isomorphism mapping x_0 onto the function $\zeta \longmapsto \zeta$. Let $y \in \mathfrak{B}$ be the element whose image is f. Since the space of maximal ideals of $\mathfrak{B}$, endowed with the Gelfand topology, is (homeomorphic to) $Sp_{\mathfrak{B}}\, x_0$, then $Sp_{\mathfrak{B}}\, y = \Delta$.

If $\{p_\nu\}$ is a sequence of polynomials converging to f uniformly on $Sp_{\mathfrak{B}}\, x$, then the sequence $\{ p_\nu(x) \}$ converges to y. Since $p_\nu\,(Sp\, x_0) = Sp\, p_\nu\,(x_0)$, then for all $\zeta \in Sp\, x_0$, $f(\zeta) \in Sp\, y$, i. e., $f\,(Sp\, x_0) \subset Sp\, y$. But then

$$\Delta = f\,(Sp\, x_0) \subset Sp\, y \subset Sp_{\mathfrak{B}}\, y = \Delta.$$

Q. E. D.

Consider the natural homomorphism of the uniform algebra $A\,(\Delta) = A\,(Sp\, y)$ onto the Banach algebra $\mathfrak{A}$. The image of $A\,(\Delta)$ lies in $\mathfrak{B}$. Let $g\,(y)$ be the image of any $g \in A\,(\Delta)$. Then $Sp\, g\,(y) = g\,(\Delta)$.

A direct application of Lemma 4 shows that, if $Sp\, x_0$ is uncountable and if condition (2) holds for all $x \in \mathfrak{B}$, there is a holomorphic function $F : D \to \mathfrak{A}$ with values in $\mathfrak{B}$, having a spectral maximum point in the domain D and such that $Sp\, F(z)$ is not independent of z.

(*) (Added in proof; December 9, 1972). According to a result of W. Zelazko, which was communicated to me by A. Browder, any complex Banach algebra $\mathcal{A}$ satisfying (2) for all $x \in \mathcal{A}$ is commutative. Hence the Banach algebra in Theorem 1 is commutative.

On the other hand, Proposition 5 of [7] implies that if the spectral maximum principle does not hold in $\mathfrak{A}$, then there is some $x \in \mathfrak{A}$ such that $Sp\, x$ contains interior points.

In conclusion we have the following theorem

THEOREM 1. *Let $\mathfrak{A}$ be a Banach algebra with unit such that* (2) *holds for all $x \in \mathfrak{A}$. Then the spectral maximum principle holds in $\mathfrak{A}$ if, and only if, $Sp\, x$ is countable for all $x \in \mathfrak{A}$.*

Let $\mathfrak{A}$ be a commutative Banach algebra with unit, and let $M_{\mathfrak{A}}$ be the space of maximal ideals of $\mathfrak{A}$, endowed with the Gelfand topology. Let $\widehat{\mathfrak{A}}$ be the subalgebra of $C(M_{\mathfrak{A}})$ consisting of the Gelfand transforms of the elements of $\mathfrak{A}$. Then (2) holds for all $x \in \mathfrak{A}$ if, and only if, $\mathfrak{A}$ is semi simple and $\widehat{\mathfrak{A}}$ is a closed subalgebra of $C(M_{\mathfrak{A}})$.

Let $\mathfrak{A}$ be a commutative Banach algebra with unit satisfying (2) for all $x \in \mathfrak{A}$, and let $F: D \to \mathfrak{A}$ be any holomorphic mapping of a domain $D \subset \mathbb{C}$ into $\mathfrak{A}$, having a spectral maximum point $z_0 \in D$. If $\operatorname{Int}(Sp\, F(z_0)) = \emptyset$, then for all $\chi \in M_{\mathfrak{A}}$ the holomorphic function $\chi \circ F: D \to \mathbb{C}$ is such that $(\chi \circ F)(D) \subset Sp\, F(z_0)$. Hence $\chi \circ F$ is constant on D. That implies the following

THEOREM 2. *Let $\mathfrak{A}$ be a commutative Banach algebra satisfying* (2) *for all $x \in \mathfrak{A}$. Then the strong spectral maximum principle and the spectral maximum principle are equivalent conditions for $\mathfrak{A}$.*

3. Let $\mathfrak{A}$ be a (uniform) function algebra on a compact Hausdorff space X. Then $\varrho(f) = \|f\|$ for all $f \in \mathfrak{A}$, and Theorems 1 and 2 apply. Since $\mathfrak{A}$ separates points, Theorem 1 implies immediately that, if $\mathfrak{A}$ satisfies the spectral maximum principle, then X is totally disconnected.

This section will provide a characterization of function algebras satisfying the spectral maximum principle.

Let X be a compact Hausdorff space. It has been shown in [5] (cf. also [6′]) that X is scattered (i. e., contains no non-empty perfect subset) if, and only if, $f(X)$ is countable for any $f \in C(X)$. Furthermore, if X is scattered, then $C(X)$ is the only function algebra on X [5, 6′].

For any compact subset $K \subset \mathbb{C}$, $R(K)$ is the closure in $C(K)$ of the subalgebra generated by the rational functions on $\mathbb{C}$ with poles off K.

LEMMA 6. *Let $\mathfrak{A}$ be a function algebra on a compact Hausdorff space X. If $Sp\, f$ has planar Lebesgue measure zero for all $f \in \mathfrak{A}$, then:*

a) $\mathfrak{A} = C(X)$;

b) *X is scattered*;

c) *$Sp\, f$ is countable for all $f \in C(X)$.*

PROOF: *a*) For any $x \in X$ let τ_x be the character of $\mathfrak{A}$ defined by $\tau_x(f) = f(x)\ (f \in \mathfrak{A})$. The mapping $\tau : x \longrightarrow \tau_x$ is a homeomorphism of X onto a closed subset of $M_{\mathfrak{A}}$. An element $f \in \mathfrak{A}$ and its Gelfand transform are related by the identity $f = \widehat{f} \circ \tau$.

Let r be a rational function on $\mathbb{C}$ with all its poles off $Sp\, f$. Then $r(f) \in \mathfrak{A}$. Since $Sp\, f$ has planar Lebesgue measure zero, by the Hartogs-Rosenthal theorem [1, p. 47], $R(Sp\, f) = C(Sp\, f)$. Therefore the continuous function $\zeta \longmapsto \overline{\zeta}$ can be uniformly approximated on $Sp\, f$ by rational functions with poles off $Sp\, f$. Hence $\overline{f} \in \mathfrak{A}$ for any $f \in \mathfrak{A}$. By the Stone Weierstrass theorem, $\mathfrak{A} = C(X)$ (1).

b) If X is not scattered, there exists some $f \in C(X)$ such that $f(X) = I = [0, 1]$. Let g be a continuous mapping of I onto Δ. Then $g \circ f \in C(X)$, while $Sp\,(g \circ f) = \Delta$ has positive planar measure.

Since X is scattered, then *b*) implies *c*). Q. E. D.

As a consequence of Lemma 6 and of Theorem 1, if $\mathfrak{A}$ satisfies the spectral maximum principle, then ($Sp\, f$ is countable for all $f \in \mathfrak{A}$ and therefore) X is scattered and $\mathfrak{A} = C(X)$. Conversely, if X is scattered then $C(X)$ is the only function algebra on X, and, for all $f \in C(X)$, $f(X) = Sp\, f$ is countable.

In conclusion we have

THEOREM 3. *Let $\mathfrak{A}$ be a function algebra on a compact Hausdorff space X. If $\mathfrak{A}$ satisfies the spectral maximum principle, the X is scattered and $\mathfrak{A} = C(X)$. Conversely if X is scattered then $\mathfrak{A}$ satisfies the spectral maximum principle.*

Let $\mathfrak{A}$ be a C^* algebra with unit and let x be a normal element of $\mathfrak{A}$. Let $\mathfrak{A}'$ be the closed C^* subalgebra of $\mathfrak{A}$ generated by x and by the unit. Then there is a $*$-isometry Φ of $C(Sp\, x)$ onto $\mathfrak{A}'$ mapping the function « 1 » onto the unit and the function « $\zeta \longmapsto \zeta$ » onto x.

If $Sp\, x$ is uncountable, then it contains a perfect set, and therefore $\mathfrak{A}'$ does not satisfy the spectral maximum principle. But for any $f \in C(Sp\, x)$, $Sp_{\mathfrak{A}}\, \Phi(f) = f\,(Sp_{\mathfrak{A}}\, x)$. Hence *if the C^* algebra $\mathfrak{A}$ contains a normal element x such that $Sp\, x$ is uncountable, then $\mathfrak{A}$ does not satisfy the spectral maximum principle.*

As a consequence of this statement we have

THEOREM 4. *The algebra of all bounded linear operators in a Hilbert space H satisfies the spectral maximum principle, if, and only if, H has finite dimension.*

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(1) This proof was inspired by that of Theorem 3, p. 201 of [3].

4. Let $\mathfrak{A}$ be a (semi-simple) regular Banach algebra with a unit.

LEMMA 7. *The regular self-adjoint Banach algebra $\mathfrak{A}$ satisfies the spectral maximum principle if, and only if, for every $f \in \mathfrak{A}$, $\mathrm{Int}(Sp\, f) = \emptyset$.*

PROOF. *a*) Let $f \in \mathfrak{A}$ be such that $\mathrm{Int}(Sp\, f) \neq \emptyset$. Let ζ_0 and ζ_1 be two distinct points of $\mathrm{Int}(Sp\, f)$. There is no restriction in assuming $\zeta_0 = 0$, $\zeta_1 = 1$. For any $\zeta \in \mathbf{C}$, $\Delta(\zeta, r)$ will indicate the closed disc with center ζ and radius r. Let r_0 and r_1 be two positive numbers such that

$$\Delta(0, r_0) \subset \mathrm{Int}(Sp\, f), \quad \Delta(1, r_1) \subset \mathrm{Int}(Sp\, f),$$

$$\Delta(0, r_0) \cap \Delta(1, r_1) = \emptyset.$$

Let ϱ and σ be two positive numbers such that

$$\Delta(0, r_0 + \varrho + \sigma) \cap \Delta(1, r_1 + \sigma) = \emptyset,$$

and that both closed discs are contained in $\mathrm{Int}(Sp\, f)$.

Let

$$F_\alpha = \hat{f}^{-1}(\Delta(\alpha, r_\alpha)), \; A_\alpha = \hat{f}^{-1}(\mathrm{Int}(\Delta(\alpha, r_\alpha + \sigma))), \; (\alpha = 0, 1).$$

Let Φ_0 and Φ_1 be two elements of $\mathfrak{A}$ such that $\hat{\Phi}_0$ and $\hat{\Phi}_1$ are real-valued functions on $M_{\mathfrak{A}}$, satisfying the following conditions

$$\hat{\Phi}_\alpha = 1 \text{ on } F_\alpha, \; \hat{\Phi}_\alpha = 0 \text{ on } M_{\mathfrak{A}} - A_\alpha,$$

$$0 \leq \hat{\Phi}_\alpha \leq 1 \text{ on } M_{\mathfrak{A}}, \; (\alpha = 0, 1).$$

The element $f_0 = \Phi_0 f$ is such that, if $B_0 = \hat{f}^{-1}(\mathrm{Int}(\Delta(0, r_0 + \varrho + \sigma)))$,

$$\mathrm{Supp}\, \hat{f}_0 \subset B_0, \; \Delta(0, r_0) \subset \hat{f}_0(M_{\mathfrak{A}}) \subset \Delta(0, r_0 + \sigma).$$

Let ψ be an element of $\mathfrak{A}$ such that $\hat{\psi}$ is a real-valued function satisfying the following conditions

$$\hat{\psi} = 0 \text{ on } \hat{f}^{-1}(\Delta(0, r_0 + \varrho + \sigma)),$$

$$\hat{\psi} = 1 \text{ on } \bar{A}_1, \; 0 \leq \hat{\psi} \leq 1 \text{ on } M_{\mathfrak{A}}.$$

The image $J = \hat{\psi}(M_{\mathfrak{A}})$ is a closed subset of $[0, 1]$ containing 0 and 1. If $f_1 = \Phi_1(f - e) + \psi$, then

$$\mathrm{Supp}\, \hat{f}_1 \subset M_{\mathfrak{A}} - B_0.$$

Being $\widehat{f_1} = f$ on F_1, then $\widehat{f_1}(M_{\mathfrak{A}}) \supset \Delta(1, r_1)$. Furthermore

$$\widehat{f_1}(M_{\mathfrak{A}}) \subset J \cup \Delta(1, r_1 + \sigma).$$

Let $h = \exp\left(\frac{4\pi}{\sqrt{2}\, r_1} f_1\right)$. Since $\Delta(1, r_1)$ contains any square with center 1 and side $\sqrt{2}\, r_1$ then $\widehat{h}(M_{\mathfrak{A}})$ contains the annulus P with center 0 and radii $e^{-2\pi\left(\frac{\sqrt{2}}{r_1}-1\right)}$, $e^{2\pi\left(\frac{\sqrt{2}}{r_1}+1\right)}$.

Let $F : \mathbb{C} \to \mathfrak{A}$ be the holomorphic mapping defined by

$$F(z) = h + z f_0 .$$

Then

$$\Delta(0, |z| r_0) \cup P \subset \widehat{F(z)}(M_{\mathfrak{A}}) \subset \Delta(0, |z|(r_0 + \sigma)) \cup \widehat{h}(M_{\mathfrak{A}}). \tag{3}$$

If z satisfies both conditions

$$e^{-2\pi\left(\frac{\sqrt{2}}{r_1}-1\right)} < |z| r_0 , \quad |z|(r_0 + \sigma) < e^{2\pi\left(\frac{\sqrt{2}}{r_1}+1\right)},$$

i. e., if

$$\frac{e^{-2\pi\left(\frac{\sqrt{2}}{r_1}-1\right)}}{r_0} < |z| < \frac{e^{2\pi\left(\frac{\sqrt{2}}{r_1}+1\right)}}{r_0 + \sigma}, \tag{4}$$

then

$$\Delta(0, |z| r_0) \cup P = \Delta(0, |z|(r_0 + \sigma)) \cup P = \Delta\left(0, e^{2\pi\left(\frac{\sqrt{2}}{r_1}+1\right)}\right).$$

Thus, by (3) $Sp\, F(z)$ is the same for all z satisfying conditions (4).

If $\sigma > 0$ is sufficiently small, then the latter inequalities are compatible and the set of points satisfying both of them is a non-empty annulus. Let $0 < r < R$ be its radii. Then $F(z)$ has a spectral maximum point in $\mathrm{Int}(\Delta(0, R))$ but $Sp\, F(z)$ is not constant.

b) If $\mathrm{Int}(Sp\, f) = \emptyset$ for all $f \in \mathfrak{A}$, then by Proposition 5 of [7], $\mathfrak{A}$ satisfies the spectral maximum principle. Q. E. D.

We will apply now Lemma 7 to the group algebra of a locally compact abelian group G. First we recall that, if $\mathfrak{A}$ is a regular Banach algebra without a unit, then also the algebra $\mathfrak{A} \times \mathbb{C}$ obtained by formally adjoining a unit to $\mathfrak{A}$ is regular, and vice-versa.

Let G be a locally compact abelian group and let Γ be the dual group of G, endowed with the locally compact topology defined by the continuous characters of G.

PROPOSITION: 8. *The group algebra* $L^1(G)$ *contains an element* $f \in L^1(G)$ *such that* $\operatorname{Int}(\widehat{f}(\Gamma)) \neq \emptyset$ *if, and only if,* G *is not compact.*

PROOF. *a*) If G is non-compact, then Γ is non-discrete. Therefore Γ contains a Cantor set C which is also a Helson set (cf., e. g. [2], Theorems 41.5 (p. 555) and 41.13 (p. 564)). So if $g: C \to \mathbb{C}$ is a continuous function on C such that $g(C) = \Delta$ there exists an element $f \in L^1(G)$ such that $\widehat{f}(x) = g(x)$ for all $x \in C$. Hence $\widehat{f}(\Gamma) \supset \Delta$.

b) Conversely, let $\operatorname{Int}(\widehat{f}(\Gamma)) \neq \emptyset$ for some $f \in L^1(G)$. If Γ is not compact the function $\widehat{f}$ vanishes at infinity. Therefore it extends uniquely to the one point compactification $\widetilde{\Gamma}$ of Γ assuming value zero at the compactifying point. Let $\widetilde{f}$ be the extended function. Then $\operatorname{Int}(\widetilde{f}(\widetilde{\Gamma})) \neq \emptyset$ and therefore $\widetilde{f}(\widetilde{\Gamma})$ contains a perfect set. Thus $\widetilde{\Gamma}$ is not scattered and *a fortiori* is not discrete.

If Γ is compact this argument applies directly to Γ, leading to the same conclusion. Q. E. D.

The following theorem is a consequence of Lemma 7 and Proposition 8.

THEOREM 4. *The group algebra* $L^1(G)$ (*or, more exactly the algebra obtained by formally adjoining a unit to* $L^1(G)$) *satisfies the spectral maximum principle if, and only if,* G *is compact.*

For example the spectral maximum principle is satisfied when G is the circle group, and is not satisfied when $G = \mathbb{R}$, $G = \mathbb{Z}$.

Furthermore *the spectral maximum principle and the strong spectral maximum principle are equivalent conditions for the group algebra* $L^1(G)$.

University of Maryland and Scuola Normale Superiore, Pisa

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