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The Cauchy problem for non linear wave equations in domains with moving boundary

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THE CAUCHY PROBLEM FOR NON LINEAR WAVE EQUATIONS IN DOMAINS WITH MOVING BOUNDARY

by JEFFERY COOPER and LUIZ A. MEDEIROS (§)

1. Introduction.

In this paper we shall consider the non linear wave equation

$$(*) \qquad u_{tt} - \Delta u + F(u) = 0$$

in a non cylindrical domain $Q \subset B = \mathbf{R}^n \times [0, T]$, with the boundary condition $u = 0$ on Σ, the lateral boundary of Q. $\Omega(t)$ will denote the intersection of P with the hyperplane at height t. We shall say that Q is monotone increasing if the $\Omega(t)$ grows with t.

In [2] Lions obtained weak solutions of $(*)$ for the special case $F(u) = |u|^\varrho\ \varrho \geq 0$, under the assumption that Q was monotone increasing. Bardos and Cooper [1] extended this result to a larger class of regions by assuming only that there is a smooth mapping $\varphi : B \to B$ such that $Q^* = \varphi(Q)$ is monotone increasing φ preserves the hyperholic character of $(*)$. Such a mapping will be called hyperbolic; a precise definition is given later.

Medeiros [4] generalized the result of Lions [2] in another direction — namely by employing: the recent convergence theorem of Strauss [6] to obtain solution of $(*)$ when Q is monotone increasing, for quite general F.

In this paper we shall combine these generalizations as follows: suppose that there exists a smoot mapping φ of B onto B (with smooth inverse ψ) such that φ is hyperbolic and $Q^* = \varphi(Q)$ is monotone increasing. Sup-

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pose that $F(x, u)$ is continuos on $\mathbf{R}^n \times \mathbf{R}$ as well as $\frac{\partial F}{\partial x_i}(x, u)$ and that

$$\left| \frac{\partial F}{\partial x_i}(x, u) \right| \leq c \mid F(x, u) \mid , \qquad i = i, 2, \ldots, n$$

for all x and u. Suppose in addition that $uF(x, u) \geq 0$. Denote by $G(x, u)$ the function such that $G_u = F$ and $G(x, 0) = 0$. Let $u_0 \in H_0^1(\Omega(0))$ such that $G(u_0) \in L^1$ and $u_1 \in L^2(\Omega(0))$ be given. Then there exists a solution of $(*)$ in the sense of distributions on Q such that

$$u \in L^\infty(0, T; H_0^1(\Omega(t)))$$

$$u_t \in L^\infty(0, T; L^2(\Omega(t)))$$

with $u(x, 0) = u_0(x)$ and $u_t(x, 0) = u_1(x)$ for $x \in \Omega(0)$.

2. Existence of Weak Solutions.

We consider the set $B \subset \mathbf{R}^{n+1}$ defined as

$$B = \{(x, t) : x \in \mathbf{R}^n , \ 0 < t < T\},$$

where $0 < T < \infty$. Let Q be an open set in B. By $\Omega(t_0)$ we denote the intersection of Q with the hyperplane $P_{t_0} = \{(x, t) : t = t_0\}$; $\Omega(0)$ (resp. $\Omega(T)$) denotes the interior of $\overline{Q} \cap P_0$ (resp. $\overline{Q} \cap P_T$). Let $\Gamma(t) = \partial\Omega(t)$ the boundary of $\Omega(t)$ and set $\Sigma = \bigcup_{0 \leq t \leq T} \Gamma(t)$. Σ is the lateral boundary of Q.

Throughout this paper we shall assume that Σ is an n-dimensional manifold of class C^1.

We shall say that Q is monotone increasing if $\Omega(t)$ grows with t. That is, if $\Omega^1(t)$ denotes the projection of $\Omega(t)$ onto P_0, then $s \leq t$ implies $\Omega'(s) \subset \Omega'(t)$.

Only real valued functions will be considered here and derivatives will be taken in the sense of disiributions.

If Ω is an open set of $\mathbf{R}^n$, then $H^1(\Omega)$ denotes the space of (classes of) functions $u \in L^2(\Omega)$ such that $\frac{\partial u}{\partial x_i} \in L^2(\Omega)$, $i = 1, 2, \ldots, n$. $H^1(\Omega)$ is a Hilbert space with the norm

$$\| u \|_{H^1}^2 = \int_\Omega \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i} \right)^2 dx + \int_\Omega u^2 \, dx$$

$H_0^1(\Omega)$ will denote the closure in $H^1(\Omega)$ of $\mathcal{D}(\Omega)$ the space of infinitely differentiable functions with compact support contained in Ω. $H^{-1}(\Omega)$ will denote the dual of $H_0^1(\Omega)$.

Now suppose that $u(x,t)$ is a measurable functions on Q. Then we say that $u \in L^\infty(0,T; L^2(\Omega(t)))$ if $u(x,t) \in L^2(\Omega(t))$ for almost all t and $\sup_{0\le t\le T} \| u(x,t)_{L^2(\Omega(t))} < \infty$. The space $L^\infty(0,T; H_0^1(\Omega(t)))$ is defined similarly.

With these definitions made we now pose the following problem. Let $F(x,u)$ be a continuous function on $\mathbf{R}^n \times \mathbf{R}$ with $uF(x,u) \ge 0$. Suppose that $\partial F/\partial x_i$ are also continuous and that for some constant c, $\left|\frac{\partial F}{\partial x_i}(x,u)\right| \le$ $\le c\,|F(x,u)|$. We denote by $G(x,u)$ one function such that $G_u = F$ with $G(x,0)=0$. Let $u_0 \in H_1(\Omega(0))$ and $u_1 \in L^2(\Omega(0))$ be given. Then we search a function u such that

(1) $$u \in L^\infty(0,T; H_0^1(\Omega(t))) \text{ and } u_t \in L^\infty(0,T; L^2(\Omega(t))),$$

(2) $$u_{tt} - \Delta u + F(x,u) = 0 \text{ in the weak sense,}$$

(3) $$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) \text{ in } \Omega(0).$$

Our method of solving (1)-(3) involves a change of variable. Let $\Phi(x,t) =$ $= (\Phi_1(x,t), \dots, \Phi_{n+1}(x,t))$ be a one to one mapping of B with derivatives of first and second order bounded on B. Let $J(x,t)$ denote the Jacobian of Φ. We shall assume that J and its derivatives are bounded away from zero so that $|J(x,t)|$ is of class C^1. We denote the inverse mapping of $\Phi(x,t)$ by $\psi(y,s)$. Next we shall assume that Φ preserves the hyperbolic character of the wave equation. We set $\nabla\Phi_i = \left(\frac{\partial\Phi_i}{\partial x_1}, \dots, \frac{\partial\Phi_i}{\partial x^n}\right)$ and assume

(4) The $n \times n$ matrix

$$a_{ij} = \langle \nabla\Phi_i, \nabla\Phi_j \rangle - \left(\frac{\partial\Phi_i}{\partial t}\right)\left(\frac{\partial\Phi_j}{\partial t}\right)$$

$i,j = 1,2,\dots,n$, is positive definite on B and bounded away from zero by a positive constant and

$$\left(\frac{\partial\Phi_{n+1}}{\partial t}\right)^2 - |\nabla\Phi_{n+1}|^2 > c_0 > 0,$$

(5) $$\frac{\partial\Phi_{n+1}}{\partial t} \ge 0 \text{ on } B \text{ and } \Phi \text{ maps } P_0 \text{ (resp. } P_T)$$

onto P_0 (resp. P_T).

Our key assumption on Q is then that

$$Q^* = \Phi(Q) \text{ is monotone increasing.} \tag{6}$$

Let $Q^*(t)$ denote $\Phi(\Omega(t))$. As was noted in [1], (4)-(6) imply that the exterior normal to Σ always lies strictly outside the forward light cone.

Before proceeding the statement of our theorem, we give a form of a recent result of Strauss [6] which we will need later.

LEMMA 1. Let Ω be a finite measure space. Let $u_j(x)$ be a sequence of measurable functions on Ω. Let $F_j(x, u)$ be a sequence of measurable fonctions on $\Omega \times \mathbf{R}$ such that

(i) $F_j(x, u)$ is uniformly bounded on $\Omega \times E$ for any bounded sub set E of $\mathbf{R}$,

(ii) $F_j(x, u_j(x))$ is measurable and

$$\int_\Omega | u_j(x) | \, | F_j(x, u_j(x)) | \, dx \leq c < \infty,$$

(iii) $| F_j(x, u_j(x)) - v(x) | \to 0$ a.e.

Then $v \in L_1(\Omega)$ and $\int_\Omega | F_j(x, u_j(x)) - v(x) | \, dx \to 0$.

we now state our existence theorem.

THEOREM 1. Let $F(x, u)$ and $G(x, u)$ be as described before. Let $u_0 \in H_0^1(\Omega(0))$ and $u_1 \in L^2(\Omega(0))$ be given and assume that $G(x, u_0(x))$ is integrable on $\Omega(0)$. Then, if (4)-(6) is satisfied, the problem (1)-(3) has a solution.

PROOF. We transform the equation (2) via the mapping $(y, s) = \Phi(x, t)$. In divergence form it then becomes (with $u(x, t) = v(\Phi(x, t))$, $D_j = \partial/\partial y_j$ and $D_s = \partial/\partial s$)

$$\begin{aligned} D_s(aD_s v - \sum_{i,j=1}^{n} D_i(a_{ij} D_j v) + \sum_{j=1}^{n} D_j(b_j D_s v) + \\ + \sum_{j=1}^{n} c_j D_j v + cD_s v + f(y, s, v) = 0, \end{aligned} \tag{7}$$

where $f(y, s, v) = F(\psi_1(y,s), \dots, \psi_n(y, s), u \cdot \psi)$ is still continuous on $\mathbf{R}^{n+1}$ and $of(y, s, v) \geq 0$. Then $g(y, s, v) = G(\psi_1(y, s), \dots, \psi_n(y, s), u \cdot \psi)$ is a primitive of f such that $g(y, s, 0$; a and a_{ij} are given in (4) and the other coefficients involve first and second order derivatives of Φ. The initial conditions become

(8) $$v(y, 0) = v_0(y) = u_0(\psi(y))$$

$$D_s v(y, 0) = v_1(y) = u_1(\psi(y) \supset \frac{\partial \psi_{n+1}}{\partial s}(y, 0) + \sum_{i=1}^{n} \frac{\partial u_0}{\partial x_i}(\psi(y)) \frac{\partial \psi_i}{\partial s}(y, 0)$$

where $\psi(y) = (\psi_1(y, 0), \dots, \psi_n(y, 0))$.

The boundary conditions of course becomes.

(9) $v = 0$ on Σ^*, the iateral boundary of Q^*.

To solve (7)-(9) we shall follow the tecnique of Straus [6] and at the same time use the penalty method. Thus we shall consider equation (7) in B with f approximated by a Lipschitz function and with the addition of the penalty term.

LEMMA 2. Let $f(y, s, v)$ be continuous on $B \times \mathbf{R}$ with $vf(y, s, v) \geq 0$. Suppose that $\frac{\partial f}{\partial s}$ is continuous on $B \times \mathbf{R}$ and that is constant such $c > 0$ $\left|\frac{\partial f}{\partial s}\right| \leq c|f|$ on $B \times \mathbf{R}$. Then there is a sequence of continuous functions $f_k(y, s, v)$ such that $vf_k(y, s, v) \geq 0$ and

(i) $|f_k(y, s, \xi) - f_k(y, s, \eta)| \leq c_k(y\ s)\,|\xi - \eta|$

where c_k is continuous on B;

(ii) $f_k \to f$ uniformy ou $B_0 \times K$ is any bounded interval of $\mathbf{R}$ and B_0 is a bounded set of B;

(iii) $\left|\frac{\partial f_k}{\partial_s}\right| \leq c|f_k|$ on $B \times R$

The proof of lemma 2 is left till the end.

Now define $g_k(y, s, v) = \int_0^v f_k(y, s, \xi)\, d\xi$. From (iii) it follows that $\left|\frac{\partial g_k}{\partial^s}\right| \leq$ $\leq c|g_k|$. After makiug this approximate the initial data. We extend $v_0(y)$ and $v_1(y)$ by zero to all of $\mathbf{R}^n$, keeping to same notation. Then $v_1(y) \in L^2(\mathbf{R}^n)$, and by the smoothness assumption on the boundary of Q we have $v_0(y) \in$ $\in H^1(\mathbf{R}^n)$. There exists a sequence $v_{0k}(y)$ such that $v_{0k}(y) \to v_0(y)$ in $H^1(\mathbf{R}^n)$ and a. e. such that v_{0k} has bounded support and $|v_{0k}(y)| \leq k$. The former is achieved by multiplying v_0 by a suitable smooth function of bounded support, and the latter by truncating at height k (see Stampacchia [5], lemma 11).

It follows from lemma 2 that $g_j \to g$ uniformely on $B_0 \times K$ where K is a bounded subset of $\mathbf{R}$, and B_0 is a bounded set of B. Then, for each fixed k, $g_j(y, 0, v_{0k}(y))$ converges to $g(y, 0, v_{0k})$ a. e. and hence in $L^1(\mathbf{R}^n)$ because the support of v_{0k} is bounded. Now because $g(y, s, v)$ is monotone increasing in $|v|$, $g(y, 0, v_{0k}(y)) \leq g(y, 0, v_0(y)) \in L^1(\mathbf{R}^n)$ and $g(v_{0k}) \to g(v_0)$ in $L^1(\mathbf{R}^n)$. Thus we may choose a subsequence of the g_j which we shall denote by g_k, such that

$$g_k(v_{0,k}) \to g(v_0) \text{ in } L^1(\mathbf{R}^n).$$

After these preliminaires, we consider the following approximate equation in B to (7)-(9):

$$D_s(aD_s V) - \sum_{i,j=1}^{n} D_j(a_{ij} D_i V) + \sum_{j=0}^{n} D_j(b_j D_s V) + \tag{10}$$

$$+ \sum_{j=1}^{n} a_j D_j V + cD_s V + f_k(y, s, V) + k M D_s V = 0$$

$$V(y, 0) = v_{0k}(y);\ D_s V(y, 0) = v_1(y). \tag{11}$$

$M(y, s)$ is a function equal to zero on Q^* and equal to one outside Q^*.

As is well known, the Galerkin method may be used to solve (10), (11) (see [3]). The only point to mention is that in multiplying by $D_s V$ in order to make the usual energy estimate, one finds that

$$f_k(y, s, V) D_s V = D_s g_k(y, s, v) - \frac{\partial g_k}{\partial^s}(y, s, V)$$

The last term may be absorbed in the estimate because $|\partial g_k/\partial s| \leq c\,|g_k|$.

Thus for each $k = 1, 2, \ldots$ there exists a solution V^k of (10), (11) such that $V^k(s)$ is weakly continuous in $H^1(\mathbf{R}^n)$ and $D_s V^k(s)$ is weakly continuous in $L^2(\mathbf{R}^n)$, $0 \leq s \leq T$. Furthermore V^k satisfies the energy inequality

$$\frac{1}{2}\int_{Rn} \Big\{ (aD_s V^k(s))^2 + \sum_{i,j=1}^{n} a_{ij} D_i V^k(s) D_j V^k(s)\, dy + \tag{12}$$

$$+ \int_{R^n} g_k(y, s, V^k(s))\, dx + k \int_0^s M(y, \xi)(D_s V^k)^2\, d\xi \leq$$

$$\leq c_1 \int_{R^n} \Big\{ \frac{1}{2}(av_1(y))^2 + \frac{1}{2}\sum_{i,j=1}^{n} a_{ij} D_i v_{0k}(y) D_j v_{0k}(y) + g_k(y, 0, v_{0k}(y)) \Big\}\, dy.$$

By our choice of the sequence v_{0k} and of the subsequence g^k, we know that the right side of (12) will converges to

$$\int_{R^n} \left\{ \frac{1}{2} (a(0) v_1(y))^2 + \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(0) D_i v_0(y) D_j v_0(y) + g(y, 0, v_0(y)) \right\} dy.$$

The left side consists only of positive terms and these must be bounded. Thus we may extract a subsequence again denoted by V^k such that

$$\text{(13)} \qquad V^k \longrightarrow V \text{ weak star in } L^\infty(0, T; H^1(\mathbf{R}^n))$$

$$D_s V^k \longrightarrow D_s V \text{ weak star in } L^\infty(0, T; L^2(\mathbf{R}^n)).$$

By a standard compactness argument (see Lions [2]) we also assume that $V^k \longrightarrow V$ a.e. in B. (12) also implies that for some constant $c_2 > 0$ we have

$$\text{(14)} \qquad k \int_0^T \int_{\mathbf{R}^n} M \,|\, D_s V^k |^2 \, dy \, ds \leq c_2$$

From the fact that Q^* is monotone increasing we may deduce that

$$M V^k(y, s) = \int_0^s M(y, \xi) \, D_s V^k(y, \xi) \, d\xi$$

and it follows that

$$\int_0^T \int_{\mathbf{R}^n} M \,|\, V^k |^2 \, dy \, ds \leq c_3 \int_0^T \int_{\mathbf{R}^n} M \,|\, D_s V^k |^2 \, dy \, dx$$

Then by Schwartz inequality and (14), we obtain

$$\text{(15)} \qquad k \int_0^T \int_{\mathbf{R}^n} M \, D_s V^k \, V^k \, dy \, ds \leq c_4$$

To obtain convergence of the non linear term we shall need (15), We multiply (10) by V^k and integrate from 0 to T. Using the weak continuity of V^k we obtain :

$$\text{(16)} \qquad (a(T) D_s V^k(T), V^k(T)), - (a(0) D_s V^k(0), V^k(0)) -$$

$$- \int_0^T (aD_s V^k, D_s V^k)\, ds + \int_0^T \sum_{i,j=1}^n (a_{ij} D_i V^k, D_j V^k)\, ds +$$

$$\int_0^T (f(V^k), V^k)\, ds + k \int_0^T (MD_s V^k, V^k)\, ds + \text{lower order terms} = 0.$$

The bounds established in (12) and (15) imply that $\int_0^T (f_k(V^k), V^k)\, ds$ is also bounded. Thus we are in a position to apply lemma 1 and we deduce that $f(V)$ is locally integrable and that $f_k(V^k) \longrightarrow f(V)$ in $L^1(D)$ for any bounded set D of B.

Let v denote the restriction of V to Q^*. Then if φ is any testing function with support in Q^*, we have $\int_0^T \int_{\mathbf{R}^n} M D_s V^k \varphi\, dy\, ds = 0$. Taking the limit as $k \longrightarrow \infty$ we will have that V satisfies (7) in $\mathcal{D}'(Q^*)$. Next we show that v satisfies (8). Let S denote the cylinder $\Omega^*(0) \times [0, T]$ which is contained in Q^* because Q^* is monotone increasing. Then of course v satisfies (7) in $\mathcal{D}'(S)$. We may deduce that the restriction of v to S is continuous (as a function of s) in $L^2(\Omega^*(0))$ and that $D_s v$ is continuous in $H^1(\Omega^*(0)) + L^1_{\text{loc}}(\Omega^*(0))$.

The usual integration by partes then implies that $v(y, 0) = v_0(y)$ and $D_s v(y, 0) = v_1(y)$ in $\Omega^*(0)$. Finally, we turn our attention to (9). Our estimate (15 and the fact that $V^k \longrightarrow V$ a.e. implies that $V = 0$ a.e. in $B \backsim Q^*$. Hence by the regularity property of the boundary of Q^* we may deduce that

$$v \in L^\infty(0, T; H_0^1(\Omega^*(s)))$$

which is to say that v satisfies (9) in a generalized sense.

Setting $u(x, t) = v(\Phi(x, t))$ and using the smoothness properties of the mapping J we find that u is a solution to our original problem (1)-(3).

Q.E.D.

PROOF OF LEMMA 2. We set

$$f_k(y, s, v) = k \int_v^{v+\frac{1}{k}} f(y, s, \xi)\, d\xi$$

for $0 \leq v \leq k$ and $-k \leq v \leq -\frac{1}{k}$ Between 0 and $-1/k$ we let f_k be linear and for $|v| \geq k$ we let f_k be the appropriate constante. We have $v f_k(v) \geq 0$ f is continuous so it clear that f_k is uniformly Lipschitz in v. Furthermore $f_k \to f$ uniformly on vounded sets. Now

$$\frac{\partial f_k}{\partial s} = k \int_v^{v+\frac{1}{k}} \frac{\partial f}{\partial s}(y, s, \xi)\, d\xi$$

so that

$$\left|\frac{\partial f_k}{\partial s}\right| \leq k \int_v^{v+\frac{1}{k}} \left|\frac{\partial f}{\partial s}\right| d\xi \leq ck \int_v^{v+\frac{1}{k}} |f|\, ds = c\,|f_k|$$

3. Final Remarks.

The importance of the condition

$$\left|\frac{\partial F}{\partial s}\right| \leq c\,|F|$$

is not yet clear. Thus if Q itself is monotone increasing, we can solve (7)-(9) with, say, $F(x, u) = e^{x_1 u} - 1$ for $u \geq 0$.

However, a change of variable would yield

$$f(y, s, v) = e^{\psi_1(y, s)\, v} - 1$$

with

$$\frac{\partial f}{\partial s} = \left(\frac{\partial v\, \psi_1}{\partial s}\right) e^{\psi_1(y, s)\, v}$$

which would not satisfy our condition.

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