ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

S. ZAIDMAN

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A REMARK ON HARMONIC ANALYSIS OF STRONGLY ALMOST-PERIODIC GROUPS OF LINEAR OPERATORS

S. ZAIDMAN (1)

1. Introduction.

Let us consider a Banach space $\mathcal{X}$, and then take a one parameter group of linear operators $G(t), -\infty < t < \infty \to \mathcal{L}(\mathcal{X}, \mathcal{X})$, which is strongly almost-periodic; this means that for any $x \in \mathcal{X}$, the $\mathcal{X}$-valued function $y(t) = G(t)x$ is (Bochner)-almost-periodic (see [2]).

It is a well-known result (see for example [1]), that for any $\mathcal{X}$-valued almost-periodic function $f(t)$, the mean value

$$\mathcal{M}(e^{-i\lambda t} f(t)) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} e^{-i\lambda\sigma} f(\sigma)\, d\sigma \tag{1.1}$$

exists for any real number λ.

Furthermore, this mean-value equals θ for all λ with the possible exception of a set $(\lambda_n)_{n=1}^{\infty}$ which is finite or countable, and is denoted by $\sigma(f)$.

A natural problem is the following (2):
Is there any strongly almost-periodic one-parameter group $G(t)$, with the property that

$$\bigcup_{x \in \mathcal{X}} \sigma(G(t)x) = \text{real line ?} \tag{1.2}$$

Answering to a letter of us, professor S. Bochner indicated a solution; this will be explained here with some more details.

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(2) It arose in connection with our paper [3].

2. We shall remember here the definition of the space $l^2[-\infty, \infty]$.

It consists of all complex-valued functions $a(\lambda)$, defined for $-\infty < \lambda < \infty$, having the property that

$$\sum_{-\infty<\lambda<\infty} |a(\lambda)|^2 < \infty \tag{2.1}$$

In fact, (2.1) means, by definition, that for a certain constant $c > 0$ we have

$$\sum_{i=1}^{n} |a(\lambda_i)|^2 < c \tag{2.2}$$

whenever arbitrary real λ_j are choosen (and for any $n = 1, 2, 3, \ldots$).

Let us remark that if $a(\lambda) \in l^2[-\infty, \infty]$ there exists a sequence $(\lambda_n)_{n=1}^{\infty}$ depending on $a(\lambda)$, such that $a(\lambda) = 0$ if $\lambda \neq \lambda_j$, $\forall j = 1,2, \ldots$

This follows becouse, if we put $\mathcal{E}_j = \left\{ \lambda \in R^1, |a(\lambda)| > \frac{1}{j} \right\}$ we see that every $\mathcal{E}_j$ is a finite set; hence $\bigcup_{j=1}^{\infty} \mathcal{E}_j = \mathcal{E}$ is a countable set, and if $a(\lambda) \neq 0$ then $\lambda \in \mathcal{E}$.

Let us denote the set $\{\lambda ; a(\lambda) \neq 0\}$ by $Sp\, a(.)$; so $Sp\, a(\cdot) \subset \mathcal{E}$ is a finite or countable set, $(\lambda_n)_1^{\infty}$, and we take a fixed ordering of it.

It can be proved that $l^2[-\infty, \infty]$ is a linear space on the complex field. We can introduce a scalar product on this space; if $a(\lambda), b(\lambda) \in l^2[-\infty, \infty]$, and $(\lambda_n)_1^{\infty} = Sp\, a(\cdot), (\mu_n)_1^{\infty} = Sp\, b(\cdot)$, then by definition $(a(\lambda), b(\lambda))_{l^2[-\infty,\infty]} = \sum_{j=1}^{\infty} a(\lambda_j)\, \overline{b}(\mu_j)$; this sum becomes finite if one of $Sp\, a(\cdot)$ or $Sp\, b(\cdot)$ is finite.

It can be proved in the usual manner that $l^2[-\infty, \infty]$ is a (complete) Hilbert space.

Let us consider now, for any real number t, the map of $l^2[-\infty, \infty]$ into itself which is defined by

$$a(\lambda) \longrightarrow e^{it\lambda} a(\lambda) \tag{2.3}$$

We shall denote this map by G_t; we see that $G_{t_1+t_2} = G_{t_1} G_{t_2}$, $G_0 = I$ for any pair t_1, t_2 of real numbers; here I is the identity operator in l^2.

Furthemore, if $(\lambda_n)_1^{\infty} = Sp\, a(\cdot)$, we have

$$\| G_t a(\cdot) \|_{l^2}^2 = \sum_{j=1}^{\infty} | e^{it\lambda_n} a(\lambda_n) |^2 = \sum_{j=1}^{\infty} | a(\lambda_n) |^2 = \| a(\cdot) \|_{l^2}^2 \tag{2.4}$$

so G_t is an isometric map of l^2, $\forall$ real t.

3. In this part of the paper we prove the following

THEOREM. *The one-parameter group G_t is strongly almost periodic in l^2*

We need for the proof several Lemmas.

Consider, for a given $\lambda_0 \in (-\infty, \infty)$, the function $\varphi_{\lambda_0}(\lambda)$ which equals 1 for $\lambda = \lambda_0$, and equals 0 for $\lambda \neq \lambda_0$. Obviously $\varphi_{\lambda_0}(\lambda) \in l^2$, and $Sp\, \varphi(\cdot) = \{\lambda_0\}$. Now, we have

LEMMA 1. *Let $a(\lambda)$ be given in l^2, and $(\lambda_n)_1^\infty = Sp\, a(\cdot)$. Then we have*

$$(3.1) \qquad a(\lambda) = \sum_{j=1}^{\infty} a(\lambda_j)\, \varphi_{\lambda_j}(\lambda), \text{ the convergence being in } l^2[-\infty, \infty]$$

Let us put in fact $b_N(\lambda) = a(\lambda) - \sum_{n=1}^{N} a(\lambda_n)\, \varphi_{\lambda_n}(\lambda)$.

It can be seen without difficulty that $Sp\, b_N(\cdot) = (\lambda_{N+1}, \lambda_{N+2}, \ldots)$. Hence $\| b_N(\lambda) \|_{l^2}^2 = \sum_{j=1}^{\infty} | b_N(\lambda_{N+j}) |^2$; but $b_N(\lambda_{N+j}) = a(\lambda_{N+j})$; hence

$$\sum_{j=1}^{\infty} | b_N(\lambda_{N+j}) |^2 = \sum_{j=1}^{\infty} | a(\lambda_{N+j}) |^2 = \sum_{k=N+1}^{\infty} | a(\lambda_k) |^2$$

This last expression tends to 0 as $N \to \infty$, because $a(\lambda) \in l^2$. This proves Lemma.

Then we remark the trivial fact that

$$(3.2) \qquad Sp\,(e^{it\lambda} a(\lambda)) = Sp\,(a(\lambda)) \text{ for any real } t.$$

Applying Lemma 1 we obtain that for any real t we have

$$(3.3) \qquad e^{it\lambda} a(\lambda) = \sum_{n=1}^{\infty} e^{it\lambda_n} a(\lambda_n)\, \varphi_{\lambda_n}(\lambda)$$

the convergence being in $l^2[-\infty, \infty]$.

Also we have the simple

LEMMA 2. *Any function $-\infty < t < \infty \to l^2[-\infty, \infty]$ which is given by*

$$(3.4) \qquad h_n(t) = e^{it\lambda_n} a(\lambda_n)\, \varphi_{\lambda_n}(\cdot) \text{ is } l^2\text{-almost-periodic.}$$

This is a particular case of the fact that if $\mathcal{X}$ is a Banach space, $x \in \mathcal{X}$, and α is a real number, the function $-\infty < t < \infty \to \mathcal{X}$, given by $e^{i\alpha t} x$

is $\mathcal{X}$ almost-periodic (in fact it is $\mathcal{X}$-periodic). In our case $\alpha = \lambda_n$, $x = a(\lambda_n)\varphi_{\lambda_n}(\lambda)$, $\mathcal{X} = l^2$.

It follows from (3.3) that $e^{it\lambda}a(\lambda)$ is l^2-almost-periodic, if we prove that the convergence in (3.3) is uniform with respect to $t \in (-\infty, \infty)$. This is done in

LEMMA 3. *The series* $\sum_{n=1}^{\infty} e^{it\lambda_n} a(\lambda_n)\varphi_{\lambda_n}(\lambda)$ *is convergent to* $e^{it\lambda}a(\lambda)$ *in* l^2-*norm, uniformly for* $-\infty < t < \infty$.

Let us consider in fact the difference

$$g_N(\lambda, t) = e^{it\lambda}a(\lambda) - \sum_{n=1}^{N} e^{it\lambda_n} a(\lambda_n)\varphi_{\lambda_n}(\lambda). \tag{3.5}$$

We see that

$$g_N(\lambda, t) = 0 \text{ if } \lambda \notin Sp\, a(\cdot) \text{ or if } \lambda \in [\lambda_1, \lambda_2, \dots, \lambda_N];$$

moreover

$$g_N(\lambda, t) = e^{it\lambda_{N+j}} a(\lambda_{N+j}) \text{ for } \lambda \in [\lambda_{N+1}, \lambda_{N+2}, \dots].$$

Consequently $Sp\, g_N(\cdot, t) = (\lambda_{N+1}, \lambda_{N+2}, \dots)$ and

$$\| g_N(\lambda, t)\|_{l^2}^2 = \sum_{j=1}^{\infty} | e^{it\lambda_{N+j}} a(\lambda_{N+j})|^2 = \sum_{k=N+1}^{\infty} | a(\lambda_k)|^2 \tag{3.6}$$

which tends to 0 as $N \to \infty$, obviously uniformly with respect to $t \in (-\infty, \infty)$. This proves the Theorem.

Let us consider $a(\lambda) = \varphi_{\lambda_0}(\lambda)$ for any fixed $\lambda_0 \in (-\infty, \infty)$. Then

$$G_t\varphi_{\lambda_0}(\lambda) = e^{i\lambda t}\varphi_{\lambda_0}(\lambda) = e^{i\lambda_0 t}\varphi_{\lambda_0}(\lambda) \text{ as easily seen.}$$

This is a l^2-valued periodic function and $\sigma(e^{i\lambda_0 t}\varphi_{\lambda_0}(\cdot)) = \{\lambda_0\}$ Hence $\bigcup_{\lambda_0 \in R^1} \sigma(G_t\varphi_{\lambda_0}(\cdot)) =$ real line R^1 and this solves the problem in the Introduction.

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