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TYPES OF ASSOCIATIVITY INHERITED BY A RING FROM A SPECIAL IDEAL

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ABSTRACT. - Let R be a power-associative ring. An ideal I of R is called *special* if (i) R/I is a finite division ring, and (ii) $x \equiv y \pmod{I}$ implies $x^q = y^q$ where q is the cardinality of R/I and $x, y \in R$. The object of the paper is to invesigate certain associator identities which are inherited by R from a special ideal I where R is power-associative and of characteristic not 2, 3, or 5.

First, it is shown that if I is anti-flexible or nearly antiflexible then R is anti-flexible or nearly anti-flexible, respectively. Next, the additional hypothesis that R is flexible is made. Then it is shown that I being non-commutative Jordan implies that R is non-commutative Jordan. A corollary of this is, that, without assuming that R is flexible, I being Jordan implies that R is Jordan. Returning to the assumption that R is flexible, it is shown that if I is alternative then so is R, and, moreover, if I is one of the recent generalizations of both Jordan and alternative rings, then so is R. The latter is established as a corollary to a very general theorem.

1. Introduction.

Let R be a power-associative ring. An ideal I of R is defined to be *special* provided

(1.1) $$R/I \text{ is a finite division ring}$$

and

(1.2) $x \equiv y \pmod{I}$ implies $x^q = y^q$ where q is the cardinality of R/I and $x, y \in R$.

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Recently the authors, in collaboration with E. C. Johnsen, proved the following [3]:

THEOREM 1.1. *Let R be a power associative ring of characteristic not 2, 3, or 5 and let I be a special ideal of R. Then*

(i) *if I is commutative, then R is commutative;*

(ii) *if I is associative, then R is associative provided R is flexible;*

(iii) *if I is commutative and associative, then R is commutative and associative.*

It is noteworthy to observe that the case $I = (0)$ of Theorem 1.1 (i) yields Wedderburn's Theorem that a finite associative division ring is commutative (except for the mild restriction on the characteristic of R), while the case $I = (0)$ of Theorem 1.1 (iii) yields Albert's Theorem that a finite power associative division ring of characteristic not 2, 3, or 5 is commutative and associative.

The object of this paper is to investigate certain associator identities which are inherited by R from a special ideal I where R is power-associative and of characteristic not 2, 3, or 5. We do this in two stages. First, we show that if I is anti-flexible or nearly anti-flexible then R is anti-flexible or nearly anti-flexible, respectively.

In the next stage, we make the additional hypothesis that R is flexible and prove in this case that if I is non-commutative Jordan, then R is non-commutative Jordan. It turns out that it is a corollary of this that, without assuming that R is flexible, I being Jordan implies that R is Jordan. Returning to the case where R is assumed to be flexible, we establish that if I is alternative then so is R. Naturally, one would expect that if I is one of the recent generalizations of both Jordan and alternative rings ([2], [4]), then so is R in the flexible case. This turns out to be true as will be established as a corollary to a very general theorem.

2. Preliminaries.

The *associator* (x, y, z) is defined by $(x, y, z) = (xy)z - x(yz)$, and the *commutator* (x, y) is defined by $(x, y) = xy - yx$. A ring is *power-associative* if $x^\gamma = x^\alpha x^\beta$ whenever $\alpha + \beta = \gamma$ where α, β, γ are positive integers and x is an element of the ring. A ring is of *characteristic not p* whenever $px = 0$ implies $x = 0$ in the ring. A ring is *flexible* if $(x, y, x) = 0$ holds in the ring. A commutative ring is *Jordan* if $(x^2, y, x) = 0$ holds in the ring. A flexible ring is *non-commutative Jordan* if $(x^2, y, x) = 0$ holds in the ring. A ring is *anti-flexible* if $(x, y, z) = (z, y, x)$ holds, and a ring is *nearly anti-flexible* if $(x, y, y) = (y, y, x)$ holds. A flexible ring is *generalized accessible* if $(z, (x, y, y)) = 0$ and

$$3((x, w), y, z) = -(w, (x, y, z)) - 2(x, (y, z, w)) + 2(y, (z, w, x)) + (z, (w, x, y)).$$

From now on, R will denote a power-associative ring of characteristic not 2, 3, or 5, I will denote a special ideal of R, and q will denote the cardinality of R/I.

Now, let $p = 2, 3$, or 5 and suppose there is an $\bar{x} \in R/I$ such that $p\bar{x} = \bar{0}$. Then $px \in I$ and hence $p^q x^q = 0$ by (1.2). But then $x^q = 0$ since R is of characteristic not p, which implies that $x \in I$ and thus $\bar{x} = \bar{0}$. Therefore R/I is a finite division ring of characteristic not 2, 3, or 5. Thus R/I is a field by Albert's theorem [1].

Since R/I is finite, there exists an element $\bar{\zeta} \in R/I$ which generates $(R/I) \setminus \{0\}$. Consider the coset $\zeta + I = \bar{\zeta}$. If $z \in \zeta + I$, then $z^q = \zeta^q$ since $z \equiv \zeta \pmod{I}$. Let $\xi = z^q = \zeta^q$. Now in R/I we have $\bar{\zeta}^q = \bar{\zeta}$, hence $\zeta^q \in \zeta + I$. Thus $\xi \in \zeta + I$ and therefore, by (1.2), $\xi^q = \xi$. Moreover, $\bar{\xi} = \xi + I$ generates $(R/I) \setminus \{0\}$, and every element in R is of the form $\varepsilon \xi^i + a$, where $a \in I$ and $\varepsilon = 0$ or $\varepsilon = 1$. Such an element ξ will be called a *distinguished element of R.*

LEMMA 2.1. *Let R be a power-associative ring of characteristic not 2, 3, or 5, let I be a special ideal of R, and let ξ be a distinguished element of R. Then for all a, a_1, $a_2 \in I$ and for all positive integers k, l, m,*

$$0 = (a, \xi^k, \xi^l) = (\xi^k, \xi^l, a) = (\xi^k, a, \xi^l), \tag{2.1}$$

$$0 = (a, \xi^m), \tag{2.2}$$

$$0 = (a_1, \xi^m, a_2), \tag{2.3}$$

and

$$0 = (a_1, a_2, \xi^m) - (\xi^m, a_2, a_1). \tag{2.4}$$

Equations (2.1) and (2.2) are established in [3], but we include their proofs since some of the identities obtained therein are needed for (2.3) and (2.4).

PROOF OF LEMMA 2.1.

Let $z \in \xi + I$. Since $\xi = z^q$, the power-associativity of R implies that

$$0 = (z, \xi^k, \xi^l) = (\xi^k, \xi^l, z) = (\xi^k, z, \xi^l) \tag{2.5}$$

and

$$0 = (z, z, \xi^k) = (z, \xi^k, z) = (\xi^k, z, z) \tag{2.6}$$

for all positive integers k, l. Furthermore, $z = \xi + a$ for some $a \in I$, and as z runs over $\xi + I$, a runs over I. Hence by (2.5), power-associativity, and the linearity of the associator, we have (2.1).

Also, using (2.1) we obtain from (2.6) that

$$0 = (a, a, \xi^k) = (a, \xi^k, a) = (\xi^k, a, a) \tag{2.7}$$

for all positive integers k and all $a \in I$. Now, by (1.2), $(a + \xi)^q = \xi^q$ for all $a \in I$, hence

$$0 = ((a + \xi)^q - \xi^q)(a + \xi) - (a + \xi)((a + \xi)^q - \xi^q)$$
$$= a\xi^q - \xi^q a = a\xi - \xi a$$

since $\xi^q = \xi$. Hence

$$0 = (a, \xi) \tag{2.8}$$

for all $a \in I$ where the commutator (x, y) is defined by $(x, y) = xy - yx$. The semi-Jacobi identity

$$(xy, z) - x(y, z) - (x, z)y = (x, y, z) - (x, z, y) + (z, x, y) \tag{2.9}$$

can be easily shown to hold in an arbitrary ring. Using (2.9) with $x = \xi$, $y = \xi^k$, $z = a$, and noting that then all of the associators in (2.9) vanish by (2.1), an easy induction based upon (2.8) yields (2.2).

Next, let b be an element of the subring of R generated by ξ. Then by the linearity of the associator we have by (2.1) and (2.7)

$$0 = (a, b^k, b^l) = (b^k, b^l, a) = (b^k, a, b^l) \tag{2.10}$$

and

$$0 = (a, a, b^k) = (a, b^k, a) = (b^k, a, a) \tag{2.11}$$

for all $a \in I$ and for all positive integers k, l. Linearizing (2.11) we obtain

$$\begin{aligned} 0 &= (a_1, a_2, b^k) + (a_2, a_1, b^k), \\ 0 &= (a_1, b^k, a_2) + (a_2, b^k, a_1), \\ 0 &= (b^k, a_1, a_2) + (b^k, a_2, a_1), \end{aligned} \tag{2.12}$$

for all $a_1, a_2 \in I$ and for all positive integers k.

Let $x = a_1$, $y = a_2$, $z = b^k$ in (2.9). Then by (2.2) and (2.12) we have

$$0 = (a_1, a_2, b^k) + (a_2, b^k, a_1) + (b^k, a_1, a_2) \tag{2.13}$$

In what follows, we will use the Teichmüller identity which holds in an arbitrary ring:

$$0 = T(w, x, y, z) = (wx, y, z) - (w, xy, z) + (w, x, yz)$$
$$- w(x, y, z) - (w, x, y) z.$$

Our next goal is to establish

$$(a_1, b^{2k}, a_2) = 2b^k (a_1, b^k, a_2) \tag{2.14}$$

for all $a_1, a_2 \in I$ and for all positive integers k. Indeed, by (2.13) and (2.12),

$$(a_1 . b^{2k}, a_2) = -(b^{2k}, a_2, a_1) + (a_1, a_2, b^{2k}). \tag{2.15}$$

If we add $0 = T(b^k, b^k, a_2, a_1) - T(a_1, a_2, b^k, b^k)$ to equation (2.15) we obtain

$$(a_1, b^{2k}, a_2) = -(b^k, b^k a_2, a_1) - b^k (b^k, a_2, a_1)$$
$$+ (a_1, a_2 b^k, b^k) + (a_1, a_2, b^k) b^k$$

upon using (2.10). But by (2.12) and (2.2),

$$(b^k, b^k a_2, a_1) = -(b^k, a_1, a_2 b^k)$$

and

$$(a_1, a_2, b^k) b^k = -b^k (a_2, a_1, b^k),$$

hence we have

$$(a_1, b^{2k}, a_2) = (b^k, a_1, a_2 b^k) + (a_1, a_2 b^k, b^k)$$
$$- b^k (b^k, a_2, a_1) - b^k (a_2, a_1, b^k).$$

Application of (2.13) yields

$$(a_1, b^{2k}, a_2) = -(a_2 b^k, b^k, a_1) + b^k (a_1, b^k, a_2),$$

but then we have

$$(a_1, b^{2k}, a_2) = -(b^k a_2, b^k, a_1) + b^k (a_1, b^k, a_2) \tag{2.16}$$

by (2.2). Adding $0 = T(b^k, a_2, b^k, a_1)$ to (2.16) and applying (2.2) and (2.10) and (2.12) we get

$$(2.17) \qquad (a_1, b^{2k}, a_2) = -(b^k, b^k a_2, a_1) - (b^k, b^k a_1, a_2) + 2b^k (a_1, b^k, a_2).$$

Next, we subtract $0 = T(b^k, b^k, a_2, a_1) + T(b^k, b^k, a_1, a_2)$ from (2.17) and use (2.10) to obtain

$$(a_1, b^{2k}, a_2) = -(b^{2k}, a_2, a_1) - (b^{2k}, a_1, a_2) + b^k (b^k, a_2, a_1) + b^k (b^k, a_1, a_2) + 2b^k (a_1, b^k, a_2)$$

from which (2.14) follows by (2.12).

Now, we expand $(a_1, (\xi^l + \xi^{l+1})^2, a_2)$ in two ways where $a_1, a_2 \in I$. First, let $b = \xi^l + \xi^{l+1}$ in (2.14) with $k = 1$. Then

$$(a_1, (\xi^l + \xi^{l+1})^2, a_2) = 2(\xi^l + \xi^{l+1})(a_1, \xi^l + \xi^{l+1}, a_2).$$

Next,

$$(a_1, (\xi^l + \xi^{l+1})^2, a_2) = (a_1, \xi^{2l} + \xi^{2l+1} + \xi^{2l+2}, a_2).$$

Now, apply (2.14) with $b = \xi$ and $k = l$ and then $k = l + 1$ to obtain

$$(a_1, (\xi^l + \xi^{l+1})^2, a_2) = 2\xi^l (a_1, \xi^l, a_2) + 2\xi^{l+1} (a_1, \xi^{l+1}, a_2) + 2(a_1, \xi^{2l+1}, a_2).$$

Comparing the two expansions of $(a_1, (\xi^l + \xi^{l+1})^2, a_2)$, and recalling that R is of characteristic not 2, we get

$$(2.18) \qquad (a_1, \xi^{2l+1}, a_2) = \xi^l (a_1, \xi^{l+1}, a_2) + \xi^{l+1} (a_1, \xi^l, a_2)$$

for all $a_1, a_2 \in I$ and for all positive integers l. By an easy induction, using (2.14) with $b = \xi$ and (2.18), we get

$$(2.19) \qquad (a_1, \xi^{m+1}, a_2) = (m+1)\xi^m (a_1, \xi, a_2) = \xi^m (a_1, (m+1)\xi, a_2)$$

for all $a_1, a_2 \in I$ and for all positive integers m.

Let $m + 1 = q$ in (2.19), to get

$$(2.20) \qquad (a_1, \xi^q, a_2) = \xi^{q-1} (a_1, q\xi, a_2).$$

Now, let p be the characteristic of the finite field R/I, then, recalling (1.1), we have $q = p^\alpha$ for some integer $\alpha \geq 1$. Since $p\xi \in I$, it follows by (1.2) that $(p\xi)^q = 0$, and hence $p^q \xi^q = 0$. But $\xi^q = \xi$ (see fourth paragraph of this section), and hence $p^q \xi = 0$. Therefore (2.20) now reduces to

$$(2.21) \qquad (a_1, \xi, a_2) = \xi^{q-1}(a_1, p^\alpha \xi, a_2).$$

Now, if $p^\alpha \xi = 0$, then $(a_1, \xi, a_2) = 0$. Next, suppose $p^\alpha \xi \neq 0$. Since, by above, $p^q \xi = 0$, there is a least positive integer σ such that $(a_1, p^\sigma \xi, a_2) = 0$; clearly $\sigma \leq q = p^\alpha$. Now, if $\sigma > 1$, then (2.21) yields

$$(a_1, p^{\sigma-1} \xi, a_2) = p^{\alpha-1} \xi^{q-1}(a_1, p^\sigma \xi, a_2) = 0,$$

contradicting the minimality of σ. Hence $\sigma = 1$. But then $(a_1, p\xi, a_2) = 0$, and hence $(a_1, p^\alpha \xi, a_2) = 0$. Therefore, by (2.21), $(a_1, \xi, a_2) = 0$. We have thus proved that, in any case,

$$(2.22) \qquad (a_1, \xi, a_2) = 0.$$

Using $0 = T(a_1, \xi, \xi^k, a_2)$ and (2.1), an easy induction based upon (2.22) yields (2.3). Finally (2.13), (2.12), and (2.3) yield (2.4).

3. Anti-flexible and nearly anti-flexible ideals.

THEOREM 3.1. *Let R be a power-associative ring of characteristic not* 2, 3 *or* 5, *and let I be a special ideal of R.* (i) *If I is antiflexible then R is anti-flexible, and* (ii) *if I is nearly anti-flexible then R is nearly anti-flexible.*

PROOF. Let ξ be a distinguished element of R. Then every element of R is of the form $a + \varepsilon \xi^j$ where $a \in I$, $\varepsilon = 0$ or $\varepsilon = 1$.

To prove (i), we wish to show that $(x, y, z) = (z, y, x)$ for all $x, y, z \in R$. Suppose $x = a_1 + \varepsilon_1 \xi^{j_1}$, $y = a_2 + \varepsilon_2 \xi^{j_2}$, $z = a_3 + \varepsilon_3 \xi^{j_3}$ where $a_i \in I$, $\varepsilon_i = 0$ or $\varepsilon_i = 1$. Then

$$(x, y, z) = (a_1 + \varepsilon_1 \xi^{j_1}, a_2 + \varepsilon_2 \xi^{j_2}, a_3 + \varepsilon_3 \xi^{j_3})$$

$$= (a_1, a_2, a_3) + (a_1, a_2, \varepsilon_3 \xi^{j_3}) + (\varepsilon_1 \xi^{j_1}, a_2, a_3)$$

by (2.1), (2.3) and the power-associativity of R. But by (2.4),

$$(a_1, a_2, \varepsilon_3 \xi^{j_3}) = (\varepsilon_3 \xi^{j_3}, a_2, a_1)$$

and

$$(\varepsilon_1 \xi^{j_1}, a_2, a_3) = (a_3, a_2, \varepsilon_1 \xi^{j_1}),$$

and

$$(a_1, a_2, a_3) = (a_3, a_2, a_1)$$

since I is anti-flexible. Hence

$$(x, y, z) = (a_3, a_2, a_1) + (\varepsilon_3 \xi^{j_3}, a_2, a_1) + (a_3, a_2, \varepsilon_1 \xi^{j_1})$$

$$= (a_3 + \varepsilon_3 \xi^{j_3}, a_2 + \varepsilon_2 \xi^{j_2}, a_1 + \varepsilon_1 \xi^{j_1})$$

$$= (z, y, x)$$

by (2.1), (2.3) and the power-associativity of R.

To establish (ii), set $z = y$ in the proof of (i).

4. Flexible rings.

LEMMA 4.1. *Let R be a flexible power associative ring of characteristic not 2, 3, or 5, let I be a special ideal of R, and let ξ be a distinguished element of R. Then for all $a_1, a_2 \in I$ and for all positive integers m,*

$$0 = (a_1, a_2, \xi^m) = (\xi^m, a_2, a_1). \tag{4.1}$$

PROOF. Linearizing $(x, y, x) = 0$ yields $(x, y, z) + (z, y, x) = 0$ which applied to (2.4) gives (4.1), since R is of characteristic not 2.

Equations (2.1), (2.2), (2.3), and (4.1) allow us to successfully study a large class of flexible rings. First we turn our attention to the Jordan case.

THEOREM 4.1. *Let R be a flexible power-associative ring of characteristic not 2, 3, or 5, and let I be a special ideal of R. If I is non-commutative Jordan, then R is non commutative Jordan.*

PROOF. Let ξ be a distinguished element of R. Then every element in R is of the form $a + \varepsilon \xi^j$ where $a \in I$, $\varepsilon = 0$ or $\varepsilon = 1$. We wish to show that $(x^2, y, x) = 0$ for all $x, y \in R$. Let $x = a_1 + \varepsilon_1 \xi^{i_1}$, $y = a_2 + \varepsilon_2 \xi^{i_2}$. Then, using (2.2),

$$(x^2, y, x) = ((a_1 + \varepsilon_1 \xi^{i_1})^2, a_2 + \varepsilon_2 \xi^{i_2}, a_1 + \varepsilon_1 \xi^{i_1})$$

$$= (a_1^2 + 2\varepsilon_1 \xi^{i_1} a_1 + \varepsilon_1 \xi^{2i_1}, a_2 + \varepsilon_2 \xi^{i_2}, a_1 + \varepsilon_1 \xi^{i_1})$$

$$= 2\varepsilon_1 (\xi^{i_1} a_1, a_2, a_1).$$

In establishing the last equality, we used the facts that R is power-associative and I is noncommutative Jordan, and equations (2.1), (2.3), and

(4.1). Now,

$$\begin{aligned} 0 &= T(\xi^{i_1}, a_1, a_2, a_1) \\ &= (\xi^{i_1} a_1, a_2, a_1) - (\xi^{i_1}, a_1 a_2, a_1) + (\xi^{i_1}, a_1, a_2 a_1) \\ &\quad - \xi^{i_1}(a_1, a_2, a_1) - (\xi^{i_1}, a_1, a_2) a_1 \\ &= (\xi^{i_1} a_1, a_2, a_1) \end{aligned}$$

using (4.1) and the fact that R is flexible. Hence $(x^2, y, x) = 0$ which completes the proof.

COROLLARY 4.1. *Let R be a power-associative ring of characteristic not 2, 3, or 5, and let I be a special ideal of R. If I is Jordan, then R is Jordan.*

PROOF. Since I is Jordan, I is commutative. Hence (2.2) together with the linearity of the commutator implies that R is commutative. But then R is flexible and it remains only to apply Theorem 4.1.

Next, we wish to study the alternative case and hence those recent generalizations which include both the alternative and Jordan cases. We do that by establishing an extremely general result.

First, we make the following definitions. Let R be a ring and let $F(X)$ be a free nonassociative ring generated by X which has R as a homomorphic image. Define $J \subset F(X)$ inductively as follows:

$$(x_1, x_2) \in J \quad \text{and} \quad (x_1, x_2, x_3) \in J \quad \text{for all} \quad x_1, x_2, x_3 \in X.$$

If $f(x_1, \dots, x_r) \in J$, then $f(y_1, \dots, y_r) \in J$ where $y_i = x_i$ or $y_i = (x_s, x_t)$ or $y_i = (x_s, x_t, x_u)$ where $x_1, \dots, x_r, x_s, x_t, x_u \in X$. Then define K to be the subring of $F(X)$ generated by J.

THEOREM 4.2. *Let R be a flexible power associative ring of characteristic not 2, 3, or 5, let I be a special ideal of R, and let $f \in K$. If $f = 0$ is an identity in I, then $f = 0$ is an identity in R.*

PROOF. Let $g(x_1, .., x_r) \in K$ where $x_1, \dots, x_r \in X$ and let $g(b_1, \dots, b_r)$ be an evaluation of g in R. Let ξ be a distinguished element of R. Then $b_i = a_i + \varepsilon_i \xi^{j_i}$ where $a_i \in I$, $\varepsilon_i = 0$ or $\varepsilon_i = 1$ for $i = 1, \dots, r$. We will show that

$$g(b_1, \dots, b_r) = g(a_1, \dots, a_r) \tag{4.2}$$

from which Theorem 4.2 follows immediately. Now,

$$(a_1 + \varepsilon_1 \xi^{j_1}, a_2 + \varepsilon_2 \xi^{j_2}) = (a_1, a_2) \tag{4.3}$$

by (2.2) and the power-associativity of R. Moreover,

$$(a_1 + \varepsilon_1 \xi^{j_1}, a_2 + \varepsilon_2 \xi^{j_2}, a_3 + \varepsilon_3 \xi^{j_3}) = (a_1, a_2, a_3) \tag{4.4}$$

by (2.1), (2.3), (4.1) and the power associativity of R. But then (4.2) follows immediately from (4.3) and (4.4) by the definition of K.

Theorem 4.2 takes care of many classes of rings. We call the reader's attention to three of them and leave it to the reader's imagination to consider others. Note that the noncommutative Jordan case is not included in Theorem 4.2.

COROLLARY 4.2. *Let R be a flexible power-associative ring of characteristic not* 2, 3, *or* 5 *and let I be a special ideal of R.*

(i) *If I is alternative, then R is alternative.*

(ii) *If I is generalized accessible, then R is generalized accessible.*

(iii) *If $(x, x, (y, z)) = ((y, z), x, x) = 0$ in I, then $(x, x, (y, z)) = ((y, z), x, x)$ in R.*

PROOF. (x_1, x_2, x_2), (x_2, x_2, x_1), $(x_3, (x_2, x_1, x_1))$,

$3((x_1, x_4), x_2, x_3) + (x_4, (x_1, x_2, x_3)) + 2(x_1, (x_2, x_3, x_4)) -$

$2(x_2, (x_3, x_4, x_1)) - (x_3, (x_4, x_1, x_2))$, $(x_1, x_1, (x_2, x_3))$,

and $((x_2, x_3), x_1, x_1)$ are all elements or K for all $x_1, x_2, x_3, x_4 \in X$.

The rings of (ii) and (iii) are each classes of rings which include both alternative and Jordan rings and were introduced by Kleinfeld, Humm Kleinfeld, and Kosier [4] and R Block [2], respectively.

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