ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3e série, tome 26, nº 2 (1972), p. 391-402
<http://www.numdam.org/item?id=ASNSP_1972_3_26_2_391_0>

ON THE HÖLDER CONTINUITY OF SOLUTIONS OF SECOND ORDER ELLIPTIC EQUATIONS IN TWO VARIABLES

by L. C. PICCININI - S. SPAGNOLO (0)

1. Introduction.

Let U be a fixed open set in R^n; we consider the differential operator

$$E = \sum_{ij\,1}^{n} D_i\,(a_{ij}(x)\,D_j) \tag{1}$$

whose coefficients are real measurable functions, defined on U, satisfying the following conditions:

$$\begin{cases} a_{ij}(x) = a_{ji}(x) \\ \lambda\,|\,\xi\,|^2 \leq \sum_{ij\,1}^{n} a_{ij}(x)\,\xi_i\,\xi_j \leq \Lambda\,|\,\xi\,|^2 \end{cases} \tag{2}$$

for every x in U and $\xi = (\xi_1, \dots, \xi_n)$ in R^n, where $\lambda > 0$. By $L = \Lambda/\lambda$ we denote the *ellipticity coefficient* of the operator E.

It is well known (see [2], [7]) that every function u, belonging to $H^1_{\text{loc}}(U)$, which satisfies on U the equation $Eu = 0$ is Hölder continuous; namely for every compact subset K of U and for all x, y in K, we get the estimate

$$|\,u(x) - u(y)\,| \leq C\,|\,x - y\,|^{\alpha} \left[\int_U u^2\,dx\right]^{1/2} \tag{3}$$

where C and α are positive constants depending only on L, n and dist $(K, \partial U)$.

Pervenuto alla Redazione il 26 Aprile 1971.

(0) Lavoro eseguito nell'ambito dei gruppi di ricerca del C. N. R. — 1971.

It is easy to see, by means of a homothety on R^n, that α, the Hölder continuity exponent, actually does not depend on $\operatorname{dist}(K, \partial U)$. Therefore, if we denote by $\alpha(L, n)$ the greatest lower bound of the values of α (when u varies in the class of all the solutions of $Eu = 0$ for some E of the type (1), (2)), we get $\alpha(L, n) > 0$.

In the particular case $n = 2$, Morrey (who, already in [5], had proved the Hölder continuity of solutions for $n = 2$) got in [6] the estimate

$$\alpha(L, 2) \geq \frac{1}{2L} .$$

Recently ([10]) Widman has proved that

$$\alpha(L, 2) \geq \frac{1}{8\sqrt{L}} ,$$

which improves the previous estimate for $L \geq 16$. On the other side, Meyers' example in [4], that we report later (example 1), implies that necessarily

$$\alpha(L, 2) \leq \frac{1}{\sqrt{L}} .$$

In this paper we show (theorem 1) that it is exactly

$$\alpha(L, 2) = \frac{1}{\sqrt{L}} .$$

For an important class of operators of type (1), (2), the *isotropic operators* (that is operators in which $a_{ij} = 0$ for $i \neq j$, $a_{jj} = a$), we prove later (theorem 2 and example 2) that the lowest Hölder continuity exponent of solutions is

$$\frac{4}{\pi} \operatorname{arctg} \frac{1}{\sqrt{L}} ,$$

a number larger than $\dfrac{4}{\pi\sqrt{L}}$, hence strictly larger than $\dfrac{1}{\sqrt{L}}$.

Passing then to the case $n \geq 3$, we shall show by an example similar to Meyers' (example 3) that

$$\alpha(L, n) \leq \frac{2}{L} .$$

2. The regularity theorem.

Theorem 1. *Let U be an open subset of R^2, and define $E = \sum_{ij}^{2} {}_{1} D_i(a_{ij}(x) D_j)$, where a_{ij} are real measurable functions on U such that $a_{ij} = a_{ji}$ and* $\lambda |\xi|^2 \leq$

$\leq \sum_{1}^{2}{}_{ij}\, a_{ij}(x)\, \xi_i\, \xi_j \leq \Lambda\, |\, \xi\,|^2$ *for all* ξ *in* R^2, *with* $\lambda > 0$. *Let* u *be a solution of the equation* $Eu = 0$ *on* U *which belongs to* $H^1_{\mathrm{loc}}(U)$. *Then for every compact subset* K *of* U, *and for all* x, y *in* K *the following estimate holds*

$$(4) \qquad |\, u(x) - u(y)\,| \leq C \cdot |\, x - y\,|^{1/\sqrt{L}} \left[\int_U u^2\, dx\right]^{1/2},$$

where $L = \Lambda/\lambda$ *and* C *is a constant depending only on* L *and* $\operatorname{dist}(K, \partial U)$.

Proof. Denoting the matrix of the coefficients of E by $A(x) = (a_{ij}(x))$, and setting $\nabla u = (D_1 u, D_2 u)$, we can write $E = \operatorname{div}(A(x)\nabla)$. Now, it is well known (see for example [1], [9]) that in order to get (4) it is enough to prove, for every x^0 in U, the inequality:

$$(5) \qquad r^{-2/\sqrt{L}}\left(\int_{|x-x^0|\leq r} |\nabla u|^2\, dx\right) \leq C \cdot \delta^{-2/\sqrt{L}}\left(\int_{|x-x^0|\leq \delta} |\nabla u|^2\, dx\right)$$

for every r, δ for which $0 < r \leq \delta < \operatorname{dist}(x^0, \partial U)$, where C is a constant depending only on L. For each x^0 fixed in U, let

$$g(r) = \int_{|x-x^0|\leq r} \langle A(x)\nabla u(x), \nabla u(x)\rangle\, dx;$$

we shall actually prove that the function $r^{-2/\sqrt{L}} g(r)$ is increasing for $r < \operatorname{dist}(x^0, \partial U)$; this fact implies immediately that (5) holds with $C = L$.

Now we remark that for every real constant k we have:

$$\operatorname{div}((u-k)\cdot A\nabla u) = (u-k)\cdot \operatorname{div}(A\nabla u) + \langle A\nabla(u-k), \nabla u\rangle = \langle A\nabla u, \nabla u\rangle .$$

Thus, making use of Green's formula (1), we can write

$$g(r) = \int_{B(r)} \operatorname{div}((u-k)\, A\nabla u)\, dx = \int_{S(r)} [(u-k)\langle A\nabla u, \underline{n}\rangle]\, d\sigma,$$

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(1) In order to use correctly Green's formula, it is convenient to reduce the problem to the case in which the coefficients of E are regular functions; this can be done in the following way: we construct, using convolution product, a sequence of matrices $A_k(x) = (a_{ij,k}(x))$, satisfying conditions (2), such that $a_{ij,k}$ are functions of class C^∞, converging to a_{ij} in $L^1_{\mathrm{loc}}(U)$ as k tends to infinity. Then, calling u_k the solution of the Dirichlet problem:

where $B(r)$ is the ball $|x - x^0| \leq r$, $S(r) = \partial B(r)$, $d\sigma$ is the one dimensional measure on $S(r)$, and $\underline{n}\; n(x)$ is the exterior normal unity vector to $S(r)$ at the point x. Now we introduce polar coordinates, namely

$$x_1 = x_1^0 + \varrho \cos \vartheta$$

$$x_2 = x_2^0 + \varrho \sin \vartheta$$

and the orthogonal matrix

$$J(\vartheta) = \begin{pmatrix} \cos \vartheta & -\sin \vartheta \\ \sin \vartheta & \cos \vartheta \end{pmatrix}.$$

We get then

$$\underline{n}(x) = J(\vartheta)\, \underline{e}_1, \qquad \text{where } \underline{e}_1 \text{ is the vector } (1,0),$$

$$\nabla u(x) = J(\vartheta)\, \overline{\nabla} u(x), \quad \text{where } \overline{\nabla} u = \left(\frac{\partial u}{\partial \varrho}, \varrho^{-1} \frac{\partial u}{\partial \vartheta}\right),$$

so that, setting $J^*(\vartheta)\, A(x)\, J(\vartheta) = P(x) = (p_{ij}(x))$,

$$g(r) = \int_{S(r)} [(u - k) \langle P \overline{\nabla} u, \underline{e}_1 \rangle]\, d\sigma = \int_{S(r)} (u - k) \left(p_{11} \frac{\partial u}{\partial \varrho} + p_{12}\, \varrho^{-1} \frac{\partial u}{\partial \vartheta}\right) d\sigma.$$

Now, since the symmetric matrix $P(x)$ has the same eigenvalues as $A(x)$, from (2) one gets the following estimates

$$\lambda \leq p_{11}(x) \leq \Lambda \tag{6}$$

$$\lambda \leq p_{22}(x) - \frac{p_{12}^2(x)}{p_{11}(x)} \leq \Lambda. \tag{7}$$

Therefore, by Schwarz's inequality and by (6), we get

$$\begin{cases} \operatorname{div}(A_k(x) \nabla u_k) = 0 \quad \text{on} \quad B_\delta = \{x \in U : |x - x^0| \leq \delta\} \\ u_k - u \in H_0^1(B_\delta), \end{cases}$$

it is easily seen (e.G. [8], § 4) that $\{u_k\}$ tends to u in $H^1(B_\delta)$; hence it is enough to prove (5) for any u_k and then take the limit as k tends to infinity.

$$g(r) = \int_{S(r)} \sqrt{p_{11}}\,(u-k)\cdot\left(\sqrt{p_{11}}\,\frac{\partial u}{\partial \varrho} + \frac{p_{12}}{\sqrt{p_{11}}}\cdot\varrho^{-1}\frac{\partial u}{\partial \vartheta}\right) d\sigma$$

$$\leq \left[\left(\int_{S(r)} p_{11}\,(u-k)^2\, d\sigma\right)\cdot\left(\int_{S(r)}\left(\sqrt{p_{11}}\,\frac{\partial u}{\partial \varrho} + \frac{p_{12}}{\sqrt{p_{11}}}\,\varrho^{-1}\frac{\partial u}{\partial \vartheta}\right)^2 d\sigma\right)\right]^{1/2}.$$

Now, for any assigned r, we can give to the constant k exactly the value $k = (2\pi r)^{-1}\left(\int_{S(r)} u\, d\sigma\right)$, hence using Wirtinger's inequality (²), and remembering (6) and (7), we have:

$$\int_{S(r)} p_{11}\,(u-k)^2\, d\sigma \leq \Lambda \int_{S(r)} (u-k)^2\, d\sigma \leq \Lambda \int_{S(r)} \left(\frac{\partial u}{\partial \vartheta}\right)^2 d\sigma$$

$$\leq \frac{\Lambda}{\lambda}\, r^2 \left(\int_{S(r)} \left(p_{22} - \frac{p_{12}^2}{p_{11}}\right)\varrho^{-2}\left(\frac{\partial u}{\partial \vartheta}\right)^2 d\sigma\right).$$

Then, since $\sqrt{|ab|} \leq \frac{1}{2}\,(|a| + |b|)$,

$$g(r) \leq \frac{\sqrt{L}}{2}\cdot r \int_{S(r)} \left[\left(p_{22} - \frac{p_{12}^2}{p_{11}}\right)\varrho^{-2}\left(\frac{\partial u}{\partial \vartheta}\right)^2 + \right.$$

$$\left. + \left(\sqrt{p_{11}}\,\frac{\partial u}{\partial \varrho} + \frac{p_{12}}{\sqrt{p_{11}}}\,\varrho^{-1}\frac{\partial u}{\partial \vartheta}\right)^2\right] d\sigma$$

$$= \frac{\sqrt{L}}{2}\cdot r \int_{S(r)} \left[p_{11}\left(\frac{\partial u}{\partial \varrho}\right)^2 + 2p_{12}\,\varrho^{-1}\frac{\partial u}{\partial \varrho}\frac{\partial u}{\partial \vartheta} + p_{22}\,\varrho^{-2}\left(\frac{\partial u}{\partial \vartheta}\right)^2\right] d\sigma$$

$$= \frac{\sqrt{L}}{2}\cdot r \int_{S(r)} \langle P\,\bar{\nabla} u, \bar{\nabla} u\rangle\, d\sigma = \frac{\sqrt{L}}{2}\cdot r \int_{S(r)} \langle A\,\nabla u, \nabla u\rangle\, d\sigma.$$

(²) For any function $w(t)$ periodic of period 2π such that $\int_0^{2\pi} w(t)\, dt = 0$, the following inequality holds $\int_0^{2\pi} w^2(t)\, dt \leq \int_0^{2\pi} [w'(t)]^2\, dt$. (See [3], § 258).

On the other side from the equality

$$g(r) = \int_{B(r)} \langle A \nabla u, \nabla u \rangle \, dx = \int_0^r dt \int_{S(t)} \langle A \nabla u, \nabla u \rangle \, d\sigma,$$

it follows that

$$g'(r) = \int_{S(r)} \langle A \nabla u, \nabla u \rangle \, d\sigma,$$

so that the estimate we have already obtained becomes

$$g(r) \leq \frac{\sqrt{L}}{2} \cdot r \cdot g'(r). \tag{8}$$

From (8) we get

$$\frac{d}{dr}[lg(r^{-2/\sqrt{L}} g(r))] = -\frac{2}{\sqrt{L}} \cdot \frac{1}{r} + \frac{g'(r)}{g(r)} \geq 0.$$

Hence the function $lg(r^{-2/\sqrt{L}} g(r))$ is increasing, so that the theorem follows.

EXAMPLE 1. (Meyers, [4] page 204).

Let E be the operator defined in polar coordinates by

$$E = L \cdot \frac{1}{\varrho} \frac{\partial}{\partial \varrho}\left(\varrho \frac{\partial}{\partial \varrho}\right) + \frac{1}{\varrho^2} \frac{\partial^2}{\partial \vartheta^2},$$

and hence in cartesian coordinates:

$$E = \sum_{1}^{2}{}_{ij} D_i (a_{ij} D_j) \quad \text{with} \quad a_{11} = (Lx_1^2 + x_2^2) \, |x|^{-2}$$

$$a_{12} = a_{21} = (L - 1) \, x_1 x_2 \, |x|^{-2}, \qquad a_{22} = (x_1^2 + Lx_2^2) \, |x|^{-2};$$

let $u(x) = |x|^{1/\sqrt{L}} \dfrac{x_1}{|x|} = \varrho^{1/\sqrt{L}} \cos \vartheta$. Then E has ellipticity coefficient L and u is a solution of $Eu = 0$, whose Hölder continuity exponent is $\dfrac{1}{\sqrt{L}}$.

3. Isotropic operators.

THEOREM 2. *Let U be an open subset of R^2, and define $E = \sum_{1}^{2}{}_i D_i (a(x) D_i)$, where $a(x)$ is a real measurable function on U, such that $\lambda \leq a(x) \leq \Lambda$, with $\lambda > 0$; let u be a solution, in $H^1_{\text{loc}}(U)$, of the equation $Eu = 0$ on U.*

Then for every compact subset K of U and for all x, y in K the following estimate holds :

$$|u(x)-u(y)| \leq C \cdot |x-y|^{\alpha}\left(\int_U |u|^2 \, dx\right)^{1/2}. \tag{9}$$

where $\alpha = \frac{4}{\pi} \operatorname{arctg} \frac{1}{\sqrt{L}}$, $L = \Lambda/\lambda$ *and* C *is a constant depending only on* L *and* $\operatorname{dist}(K, \partial U)$.

PROOF. Using the same method and the same notations as in the proof of theorem 1 (remembering that, since $A(x) = a(x) \cdot I$, $P(x) = A(x)$), we get the expression

$$g(r) = \int_{S(r)} \left[a(x)(u(x)-k) \cdot \frac{\partial u}{\partial \varrho}(x)\right] d\sigma ;$$

hence, by Schwarz's inequality :

$$g(r) \leq \left\{\int_{S(r)} [a(u-k)^2] \, d\sigma \cdot \int_{S(r)} \left[a\left(\frac{\partial u}{\partial \varrho}\right)^2\right] d\sigma\right\}^{1/2}.$$

At this point, instead of using Wirtinger's inequality, we use the following one, that will be proved in lemma 1 :

$$\int_{S(r)} a(x) \, |u(x)-k|^2 \, d\sigma \leq \left(\frac{4}{\pi} \operatorname{arctg} \frac{1}{\sqrt{L}}\right)^{-2} \cdot \int_{S(r)} a(x) \left[\left(\frac{\partial u}{\partial \vartheta}\right)(x)\right]^2 d\sigma$$

where $k = (2\pi r)^{-1} \int_{S(r)} a(x) \, u(x) \, d\sigma$. We get then, as in the proof of theorem 1,

$$g(r) \leq \frac{1}{2} \cdot \left(\frac{4}{\pi} \operatorname{arctg} \frac{1}{\sqrt{L}}\right)^{-1} \cdot r \cdot g'(r), \qquad \text{and hence (9).}$$

LEMMA 1. *Let $a(t)$ be a real measurable function, periodic of period 2π, such that $1 \leq a(t) \leq L$; let $w(t)$ be a function, belonging to $H^1_{\mathrm{loc}}(-\infty, +\infty)$, periodic of period 2π, such that $\int_0^{2\pi} a(t) \, w(t) \, dt = 0$. Then the following inequality holds :*

$$\int_0^{2\pi} a(t) \, w^2(t) \, dt \leq \left(\frac{4}{\pi} \operatorname{arctg} \frac{1}{\sqrt{L}}\right)^{-2} \int_0^{2\pi} a(t) \, [w'(t)]^2 \, dt. \tag{10}$$

This becomes an equality only if $a(t) = a_0(t + \Phi)$, $w(t) = C \cdot w_0(t + \Phi)$ *where* C *and* Φ *are real constants and*

$$a_0(t) = \begin{cases} 1 & \text{for } 0 \leq t < \frac{\pi}{2}, \quad \pi \leq t < \frac{3}{2}\pi \\ L & \text{for } \frac{\pi}{2} \leq t < \pi, \quad \frac{3}{2}\pi \leq t < 2\pi \end{cases} \tag{11}$$

$$w_0(t) = \begin{cases} \sin\left[\sqrt{\lambda}\left(t - \frac{\pi}{4}\right)\right] & \text{for } 0 \leq t \leq \frac{\pi}{2} \\ \frac{1}{\sqrt{L}} \cos\left[\sqrt{\lambda}\left(t - \frac{3}{4}\pi\right)\right] & \text{for } \frac{\pi}{2} \leq t \leq \pi \\ -\sin\left[\sqrt{\lambda}\left(t - \frac{5}{4}\pi\right)\right] & \text{for } \pi \leq t \leq \frac{3}{2}\pi \\ -\frac{1}{\sqrt{L}} \cos\left[\sqrt{\lambda}\left(t - \frac{7}{4}\pi\right)\right] & \text{for } \frac{3}{2}\pi \leq t \leq 2\pi \end{cases} \tag{12}$$

where $\lambda = \left(\frac{4}{\pi} \operatorname{arctg} \frac{1}{\sqrt{L}}\right)^2$.

PROOF. Consider for any $a(t)$ such that $1 \leq a(t) \leq L$, the eigenvalue problem

$$\begin{cases} (aw')' + \lambda a w = 0 \\ w \text{ periodic of period } 2\pi. \end{cases} \tag{13}$$

It is easy to prove that the λ's for which this problem has not constant solutions build a countable sequence $\{\lambda_n\}$ with $0 < \lambda_1 < \lambda_2 < \dots$, and that for any function $w(t)$, periodic of period 2π, such that $\int_0^{2\pi} a(t)\, w(t)\, dt = 0$, the following estimate holds

$$\int_0^{2\pi} (a \cdot w^2)\, dt \leq \frac{1}{\lambda_1} \cdot \int_0^{2\pi} [a \cdot (w')^2]\, dt.$$

Therefore, in order to prove (10) it is sufficient to show that, if $\lambda \neq 0$ and $w(t) \not\equiv 0$ satisfy (13), then necessarily

$$\lambda \geq \left(\frac{4}{\pi} \operatorname{arctg} \frac{1}{\sqrt{L}}\right)^2.$$

It is earily seen that a solution of (13) in each period has at least two zero's, and that between any couple of zero's of the function there is one and only one zero of its derivative. Let t_0, t_2, t_4 be three consecutive zero's of w and let t_1 and t_3 be two zero's of w' in such a way that $t_0 < t_1 < t_2 < t_3 < t_4$; without loss of generality we may suppose that $w(t_1) > 0$ and $w(t_3) < 0$. It is obvious that

(14) $$t_4 - t_0 \leq 2\pi.$$

We define, for $t_0 < t \leq t_1$, the function

(15) $$f(t) = \frac{a(t)\, w'(t)}{w(t)};$$

according to (13) this function satisfies the following first order differential equation:

(16) $$f'(t) = -\lambda\, a(t) - \frac{f^2(t)}{a(t)}.$$

We remark that, since $f' < 0$, f is strictly decreasing and further $\lim_{t \to t_0^+} f(t) = +\infty$, $f(t_1) = 0$; there is one and only one point, say τ, in the interval (t_0, t_1) such that $f(\tau) = \sqrt{\lambda L}$. Since f is decreasing, the following inequalities hold:

$$\begin{cases} -\lambda\, a(t) - \dfrac{f^2(t)}{a(t)} \geq -\lambda - f^2(t) & \text{for } t_0 < t \leq \tau \\ -\lambda\, a(t) - \dfrac{f^2(t)}{a(t)} \leq -\lambda L - \dfrac{f^2(t)}{L} & \text{for } \tau \leq t \leq t_1. \end{cases}$$

Thus, calling $f_0(t)$ the solution of the following Cauchy problem

(17) $$\begin{cases} f_0'(t) = \begin{cases} -\lambda - f_0^2(t) & \text{for } t_0 < t \leq \tau \\ -\lambda L - \dfrac{f_0^2(t)}{L} & \text{for } \tau \leq t < t_1 \end{cases} \\ f_0(\tau) = \sqrt{\lambda L}, \end{cases}$$

we get

$$\begin{cases} f_0(t) \geq f(t) & \text{for } t \leq \tau \\ f_0(t) \leq f(t) & \text{for } t \geq \tau. \end{cases}$$

Writing down explicitely the solution of (17) we have

$$f_0(t) = \begin{cases} \sqrt{\lambda}\,\mathrm{tg}[\sqrt{\lambda}(\tau - t) + \mathrm{arctg}\sqrt{L}] & \text{for} \quad t \leq \tau \\ \sqrt{\lambda}\, L \left[\mathrm{tg}\sqrt{\lambda}(\tau - t) + \mathrm{arctg}\dfrac{1}{\sqrt{L}}\right] & \text{for} \quad t \geq \tau . \end{cases}$$

Therefore $f_0(t)$ tends to infinity for t-converging to $\tau - \frac{1}{\sqrt{\lambda}}\left(\frac{\pi}{2} - \mathrm{arctg}\sqrt{L}\right)$ and vanishes for t-equal to $\tau + \frac{1}{\sqrt{\lambda}}\,\mathrm{arctg}\frac{1}{\sqrt{L}}$. It follows that

$$t_1 - t_0 \geq \frac{1}{\sqrt{\lambda}}\left(\frac{\pi}{2} - \mathrm{arctg}\sqrt{L}\right) + \frac{1}{\sqrt{\lambda}}\cdot\mathrm{arctg}\frac{1}{\sqrt{L}} = \frac{2}{\sqrt{\lambda}}\,\mathrm{arctg}\frac{1}{\sqrt{L}} .$$

In a similar way we can prove that

$$t_{i+1} - t_i \geq \frac{2}{\sqrt{\lambda}}\,\mathrm{arctg}\frac{1}{\sqrt{L}} \qquad \text{for} \quad i = 1, 2, 3 ;$$

hence adding the relations we have obtained we get at last $t_4 - t_0 \geq \geq \frac{8}{\sqrt{\lambda}}\,\mathrm{arctg}\frac{1}{\sqrt{L}}$. So, recalling (14), we can state

$$2\pi \geq \frac{8}{\sqrt{\lambda}}\,\mathrm{arctg}\frac{1}{\sqrt{L}} , \quad \text{that is } \lambda \geq \left(\frac{4}{\pi}\,\mathrm{arctg}\frac{1}{\sqrt{L}}\right)^2$$

The inequalities we have just proved hold strictly, unless $f(t) = f_0(t)$ for all t, which corresponds to the choice (11), (12).

We are now able to prove, by the following example, that the worst Hölder continuity exponent for solutions of isotropic operators (see theorem 2) is exactly $\frac{4}{\pi}\,\mathrm{arctg}\frac{1}{\sqrt{L}}$.

EXAMPLE 2. Let

$$F = \sum_1^2{}_i\, D_i(a_0(\vartheta)\, D_i)$$

$$u(x) = |x|^{(4/\pi)\cdot\mathrm{arctg}(1/\sqrt{L})}\, w_0(\vartheta)$$

where $\vartheta = \mathrm{arctg}(x_2/x_1)$ and $a_0(\vartheta)$, $w_0(\vartheta)$ are the functions defined by (11) and (12). In this case the ellipticity coefficient of E is equal to L, u satisfies the equation $Eu = 0$ and is Hölder continuous with exponent $\frac{4}{\pi}\,\mathrm{arctg}\frac{1}{\sqrt{L}}$.

4. Some remarks in the case $n \geq 3$.

When n (dimension of the space) is larger than 2, the method we followed to prove theorem 1 still allows us to conclude that there is a number $\mu > 0$ such that the function

$$\varrho^{-\mu} \int_{|x-x_0| \leq \varrho} \langle A(x) \nabla u(x), \nabla u(x) \rangle \, dx$$

is increasing for $0 < \varrho < \operatorname{dist}(x^0, \partial U)$. But, generaly, $\mu \leq n-2$, then it is not possible to deduce from this fact the Hölder continuity of u.

Nevertheless we can remark that for $n \geq 3$ the worst Hölder exponent $\alpha(L, n)$ is infinitesimal, as $L \to +\infty$, of order greater than or equal to $\frac{1}{L}$, and not of order $\frac{1}{\sqrt{L}}$ as in the case $n = 2$. Namely, by the following example, we get

$$\alpha(L, n) \leq \frac{1}{2}\left[\sqrt{(n-2)^2 + \frac{4(n-1)}{L}} - (n-2)\right],$$

hence, in particular,

$$\alpha(L, n) \leq \frac{n-1}{n-2} \cdot \frac{1}{L} \leq \frac{2}{L}.$$

EXAMPLE 3.

We use the polar coordinates ϱ and $\vartheta = (\vartheta_1, \dots, \vartheta_{n-1})$, and denote by Δ_ϑ the Laplace-Beltrami operator on the unit ball S_{n-1}. Let

$$E = \varrho^{-n+1} \cdot \frac{\partial}{\partial \varrho}\left(L \varrho^{n-1} \frac{\partial}{\partial \varrho}\right) + \varrho^{-2} \Delta_\vartheta \; (^3)$$

and

$$u(x) = |x|^\alpha \cdot \frac{x_1}{|x|}$$

where $\alpha = \frac{1}{2}\left[\sqrt{(n-2)^2 + \frac{4(n-1)}{L}} - (n-2)\right]$.

Then E has ellipticity coefficient L; in fact in cartesian coordinates we get $E = \operatorname{div}(A(x)\nabla)$, where $A(x) = J(\vartheta) \cdot P \cdot J^*(\vartheta)$; $J(\vartheta)$ represents the

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(3) For $L = 1$, E represents the Laplace operator Δ.

orthogonal matrix associated to the change of coordinates and $P = (p_{ij})$ is given by $p_{ij} = 0$ for $i \neq j$, $p_{11} = L$, $p_{jj} = 1$ for $j \neq 1$. Further $Eu = 0$; in fact, setting $w(\vartheta) = x_1 / |x|$, we can verify that $\Delta_\vartheta w = -(n-1) w$, and hence $Eu = |x|^{\alpha-2} w(\vartheta)(\alpha(\alpha + n - 2) L - (n-1)) = 0$, since α is a solution of equation $\alpha(\alpha + n - 2) L - (n-1) = 0$.

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