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A NOTE ON ALGEBRAIC DIFFERENTIAL EQUATIONS WHOSE COEFFICIENTS ARE ENTIRE FUNCTIONS OF FINITE ORDER (1)

by Steven Bank

1. Introduction.

In [1], we investigated the problem of determining the rate of growth of entire functions which are solutions of first order algebraic differential equations whose coefficients are arbitrary entire functions (i. e. of the form

$$\Omega(z, y, dy/dz) = 0, \quad \text{where} \quad \Omega(z, y, dy/dz) = \Sigma f_{kj}(z)\, y^k (dy/dz)^j$$

is a polynomial in y and dy/dz, whose coefficients $f_{kj}(z)$ are entire functions). In the special case where the coefficients $f_{kj}(z)$ are entire functions of finite order, it was shown [1; § 3] that the growth of an entire solution $h(z)$ of such an equation is restricted in the following natural way: There exists a positive real number b such that the Nevanlinna characteristic $T(r, h)$ of $h(z)$ satisfies the inequality, $T(r, h) \leq \exp(r^b)$ for all r greater than some number r_0. As a corollary of this result about entire solutions, it was shown [1; § 5] that the same growth estimate must hold for certain meromorphic solutions of such equations, e. g. those meromorphic solutions $h(z)$ for which there is a value of λ (finite or infinity) such that the sequence of moduli of the non zero roots of the equation $h(z) = \lambda$ has a finite exponent of convergence [4; p. 188]. (This follows by applying the ideas of [1; § 5] to the function $h(z) - \lambda$ (or to $h(z)$ if $\lambda = \infty$), and in fact, by using a similar argument, it is not difficult to see that the same growth estimate must hold for a meromorphic solution $h(z)$ if there exists a meromorphic function $\psi(z)$

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of finite order such that the sequence of moduli of the non-zero roots of the equation $h(z) = \psi(z)$ has a finite exponent of convergence). A natural question is thus raised, namely, does the same growth estimate (or, in fact, does *any* uniform growth estimate) hold for arbitrary meromorphic solutions of first order algebraic differential equations whose coefficients are entire functions of finite order? In this paper, we answer this question in the *negative* by proving the following result: Given *any* positive real-valued function $\Phi(r)$ on the interval $(0, +\infty)$, there exists a meromorphic function $h(z)$ in the plane such that $h'(z)/h(z)$ is of finite order of growth and $T(r, h) > \Phi(r)$ for a sequence of r tending to $+\infty$. Since $h'(z)/h(z)$ can then be written as the quotient $\varphi_1(z)/\varphi_2(z)$ of two entire functions of finite order [5; pp. 40-42], the function $h(z)$ is a solution of the first order equation $\varphi_2(z)y' - \varphi_1(z)y = 0$, whose coefficients are entire functions of finite order, and hence no uniform growth estimate exists for arbitrary meromorphic solutions of such equations.

2. We now state our main result. The proof will be given in § 4.

THEOREM: Given any positive real-valued function $\Phi(r)$ on the interval $(0, +\infty)$, there exists a meromorphic function $h(z)$ in the plane such that $h'(z)/h(z)$ is of finite order of growth and $T(r, h) > \Phi(r)$ for a sequence of r tending to $+\infty$.

In addition, the function $h(z)$ can be constructed so that its zeros and poles lie on any given sequence of circles $|z| = r_n$ for which the sequence of radii $\{r_n\}$ is a strictly increasing sequence in $(1, +\infty)$ having a finite exponent of convergence. If σ is the exponent of convergence of the sequence $\{r_n\}$, then for the $h(z)$ that we construct, the order of $h'(z)/h(z)$ is at most $\max\{1, \sigma\}$.

3. NOTATION: For a meromorphic function $h(z)$, we will use the standard notation for the Nevanlinna functions $m(r, h)$, $n(r, h)$, $N(r, h)$ and $T(r, h)$ introduced in [5; pp 6,12].

4. PROOF OF THE THEOREM: Let $\Phi(r)$ be given and let $\{r_n\}$ be strictly increasing in $(1, +\infty)$ having a finite exponent of convergence σ. For each $n = 1, 2, \ldots$, choose a positive integer μ_n such that

$$\mu_n > 2\Phi(2r_n). \tag{1}$$

Choose a strictly increasing sequence of positive integers $\{p_n\}$ in $[2, +\infty)$ such that

$$p_n > \log(n^2 \mu_n) \text{ for each } n. \tag{2}$$

Now set

$$\varphi_n = ((p_n!)\,\mu_n\, 2^{p_n})^{-1} \quad \text{for } n = 1, 2, \dots . \tag{3}$$

Choose two sequences of complex numbers $\{z_n\}$ and $\{w_n\}$ such that for each n,

$$|z_n| = |w_n| = r_n \quad \text{and} \quad 0 < |z_n - w_n| \leq \varphi_n . \tag{4}$$

Now define $f(z) = \prod_{n=1}^{\infty} (E(z/z_n, p_n))^{\mu_n}$, where $E(z,p) = (1-z)\exp(z + z^2/2 + \dots + z^p/p)$. In view of (2), it follows from [3; pp. 228-229] that $f(z)$ represents an entire function having zeros of multiplicity μ_n at the points z_n (and no other zeros), and it follows by simple calculation (using [6; p. 292]) that for $z \notin \{z_n : n \geq 1\}$,

$$f'(z)/f(z) = \sum_{n=1}^{\infty} \mu_n\, L(z, z_n), \tag{5}$$

where

$$L(z, z_n) = (1/z_n) + (z/z_n^2) + \dots + (z^{p_n-1}/z_n^{p_n}) + (z - z_n)^{-1} . \tag{6}$$

Similarly, we set $g(z) = \prod_{n=1}^{\infty} (E(z/w_n, p_n))^{\mu_n}$, so

$$g'(z)/g(z) = \sum_{n=1}^{\infty} \mu_n\, L(z, w_n) \quad \text{for} \quad z \notin \{w_n : n \geq 1\}. \tag{7}$$

Finally, we set $h = f/g$, and we will prove that h has the desired properties.

First, since h has a pole of order μ_m at $z = w_m$, we have $n(r_m, h) \geq \mu_m$ for each m. But since

$$N(r, h) \geq \int_{r/2}^{r} (n(t, h)/t)\, dt, \quad \text{we have} \quad N(r, h) \geq (1/2)\, n(r/2, h)$$

for each r. Thus $N(2r_m, h) \geq (1/2)\,\mu_m$ for each m. In view of (1), it follows that $T(2r_m, h) > \Phi(2r_m)$ for each m, thus proving that $T(r, h) > \Phi(r)$ for a sequence of r tending to $+\infty$.

Now let $\varepsilon > 0$ and set $\delta = \max\{1, \sigma + \varepsilon\}$. We will show that $h'(z)/h(z)$ is of order $\leq \delta$.

Let D be the domain obtained by removing from the plane all the disks $|z - z_n| < 2^{-p_n}$ and $|z - w_n| < 2^{-p_n}$. Since $h'/h = (f'/f) - (g'/g)$, it follows from (5) and (7) that

$$h'(z)/h(z) = \sum_{n=1}^{\infty} \mu_n (L(z, z_n) - L(z, w_n)) \quad \text{for } z \text{ in } D. \tag{8}$$

Now let z be a point of D. We may write (8) as,

$$h'(z)/h(z) = \Sigma_1 + \Sigma_2, \tag{9}$$

where Σ_1 is the summation in (8) extended over those n for which $|z| \leq (1/2) r_n$, while Σ_2 is the summation in (8) extended over those n for which $|z| > (1/2) r_n$. (Clearly Σ_2 is a finite sum).

We first consider Σ_1. For an n in this summation, we have $|z/z_n| \leq 1/2$ and $|z/w_n| \leq 1/2$. Hence by considering the power series expansions of $(z - z_n)^{-1}$ and $(z - w_n)^{-1}$ around the point $z = 0$, it is clear from (6) that

$$L(z, z_n) = - \sum_{q=p_n}^{\infty} (z^q / z_n^{q+1}), \tag{10}$$

and similarly,

$$L(z, w_n) = - \sum_{q=p_n}^{\infty} (z^q / w_n^{q+1}). \tag{11}$$

We introduce the following notation for a non-negative integer q:

$$S_q(z, w) = \sum_{j=0}^{q} z^j w^{q-j}. \tag{12}$$

Hence clearly, for non-zero z and w we have,

$$(1/w^{q+1}) - (1/z^{q+1}) = (z - w)(wz)^{-(q+1)} S_q(z, w). \tag{13}$$

In view of (10) and (11), we thus obtain,

$$L(z, z_n) - L(z, w_n) = \sum_{q=p_n}^{\infty} z^q (z_n - w_n)(w_n z_n)^{-(q+1)} S_q(z_n, w_n). \tag{14}$$

Since $|z_n| = |w_n| = r_n$, clearly $|S_q(z_n, w_n)| \leq (q+1) r_n^q$. Hence in view of (4) and the fact that $|z| \leq (1/2) r_n$, we obtain from (14)

$$|L(z, z_n) - L(z, w_n)| \leq \sum_{q=p_n}^{\infty} (1/2)^q (q+1) \varphi_n r_n^{-2}. \tag{15}$$

Since $r_n > 1$ and $p_n \geq 2$, we thus have

$$(16) \qquad | L(z, z_n) - L(z, w_n) | \leq 2\varphi_n \text{ if } n \text{ appears in } \Sigma_1 .$$

Since by (3), $\mu_n \varphi_n = ((p_n!) 2^{p_n})^{-1}$, it clearly follows that $|\Sigma_1| \leq \sum_{n=1}^{\infty} ((p_n!) 2^{p_n-1})^{-1}$. Since $\{p_n\}$ is a strictly increasing sequence of positive integers in $[2, +\infty)$, we have $p_n \geq n+1$ and hence we obtain

$$(17) \qquad | \Sigma_1 | \leq e^{1/2} .$$

We now consider Σ_2. In view of (6) (and the corresponding expression for $L(z, w_n)$), we have,

$$(18) \qquad L(z, z_n) - (L(z, w_n) = (z_n - w_n) R(z) + \sum_{q=0}^{p_n-1} z^q (z_n^{-(q+1)} - w_n^{-(q+1)}),$$

where $(R(z) = ((z - z_n)(z - w_n))^{-1}$. Since z is in D, $| R(z) | \leq 2^{2p_n}$. In view of (13) and the fact that $| z_n | = | w_n | = r_n$ (so that $S_q(w_n, z_n) | \leq (q+1) r_n^q$), we thus obtain

$$(19) \qquad | L(z, z_n) - L(z, w_n) | \leq | z_n - w_n | \left(2^{2p_n} + \sum_{q=0}^{p_n-1} | z |^q (q+1) r_n^{-(q+2)} \right).$$

Now the summation appearing on the right side of (19) can be written $| z |^{p_n-1} \sum_{q=0}^{p_n-1} | z |^{q-(p_n-1)} (q+1) r_n^{-(q+2)}$, which is less than or equal to $| z |^{p_n-1} \cdot$ $\cdot 2^{p_n-1} \sum_{q=0}^{p_n-1} 2^{-q} (q+1) r_n^{-(p_n+1)}$, since $| z | > (1/2) r_n$ for an n appearing in Σ_2. Since $r_n > 1$, we see, therefore, that the summation on the right side of (19) is at most $| z |^{p_n-1} 2^{p_n+1}$, and hence by (19),

$$(20) \qquad | L(z, z_n) - L(z, w_n) | \leq | z_n - w_n | (2^{2p_n} + | z |^{p_n-1} 2^{p_n+1}).$$

Since $| z_n - w_n | \leq \varphi_n$ and since $\mu_n \varphi_n = ((p_n!) 2^{p_n})^{-1}$, it follows that

$$(21) \qquad | \Sigma_2 | \leq \sum_{n=1}^{\infty} (2^{p_n}/p_n!) + 2 \sum_{n=1}^{\infty} (| z |^{p_n-1} / p_n!).$$

Since the sequence $\{p_n\}$ is a strictly increasing sequence of positive integers, we clearly obtain,

$$(22) \qquad | \Sigma_2 | \leq e^2 + 2e^{| z |} .$$

Hence from (9), (17) and (22), we have,

$$(23) \qquad |h'(z)/h(z)| \leq e^{1/2} + e^2 + 2e^{|z|} \quad \text{for } z \text{ in } D.$$

Now let E be the union of all the intervals, $(r_n - 2^{-p_n}, r_n + 2^{-p_n})$.

Clearly if r is a positive real number which is not in E, then every point on the circle $|z| = r$ must lie in D, and hence (23) is valid on $|z| = r$. In view of the properties of the function $\log^+ x$ [5; p. 14], it thus follows from (23) that there exists a real number $\alpha_0 > 1$ such that,

$$(24) \qquad m(r, h'/h) \leq 2r \quad \text{if} \quad r \notin E \quad \text{and} \quad r > \alpha_0.$$

Now h'/h has poles only at the points, $z_1, w_1, z_2, w_2, \ldots$ and each pole is simple. Since the sequence $(r_1, r_1, r_2, r_2, \ldots)$ has the same exponent of convergence as $\{r_n\}$, namely σ, and since $\varepsilon > 0$, it follows from [2; p. 25], that

$$(25) \qquad \int_0^\infty (N(t, h'/h)/t^{A+1})\, dt < +\infty \quad \text{where} \quad A = \sigma + (\varepsilon/2).$$

Hence there exists $\alpha_1 > 0$ such that for $r \geq \alpha_1$, we have $\int_r^\infty (N(t,h'/h)/t^{A+1})dt < 1$.

Since $N(t, h'/h)$ is increasing, we thus obtain,

$$(26) \qquad N(r, h'/h) \leq A r^A \quad \text{for } r > \alpha_1.$$

Setting $\alpha_2 = \max\{\alpha_0, \alpha_1\}$ and recalling that $\delta = \max\{1, \sigma + \varepsilon\}$, we have from (24) and (26) that,

$$(27) \qquad T(r, h'/h) \leq (2 + A)\, r^\delta \quad \text{if} \quad r \notin E \quad \text{and} \quad r > \alpha_2.$$

Now the set E is of measure at most $2\, \Sigma\, 2^{-p_n}$ which is certainly less than or equal to 2 since $\{p_n\}$ is a strictly increasing sequence of positive integers. Thus if $r > \alpha_2$, then $[r, r+3]$ cannot be contained in E, so there exists t in $[r, r+3]$ with t not in E. Hence by (27),

$$(28) \qquad T(t, h'/h) \leq (2 + A)\, t^\delta.$$

Since $T(-, h'/h)$ is increasing and since $t \leq r + 3$, it follows that $T(r, h'/h) \leq$ $\leq (2 + A)(2r)^\delta$ if $r > \max\{\alpha_2, 3\}$. Thus h'/h is of order at most $\delta = \max$ $\{1, \sigma + \varepsilon\}$ for every $\varepsilon > 0$. Hence it follows that h'/h is of order at most $\max\{1, \sigma\}$ which concludes the proof of the theorem.

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