

ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

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for spaces of constant curvature**

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3^e série, tome 26,
n° 1 (1972), p. 233-245

http://www.numdam.org/item?id=ASNSP_1972_3_26_1_233_0

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PROPERTIES OF HYPERSURFACES
WHICH ARE CHARACTERISTIC FOR SPACES
OF CONSTANT CURVATURE

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If it is well-known that the spaces of constant curvature are characterized among all Riemannian spaces by the following property: *for any $(n - 1)$ -dimensional linear element E_{n-1} of a Riemann space N ($\dim N = n \geq 3$) there is a totally geodesic hypersurface $M \subset N$ which is tangent to E_{n-1} .* (Cf. [1]). The purpose of this Note is to present a number of theorems of the above type; only the requirement that our hypersurfaces should be totally geodesic will be replaced by another geometrical or analytical postulates. Umbilical points of hypersurfaces and so called «normal Bianchi identity» play the leading part here.

Throughout the paper we shall keep all the notations and conventions of the famous book by Kobayashi and Nomizu ([2], [3]).

* * *

Let N be a Riemannian manifold of dimension $n \geq 3$, g the corresponding Riemann metric on N and $M \subset N$ a hypersurface. Because all the postulates put on hypersurfaces will be purely local, we can suppose (if not otherwise stated) that M is a «small» hypersurface, diffeomorphic with an open region in \mathbb{R}^{n-1} .

Denote by ∇, ∇' the covariant differentiation on N, M respectively. Let ξ be a field of unit normal vectors and X, Y tangent vector fields on M . The formulas of Gauss and Weingarten are given by

$$(1) \quad \nabla_X Y = \nabla'_X Y + h(X, Y) \xi,$$

$$(2) \quad \nabla_X \xi = -AX.$$

Here $h(X, Y)$ is the second fundamental form and A is a symmetric transformation on each tangent space $T_x(M)$, $x \in M$. Moreover, we have $g(AX, Y) = g(X, AY) = h(X, Y)$.

A point $x \in M$ is called an *umbilical point* if $A = \lambda \cdot I$ on $T_x(M)$, where λ is a scalar and I is an identity transformation.

DEFINITION 1. A hypersurface $M \subset N$ is called a *U-sphere* if $A = \lambda \cdot I$ on the tangent bundle $T(M)$, where λ is a constant, $\lambda \neq 0$.

Denote by R, R' the Riemann curvature tensors of N, M respectively. At any point $x \in M$ we have

$$(3) \quad R(X, Y)\xi = (V'_Y A)(X) - (V'_X A)(Y),$$

$$(4) \quad \text{proj}_{T(M)} R(X, Y)Z = R'(X, Y)Z + g(AX, Z)AY - g(AX, Z)AX,$$

where $X, Y, Z \in T_x(M)$ and ξ is a normal vector to M at x . (Cf. [3]).

We introduce new tensors $A(X, Y), B(X, Y)Z$ on M by

$$(5) \quad A(X, Y) = (V'_Y A)(X) - (A'_X A)(Y),$$

$$(6) \quad B(X, Y)Z = g(AX, Z)AY - g(AY, Z)AX.$$

Let us remind the first Bianchi identity $\sigma(R'(X, Y)Z) = 0$ and the second Bianchi identity $\sigma((V'_X R')(Y, Z)) = 0$, where X, Y, Z are tangent vector fields on M and σ denotes the cyclic sum with respect to X, Y, Z .

The tensor $B(X, Y)Z$ defined by (6) also satisfies the first Bianchi identity but not the second one, in general.

DEFINITION 2. We say that a hypersurface $M \subset N$ satisfies the *normal Bianchi identity* if $\sigma((V'_X B)(Y, Z)) = 0$ for any vector fields X, Y, Z on M .

REMARK. This definition is independent of the choice of a normal unit vector field ξ .

A routine calculation leads to the following

PROPOSITION 1. *The normal Bianchi identity holds on a hypersurface $M \subset N$ if and only if*

$$(7) \quad A(X, Y) \wedge AZ + A(Y, Z) \wedge AX + A(Z, X) \wedge AY = 0$$

in the vector bundle $\Lambda^2 T(M)$.

If N is a Riemann manifold with the constant curvature C , then

$$(8) \quad R(X, Y)Z = C\{g(Y, Z)X - g(X, Z)Y\}.$$

Hence and taking into account (3) - (6) we obtain

$$(9) \quad \Delta(X, Y) = (V'_X A)(Y) - (V'_Y)(X) = 0 \quad (\text{Equation of Codazzi}),$$

$$(10) \quad B(X, Y)Z = C\{g(Y, Z)X - g(X, Z)Y\} - R'(X, Y)Z$$

(Equation of Gauss).

Finally, (7) and (9) imply

PROPOSITION 2. *If N is a space of constant curvature then any hypersurface $M \subset N$ satisfies the Codazzi equation $\Delta(X, Y) = 0$ and also the normal Bianchi identity.*

In order to prove a converse we shall remind some well-known definitions.

For each plane p in the tangent space $T_x(N)$, i. e., for any 2-dimensional subspace of $T_x(N)$, the sectional curvature $K(p)$ is defined by $K(p) = R(e_1, e_2, e_1, e_2) = g(R(e_1, e_2)e_2, e_1)$, where $\{e_1, e_2\}$ is an orthonormal basis of p . (In the following we shall put $K(e_1, e_2) = R(e_1, e_2, e_1, e_2)$ for abbreviation).

A point $x \in N$ is called *isotropic* if the sectional curvature $K(p)$ is the same for any plane $p \subset T_x(N)$.

Now we shall present a number of lemmas the statements of which are well-known.

LEMMA 1. *Suppose that there is an orthonormal basis $\{e_1, \dots, e_n\}$ in $T_x(N)$ and a constant C such that $K(e_i, e_j) = C$ for any $i, j = 1, \dots, n, i \neq j$. Then x is an isotropic point of N .*

LEMMA 2. *Suppose that $g(R(e_i, e_j)e_k, e_j) = 0$ for any orthonormal triplet $\{e_i, e_j, e_k\}$ of vectors of $T_x(N)$. Then x is an isotropic point of N .*

PROOF. Consider an orthonormal basis $\{e_1, \dots, e_n\}$ of $T_x(N)$, and for any triplet of indices $\{i, j, k\} \subset \{1, \dots, n\}$ put $f_i = \frac{e_i + e_k}{\sqrt{2}}, f_j = e_j, f_k = \frac{e_i - e_k}{\sqrt{2}}$. The equality $g(R(f_i, f_j)f_k, f_j) = 0$ implies $K(e_i, e_j) = K(e_k, e_j)$ and we can use Lemma 1.

LEMMA 3. (Schur's lemma). *Let N be a Riemannian manifold of dimension $n \geq 3$. If all points of N are isotropic, then N is a space of constant curvature. (Cf [2]).*

Now we can derive

PROPOSITION 3. *Let N be a Riemannian manifold of dimension $n \geq 3$ with the following property: to any point $x \in N$ and any hyperplane $E_{n-1} \subset T_x(N)$ there is a hypersurface $M \subset N$ such that*

a) $M \ni x, T_x(M) = E_{n-1}$, b) M satisfies the Codazzi equation at x . Then N is a space of constant curvature.

PROOF. Let a point $x \in N$ and an orthonormal triplet $\{e_i, e_j, e_k\}$ of vectors of $T_x(N)$ be given. Denote $E_{n-1} = \{X \in T_x(N) \mid g(X, e_k) = 0\}$. Let $M \subset N$ be a hypersurface satisfying the conditions a), b) of the Proposition with respect to E_{n-1} . According to (3) we obtain $R(e_i, e_j)e_k = 0$. Now we apply Lemma 3 to complete the proof.

Propositions 2 and 3 give us

THEOREM 1. *Let N be a Riemannian manifold of dimension $n \geq 3$. Then the following two statements are equivalent:*

- (i) *Any hypersurface $M \subset N$ satisfies the Codazzi equation $\Delta(X, Y) = 0$.*
- (ii) *N is a space of constant curvature.*

We are in a position to prove also the following

THEOREM 2. *Let N be a Riemannian manifold of dimension $n \geq 4$. Then the following two statements are equivalent:*

- (i) *Any hypersurface $M \subset N$ satisfies the normal Bianchi identity.*
- (ii) *N is a space of constant curvature.*

PROOF. The implication (ii) \implies (i) was stated in Proposition 2. Let us prove the converse. Let $x \in N$ be a fixed point and consider a system of normal coordinates $\{x^1, \dots, x^n\}$ in a neighbourhood V_x of x . In this way the neighbourhood V_x of x can be represented as an open region U_x of a coordinate space $\mathbb{R}^n(x^1, \dots, x^n)$; the point x is mapped onto the origin $0 \in \mathbb{R}^n$. The space \mathbb{R}^n provided with its canonical euclidean metric is called the osculating euclidean space of N at the point x . (Cf. [1]). We have a canonical isomorphism of $T_x(N)$ onto $T_0(\mathbb{R}^n)$. Suppose that $\{e_i, e_j, e_k\}$ is an orthonormal triplet of vectors of $T_x(N)$ and $\{e'_i, e'_j, e'_k\}$ the corresponding orthonormal triplet of $T_0(\mathbb{R}^n)$.

Let us construct a small piece M' of a cylinder $\mathbb{S}^1 \times \mathbb{R}^{n-2} \subset \mathbb{R}^n$ with the following properties:

- a) M' passes through the origin $0 \in \mathbb{R}^n$ and $M' \subset U_x$,
- b) M' is normal to e'_k at 0 ,
- c) with respect to the canonical identification

$$T_0(M') = T_0(\mathbb{S}^1) \dot{+} T_0(\mathbb{R}^{n-2}) \quad (\text{orthogonal decomposition!})$$

we have $e'_i, e'_j \in T_0(\mathbb{R}^{n-2})$.

Let Z be a vector of $T_x(N)$ corresponding to a vector $Z' \in T_0(\mathbb{S}^1)$.

If M is a hypersurface of N corresponding to M' , then it is well-known that the second fundamental form of $M \subset N$ at x is the same as the second fundamental form of $M' \subset \mathbb{R}^n$ at 0 . Hence $Ae_i = Ae_j = 0$, $AZ = \lambda \cdot Z$. Formula (7) implies $\Delta(e_i, e_j) = \mu \cdot Z$ and from (3) we obtain $R(e_i, e_j)e_k = \mu \cdot Z$. Hence $g(R(e_i, e_j)e_k, e_j) = 0$ and x is an isotropic point of N according to Lemma 2. Now the Schur's lemma completes our proof.

REMARK. If N is a Riemannian manifold of dimension 3, then the normal Bianchi identity is trivially satisfied on each surface $M \subset N$.

The rest of this paper is devoted mainly to the study umbilical points and U spheres. The following theorem of the linear algebra is well-known and it will be useful for our further calculations:

LEMMA 4. Let g, h be two quadratic forms in a real vector space \mathbb{R}^n and let g be positively definite. Then there is a basis $\{e_1, \dots, e_n\}$ of \mathbb{R}^n such that $g = \sum_{i=1}^n (\xi^i)^2$, $h = \sum_{i=1}^n \lambda_i (\xi^i)^2$ with respect to the dual basis $\{\xi^1, \dots, \xi^n\}$ of $(\mathbb{R}^n)^*$.

CONVENTION: $N(C)$ will denote a Riemannian manifold with the constant curvature C .

PROPOSITION 4. Let $N(C)$ be a space of dimension $n \geq 4$ and $M \subset N(C)$ a hypersurface. Then the following two statements are equivalent:

- (i) M is a U -sphere.
- (ii) M is a space $M(C')$, where $C' > C$.

PROOF. Let us multiply (6) and (10) by the vector X to the right and put $Z = Y$. We obtain $g(AX, Y)^2 - g(AX, X)g(AY, Y) = C\{g(X, X)g(Y, Y) - g(X, Y)^2\} - g(R'(X, Y)Y, X)$, where $g(AX, Y) = h(X, Y)$ is the second fundamental form of the hypersurface $M \subset N$. Denote by K_M the

sectional curvature of M . if X, Y are orthonormal vectors of $T_x(M)$, it follows

$$(11) \quad g(AX, Y)^2 - g(AX, X)g(AY, Y) = C - K_M(X, Y).$$

The proof of (i) \implies (ii): If M is a U -sphere, $A = \lambda \cdot I$, $\lambda \neq 0$, we obtain $K_M(X, Y) = C + \lambda^2 = \text{const.}$, *q. e. d.*

The proof of (ii) \implies (i): Let us re-write (11) in the form $h(X, X)h(Y, Y) - h(X, Y)^2 = K_M(X, Y) - C$. If (ii) is satisfied, we obtain

$$(12) \quad h(X, X)h(Y, Y) - h(X, Y)^2 = C' - C > 0$$

for any orthonormal pair $\{X, Y\}$ of $T_x(M)$.

Let $\{e_1, \dots, e_{n-1}\}$ be an orthonormal basis of $T_x(M)$ such that $h(X, Y)$ assumes a diagonal form with respect to this basis (Lemma 4). Put $h(e_i, e_i) = \lambda_i$ for $i = 1, \dots, n-1$; then (12) implies $\lambda_i \lambda_j = C' - C > 0$ for $i, j = 1, \dots, n-1 \geq 3$ as required, we obtain $\lambda_1 = \lambda_2 = \dots = \lambda_{n-1} = \pm \sqrt{C' - C}$ and $h = \pm \sqrt{C' - C} \cdot g$ on M . Hence follows $A = \pm \sqrt{C' - C} \cdot I$ on M ; the sign depends on the orientation of M .

REMARK. If $\dim N(C) = 3$, the implication (ii) \implies (i) is false as the following example shows: let M be a small piece of a sphere S^2 in the euclidean space E^3 ; consider a non-trivial isometric deformation of M in E^3 . The deformed surface is a space of constant curvature but not a U -sphere.

Let N be a Riemannian manifold. To any point $x \in N$ and any sufficiently small number $r > 0$ the locus of all points at the distance r from x is a regular submanifold $S(x; r)$ of dimension $n-1$, called a *metric sphere with center x and radius r* .

PROPOSITION 5. *Let $N(C)$ be a space of dimension $n \geq 3$. Then any sufficiently small metric sphere in $N(C)$ is a space form $N(C')$ with $C' > C$.*

PROOF. It suffices to cite some classical results only (cf. [5]).

For $C = 0$ the assertion is trivial.

For $C > 0$ (elliptic geometry) we put $k = 1/\sqrt{C}$. The following result is well-known: *The metric of the space $N(1/k^2)$ induces on a sphere of radius r the metric of a euclidean sphere of radius $k \cdot \sin(r/k)$.*

For $C < 0$ (hyperbolic geometry) we put $k = 1/\sqrt{-C}$. Let us remember the following theorem: *The metric of the space $N(-1/k^2)$ induces on a sphere of radius r the metric of a euclidean sphere of radius $k \cdot \text{Sh}(r/k)$.* Hence our Proposition follows.

PROPOSITION 6. *Let $N(C)$ be a space of dimension $n \geq 3$. Then any sufficiently small metric sphere in $N(C)$ is a U -sphere.*

PROOF. If the dimension $n \geq 4$, the result follows from Propositions 4 and 5. For $n = 3$, we can identify $N(C)$ with a totally geodesic hypersurface of a space form $N'(C)$ of dimension 4. Any metric sphere in $N(C)$ can be represented as an intersection of a metric sphere in $N'(C)$ by $N(C)$. Hence our result follows.

REMARK. The converse of Proposition 6 is true in the euclidean and the elliptic case only. As for a hyperbolic space $N(-1/k^2)$, any U -sphere in N is locally either a metric sphere (for $|\lambda| > 1/k$), or a limit hypersurface bearing a euclidean metric (for $|\lambda| = 1/k$), or finally an equidistant hypersurface (for $0 < |\lambda| < 1/k$).

PROPOSITION 7. *Let N be a Riemannian space of dimension $n \geq 3$. Suppose that to any linear $(n-1)$ -dimensional element E_{n-1} of N there is a U -sphere tangent to E_{n-1} . Then N is a space of constant curvature.*

PROOF. Let M be a U -sphere tangent to E_{n-1} . We have $A = \lambda \cdot I$ along M , $\lambda = \text{const}$. Hence we see that the Codazzi equation $(V'_Y A)(X) - (V'_X A)(Y) = 0$ holds at the base point x of E_{n-1} . Now we can apply Proposition 3.

PROPOSITION 8. *Let N be a Riemannian space of dimension n . Then to any linear $(n-1)$ -dimensional element E_{n-1} of N there is a metric sphere M tangent to E_{n-1} .*

PROOF. Let x_0 be the base point of E_{n-1} . According to [2], Theorem 8.7. there is a spherical normal coordinate neighbourhood $U(x_0; \varrho)$ such that each point of $U(x_0; \varrho)$ has a normal coordinate neighbourhood containing $U(x_0; \varrho)$. Let γ be a geodesic emanating from x_0 and orthogonal to E_{n-1} at x_0 . Choose a point y on γ at the distance $d < \frac{\varrho}{2}$ from x_0 . Then the metric sphere $S(y; d)$ satisfies our Proposition.

THEOREM 3. *Let N be a Riemannian manifold of dimension $n \geq 3$. Then the following statements (i) - (iii) are equivalent:*

- (i) *To any $(n-1)$ -dimensional linear element E_{n-1} of N there is a U -sphere tangent to E_{n-1} .*
- (ii) *Any sufficiently small metric sphere of N is a U -sphere.*
- (iii) *N is a space of constant curvature.*

PROOF: (iii) \implies (ii) - Proposition 6,
 (i) \implies (iii) - Proposition 7,
 (ii) \implies (i) - Proposition 8.

Now, let us remind some further definitions. *The Ricci tensor field* is a covariant tensor field of degree 2 on N defined as follows: for any vectors $X, Y \in T_x(N)$ and any orthonormal basis $\{e_1, \dots, e_n\}$ of $T_x(N)$ we put $S(X, Y) = \sum_{i=1}^n g(R(e_i, X)Y, e_i)$. A Riemannian manifold N is called an *Einstein manifold* if $S = \lambda \cdot g$ on N , where λ is a constant.

REMARK. From a more general point of view: whenever we speak about an Einstein manifold, we mean an Einstein manifold the signature of which is zero (cf. [4]).

It is well-known that an Einstein manifold of dimension $n \leq 3$ is a space of constant curvature. We can characterize Einstein manifolds of dimension 4 as follows:

LEMMA 5. *A 4-dimensional Riemannian manifold N is Einsteinian if and only if the following property is satisfied: For any point $x \in N$ and any orthogonal decomposition of $T_x(N)$ into two planes p, p' we have $K(p) = K(p')$.*

PROOF. 1) Let N be Einsteinian, $S = \lambda \cdot g$, and let us have an orthogonal decomposition $T_x(N) = p \dot{+} p'$. Choose an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ such that $p = (e_1, e_2), p' = (e_3, e_4)$. Then $S(e_i, e_i) = \lambda \cdot g(e_i, e_i) = \lambda$ for $i = 1, \dots, 4$; on the other hand $S(e_i, e_i) = \sum_{j=1}^4 g(R(e_j, e_i)e_i, e_j) = \sum_{j=1}^4 K(e_i, e_j)$ ($j \neq i$). We obtain easily $K(p) - K(p') = K(e_1, e_2) - K(e_3, e_4) = (1/2)[S(e_1, e_1) + S(e_2, e_2) - S(e_3, e_3) - S(e_4, e_4)] = (1/2)(2\lambda - 2\lambda) = 0$, q. e. d.

2) Let us accept the implication $T_x(N) = p \dot{+} p' \implies K(p) = K(p')$. On a fixed space $T_x(N)$ the Ricci tensor S is a symmetric bilinear form and g is positively definite. Thus, according to Lemma 4, there is an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of $T_x(N)$ in which S takes on a diagonal form. It means that $S(e_i, e_j) = 0$ for $i \neq j$. Further, $S(e_i, e_i) = \sum_{j=1}^4 K(e_i, e_j)$ for $i = 1, \dots, 4$, and because $K(e_1, e_2) = K(e_3, e_4), K(e_1, e_3) = K(e_2, e_4), K(e_1, e_4) = K(e_2, e_3)$, we obtain easily $S(e_1, e_1) = \dots = S(e_4, e_4)$. Hence we

get $S = \lambda \cdot g$ on $T_x(N)$. According to [2], p. 292, the coefficient λ is independent of the point $x \in N$. Consequently, N is an Einstein space.

The following Theorem is added to complete our theory of U -spheres:

THEOREM 4. *Let N be an Einstein manifold of dimension $n \geq 3$ and $M \subset N$ a hypersurface all points of which are umbilical. Then M is a U -sphere, or a totally geodesic submanifold.*

PROOF. We suppose that $A = \mu I$ on M where μ is a real function. Then

$$(13) \quad A(X, Y) = (V'_Y A)(X) - (V'_X A)(Y) = (Y\mu)X - (X\mu)Y.$$

Let $\{e_1, \dots, e_{n-1}\}$ be an orthonormal basis of $Z_x(M)$ and e_n a unit vector of $T_x(N)$ which is normal to $T_x(M)$. According to (3) and (13) $R(e_i, e_j)e_n = (e_j\mu)e_i - (e_i\mu)e_j$ for $i, j = 1, \dots, n-1$ and $S(e_j, e_n) = \sum_{i=1}^n g(R(e_i, e_j)e_n, e_i) = \sum_{i=1, i \neq j}^{n-1} g(R(e_i, e_j)e_n, e_i) = (n-2)(e_j\mu)$. On the other hand, $S(e_j, e_n) = \lambda \cdot g(e_j, e_n) = 0$ because N is Einsteinian. Hence we obtain $e_j\mu = 0$ for any $j = 1, \dots, n-1$ or, what is the same, the differential $(d\mu)_x = 0$. Consequently, $d\mu = 0$ on M and $\mu = \text{const}$.

In the end we shall study the normal Bianchi identity at the umbilical points of a hypersurface. We start with

LEMMA 6. *Let N be a Riemannian manifold of dimension $n \geq 3$. Then to any $(n-1)$ -dimensional linear element E_{n-1} of N there is a hypersurface $M \subset N$ tangent to E_{n-1} at its base point x and such that $A = I$ at x .*

PROOF. We employ the concept of an osculating euclidean space as in the proof of Theorem 2.

PROPOSITION 9. *Let N be a Riemannian manifold of dimension $n \geq 4$ and $x \in N$ a fixed point. Then the following two statements are equivalent:*

(i) *Any hypersurface $M \subset N$ for which x is an umbilical point satisfies the normal Bianchi identity at x .*

(ii) *For any two totally orthogonal planes p, p' of $T_x(N)$ we have $K(p) = K(p')$.*

PROOF. (i) \implies (ii). Let e_1, e_2, e_3, e_4 be orthonormal vectors of $T_x(N)$ such that $p = (e_1, e_2), p' = (e_3, e_4)$. Put $E_{n-1} = \{X \in T_x(N) \mid g(X, e_4) = 0\}$. Ac-

ording to Lemma 6 there is a hypersurface $M \subset N$ such that $T_x(M) = E_{n-1}$ and x is a «unit» umbilic : $AX = X$ for any $X \in E_{n-1}$. Formula (7) takes on the form

$$(14) \quad \Delta(X, Y) \wedge Z + \Delta(Y, Z) \wedge X + \Delta(Z, X) \wedge Y = 0 \quad (X, Y, Z \in E_{n-1}).$$

Put $X = e_1, Y = e_2, Z = e_3, \xi = e_4$, then (3) and (14) imply

$$(15) \quad R(e_1, e_2) e_4 \wedge e_3 + R(e_2, e_3) e_4 \wedge e_1 + R(e_3, e_1) e_4 \wedge e_2 = 0.$$

According to the Cartan's lemma there are numbers a, b, c, d, f, g such that

$$R(e_1, e_2) e_4 = a \cdot e_3 + b \cdot e_1 + c \cdot e_2$$

$$(16) \quad R(e_2, e_3) e_4 = b \cdot e_3 + d \cdot e_1 + f \cdot e_2$$

$$R(e_3, e_1) e_4 = c \cdot e_3 + f \cdot e_1 + g \cdot e_2$$

Particularly, we have $g(R(e_1, e_2) e_4, e_1) = g(R(e_2, e_3) e_4, e_3) = b$. Now, the orthonormal quadruple $\{e_1, e_2, e_3, e_4\}$ was arbitrary and it can be replaced by the quadruple $\{f_1, f_2, f_3, f_4\}$, where

$$(17) \quad f_1 = e_1, f_2 = \frac{e_2 + e_4}{\sqrt{2}}, f_3 = e_3, f_4 = \frac{e_2 - e_4}{\sqrt{2}}.$$

We get $g(R(f_1, f_2) f_4, f_1) + g(R(f_3, f_2) f_4, f_3) = 0$, whence

$$K(e_1, e_2) + K(e_3, e_2) = K(e_1, e_4) + K(e_3, e_4).$$

Similarly, we obtain

$$K(e_2, e_3) + K(e_4, e_3) = K(e_2, e_1) + K(e_4, e_1).$$

By the subtraction we get finally

$$K(p) = K(e_1, e_2) = K(e_3, e_4) = K(p').$$

(ii) \implies (i): Suppose first $\dim N = 4$. Let $M \subset N$ be a hypersurface having an umbilic at x . Denote by e_4 a unit vector of $T_x(N)$ normal to $E_3 = T_x(M)$. It suffices to prove (15), or equivalently, (16) for any orthonormal triplet $\{e_1, e_2, e_3\}$ of E_3 .

Now, each $R(e_i, e_j)e_4$ ($i, j = 1, 2, 3$) is normal to e_4 and hence it is a linear combination of e_1, e_2, e_3 . We have only to show the relation

$$g(R(e_1, e_2)e_4, e_1) + g(R(e_3, e_2)e_4, e_3) = 0$$

for any orthonormal triple $\{e_1, e_2, e_3\}$ of E_3 . For this purpose, take the quadruple $\{f_1, f_2, f_3, f_4\}$ given by (17) and write up the identity $K(f_1, f_2) = K(f_3, f_4)$. Taking into account $K(e_1, e_2) = K(e_3, e_4)$, $K(e_1, e_3) = K(e_1, e_4)$, we obtain the wanted relation.

Consider now the case $\dim N \geq 5$. Then, according to the assumption (ii) and Lemma 1, x is an isotropic point of N . Formula (8) shows that (15) is satisfied trivially for any orthonormal quadruple $\{e_1, e_2, e_3, e_4\}$ of $T_x(N)$. This completes the proof.

Now we can formulate two theorems, which are consequences of Lemma 5, Proposition 9, Lemma 1 and Lemma 3.

THEOREM 5. *Let N be a Riemannian manifold of dimension $n = 4$. Then the following two statements are equivalent:*

- (i) *Any hypersurface $M \subset N$ satisfies the normal Bianchi identity at its umbilical points.*
- (ii) *N is an Einstein space.*

THEOREM 6. *Let N be a Riemannian manifold of dimension $n \geq 5$. Then the following two statements are equivalent:*

- (i) *Any hypersurface $M \subset N$ satisfies the normal Bianchi identity at its umbilical points.*
- (ii) *N is a space of constant curvature.*

REMARK. Theorem 6 is closely related to a theorem by T. Y. Thomas: *If $M \subset E^n$, $n \geq 5$, is a hypersurface the type number of which is ≥ 4 at each point, then the Codazzi equations are algebraic consequences of the Gauss equations.* (Cf. [6]).

Appendix.

In Theorem 4 we have characterized space of constant curvature as Riemannian manifolds containing «sufficiently many» U -spheres. A direct generalization of a U -sphere is an *umbilical hypersurface*, i. e., a hypersurface $M \subset N$ all points of which are umbilical. One can ask about the nature of spaces containing «sufficiently many» umbilical hypersurfaces. The answer is given by the following Theorem:

THEOREM 7. *Let N be a Riemannian manifold of dimension $n \geq 4$. Then the following two statements are equivalent:*

(i) *To any $(n - 1)$ dimensional linear element E_{n-1} of N there is an umbilical hypersurface tangent to E_{n-1} .*

(ii) *N is locally conformally flat.*

PROOF. (ii) \implies (i). Remind the well-know fact that all umbilical points of a submanifold $M \subset N$ remain umbilical under conformal transformations of the metric g on N (See e. g. [7]). Thus, we can always use an auxiliary euclidean neighbourhood to realize a geometrical construction required by (i).

(i) \implies (ii). Let $x \in N$, $E_{n-1} \subset T_x(N)$ be given, and let $M \subset N$ be a hypersurface such that $T_x(M) = E_{n-1}$, $A = \mu \cdot I$ on M . Similarly as in the proof of Theorem 4 we obtain the relation $R(e_i, e_j)e_n = (e_j \mu)e_i - (e_i \mu)e_j$ for any $e_i, e_j \in E_{n-1}$ and a unit e_n normal to E_{n-1} . Hence $e_j \mu = g(R(e_i, e_j)e_n, e_i) = g(R(e_k, e_j)e_n, e_k)$ for any orthonormal quadruple $\{e_i, e_j, e_k, e_n\}$ of $T_x(N)$, $e_n \perp E_{n-1}$. Because the element $E_{n-1} \subset T_x(M)$ can be arbitrary, we obtain $g(R(e_1, e_2)e_4, e_1) = g(R(e_3, e_2)e_4, e_3)$ for any orthonormal quadruple $\{e_1, e_2, e_3, e_4\}$ of $T_x(N)$. Using a similar argument as in the proof of Proposition 9 (Formula (17)), we obtain finally:

$$K(e_1, e_2) + K(e_3, e_4) = K(e_1, e_4) + K(e_2, e_3)$$

for any orthonormal quadruple $\{e_1, e_2, e_3, e_4\}$ of $T_x(N)$.

As was shown by R. S. KULKARNI, [8], this last condition is equivalent to the requirement that *the conformal curvature tensor C vanishes at the point x* . Consequently, the Riemannian space N is locally conformally flat.

As a consequence of Theorem 7 we can state

THEOREM 8. *Let N be a Riemannian manifold of dimension $n \geq 4$. Suppose that any sufficiently small metric sphere of N consists of umbilical points. Then N is locally conformally flat.*

PROBLEM. Does a converse of Theorem 8 hold, too?

Added in Proof, March 15, 1972: It has been pointed to me by R. S. Kulkarni that the conclusion of Theorem 8 may be strenghtened to the effect that N be a space of constant curvature.

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