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INVARIANT SUBMANIFOLDS OF CODIMENSION 2 OF ALMOST CONTACT MANIFOLDS

SAMUEL I. GOLDBERG [1]

1. Introduction.

In his dissertation, Smyth [8] classified the complex hypersurfaces M of the simply connected complex space forms $\widetilde{M}$ under the conditions that in the induced metric they are complete Einstein spaces. M is then a totally geodesic submanifold, or else the holomorphic sectional curvature of $\widetilde{M}$ is positive and M is a complex hypersphere. A local analogue for odd dimensional manifolds was subsequently obtained by Yano and Ishihara [9]. They proved that if M is an invariant submanifold of codimension 2 of a normal contact Riemannian manifold $\widetilde{M}$ of constant sectional curvature and if in the induced metric M is an Einstein space, then M is a totally geodesic submanifold of $\widetilde{M}$. Observe that the exceptional part of Smyth's result does not occur, that is positive curvature yields the same result in all cases.

Consider either a $(2n+1)$-dimensional normal contact Riemannian manifold or a cosymplectic space and let M be an invariant submanifold immersed as an orientable hypersurface (M, j) of a hypersurface (P, i) along which the fundamental vector field of $\widetilde{M}$ is tangent. Then, if the induced f-structure on P (of rank $2n-2$) is normal, or, if the unit normal field of $j(M)$, with respect to the induced Riemannian metric, is a Killing vector field, M is a totally geodesic submanifold of $\widetilde{M}$. This is an odd dimensional analogue of a result on complex hypersurfaces of Kaehler manifolds obtained in [3].

As in [3], no assumption on the metric structure of $\widetilde{M}$ is made. Indeed, it is not assumed that the ambient space is a space form or that the submanifold is an Einstein space.

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2. Hypersurfaces of almost contact manifolds.

Let $\widetilde{M}$ be an almost contact metric manifold of dimension $2n+1$, $n \geqq 2$, with fundamental affine collineation $\widetilde{\varphi}$, fundamental vector field $\widetilde{E}$, compatible metric $\widetilde{g}$ and contact form $\widetilde{\eta}$, where

$$\widetilde{\eta} = \widetilde{g}\,(\widetilde{E}, \cdot\,).$$

Let $\widetilde{N}$ be the field of unit normals to $i\,(P)$ with respect to $\widetilde{g}$. Consider a $2n$-dimensional hypersurface P immersed in $\widetilde{M}$ with immersion $i : P \longrightarrow \widetilde{M}$ having the property

(T): *For each $p \in P$, the vector $\widetilde{E}_{i(p)}$ belongs to the tangent hyperplane of $i\,(P)$.*

Then,

$$\widetilde{\varphi}\, i_* X = i_* f X + \alpha\,(X)\,\widetilde{N}, \tag{2.1}$$

$$\widetilde{\varphi}\,\widetilde{E} = 0, \tag{2.2}$$

$$\widetilde{\eta}\,(\widetilde{N}) = 0, \tag{2.3}$$

where f and α are tensor fields on P of types (1,1) and (0,1), respectively, i_* is the induced tangent map and $X \in \mathfrak{X}\,(P)$ — the module of C^∞ vector fields on P. Since i is a regular map, there is a vector field E' on P such that

$$\widetilde{E} = i_*\, E'\,. \tag{2.4}$$

Hence, by (2.1) and (2.2), $fE' = 0$ and $\alpha\,(E') = 0$. Putting $\eta' = i^* \widetilde{\eta}$, we have

$$\eta'\,(E') = 1. \tag{2.5}$$

Since $\widetilde{\varphi}\,\widetilde{N}$ is orthogonal to $\widetilde{N}$ with respect to $\widetilde{g}$, it is tangent to the hypersurface, so there is a vector field A on P such that

$$\widetilde{\varphi}\,\widetilde{N} = -\, i_*\, A. \tag{2.6}$$

Applying $\widetilde{\varphi}$ to both sides of (2.1) gives $f^2 X = -X + \eta'\,(X)\,E' + \alpha\,(X)\,A$ and $\alpha\,(fX) = 0$.

Applying $\widetilde{\varphi}$ to both sides of (2.6) yields $fA = 0$ and $\alpha(A) = 1$. Summarizing, we have the following result established in [5].

PROPOSITION 1. *Let P be a 2n-dimensional hypersurface immersed in the almost contact manifold $\widetilde{M}$ with immersion i. Then, there exist tensor fields f, E', η', A and α on P satisfying the relations*

$$f^2 = -I + \eta' \otimes E' + \alpha \otimes A, \tag{2.7}$$

$$\eta' \circ f = 0, \ \alpha \circ f = 0, \tag{2.8}$$

$$fE' = 0, \ fA = 0, \tag{2.9}$$

$$\eta'(E') = 1, \ \eta'(A) = 0, \tag{2.10}$$

$$\alpha(E') = 0, \ \alpha(A) = 1, \tag{2.11}$$

where I is the identity transformation of P_p, that is the induced structure on P is a globally framed f-structure of rank $2n-2$.

Let $\widetilde{\nabla}$ be the Riemannian connection of $(\widetilde{M}, \widetilde{g})$ and let D be the induced connection on P, that is, the Riemannian connection of $G = i^* \widetilde{g}$. Then, the equations of Gauss and Weingarten are

$$\widetilde{\nabla}_{i_* X} i_* Y = i_* D_X Y + h(X, Y) \widetilde{N} \tag{2.12}$$

and

$$\widetilde{\nabla}_{i_* X} \widetilde{N} = - i_* HX, \tag{2.13}$$

respectively, where h and H are the second fundamental tensors of the immersion of types (0,2) and (1,1), respectively and

$$h(X, Y) = G(HX, Y).$$

If the structure on $\widetilde{M}$ is *normal*, that is, if the almost complex structure $\widetilde{J}$ on $\widetilde{M} \times R$ defined by

$$\widetilde{J}\left(\widetilde{X}, \varrho \frac{d}{dt}\right) = \left(\widetilde{\varphi}\widetilde{X} - \varrho \widetilde{E}, \widetilde{\eta}(\widetilde{X}) \frac{d}{dt}\right),$$

where ϱ is a C^∞ real valued function and $\widetilde{X}$ is a C^∞ vector field on $\widetilde{M}$, gives rise to a complex structure on $\widetilde{M} \times R$, then the tensor field $[\widetilde{\varphi}, \widetilde{\varphi}] + d\widetilde{\eta} \otimes \widetilde{E}$ (of type (1,2)) vanishes, where $[\widetilde{\varphi}, \widetilde{\varphi}](\widetilde{X}, \widetilde{Y}) = [\widetilde{\varphi}\widetilde{X}, \widetilde{\varphi}\widetilde{Y}] - \widetilde{\varphi}[\widetilde{\varphi}\widetilde{X}, \widetilde{Y}] - \widetilde{\varphi}[\widetilde{X}, \widetilde{\varphi}\widetilde{Y}] + \widetilde{\varphi}^2[\widetilde{X}, \widetilde{Y}], \widetilde{X}, \widetilde{Y} \in \mathcal{X}(\widetilde{M})$.

An almost contact metric structure is called *quasi-Sasakian* if it is normal and its fundamental form $\widetilde{\Phi}$ is closed, where $\widetilde{\Phi}(\widetilde{X}, \widetilde{Y}) = \widetilde{g}(\widetilde{\varphi}\widetilde{X}, \widetilde{Y})$. Thus, Sasakian ($\widetilde{\Phi} = d\eta$) and cosymplectic ($d\widetilde{\eta} = 0$) manifolds are quasi-Sasakian [1].

The hypersurface P carries an almost hermitian structure. To see this, we set

$$J = f + \eta' \otimes A - \alpha \otimes E'. \tag{2.14}$$

Then, from (2.7) (2.11), it is seen that J is an almost complex structure, that is $J^2 = -I$. From (2.1), (2.4) and (2.6), we see that

$$\eta' = G(E', \cdot), \quad \alpha = G(A, \cdot).$$

The fields E' and A are therefore orthonormal by (2.10) and (2.11). Observe that

$$JE' = A.$$

By (2.1), since $\widetilde{\varphi}$ is skew symmetric with respect to $\widetilde{g}$, f is skew symmetric with respect to G, We put $F(X, Y) = G(fX, Y)$, that is $F = i^* \widetilde{\Phi}$. Then, from (2.14),

$$G(JX, Y) = F(X, Y) + \eta'(X)\,\alpha(Y) - \alpha(X)\,\eta'(Y),$$

from which J is skew symmetric with respect to G.

Putting $\Omega(X, Y) = G(JX, Y)$, we obtain

$$\Omega = F + 2\eta' \wedge \alpha. \tag{2.15}$$

Observe that if $\widetilde{M}$ is quasi-Sasakian, then the 2 form F is closed.

If the ambient space is cosymplectic, η' is also closed. The following result was obtained in [5].

PROPOSITION 2. *If the ambient space is cosymplectic,*

$$(D_X f)\, Y = \alpha(Y)\, HX - h(X, Y)\, A,$$

$$D_X E' = 0, \quad D_X A = fHX,$$

$$D_X \eta' = 0, \quad (D_X \alpha)(Y) = -h(X, fY),$$

$$h(X, E') = \eta'(HX) = 0, \quad h(X, A) = \alpha(HX).$$

When the vector bundle over P, with fibre the vector space spanned by E' and A at each point of P, is endowed with an affine connection γ, it admits an almost complex structure $\widehat{J}$. If $\widehat{J}$ is integrable, the globally framed f-structure is *normal* [7]. By defining γ in such a way that E' and A are parallel fields, it has zero curvature. The f structure is then normal if $[f,f] + d\eta' \otimes E' + d\alpha \otimes A$ vanishes. The following result was also obtained in [5].

PROPOSITION 3. *If the ambient space is cosymplectic, then a necessary and sufficient condition for the induced globally framed f-structure on P to be normal is that*

$$fH - Hf = \alpha \otimes D_A A.$$

3. Hypersurfaces of almost complex manifolds.

Let M be an immersed orientable hypersurface of P. We denote by j the immersion and by N the field of unit normals to $j(M)$ with respect to G (with orientation determined by P). Let ∇ be the Riemannian connection of (M, g), $g = j^* G$. Then,

$$D_{j_* x} j_* y = j_* \nabla_x y + k(x, y) N \tag{3.1}$$

and

$$D_{j_* x} N = -j_* Kx, \tag{3.2}$$

where k and K are the second fundamental tensors of the immersion j, of types (0,2) and (1,1), respectively, and $x, y \in \mathfrak{X}(M)$. We set

$$\eta(x) = G(Jj_* x, N) \tag{3.3}$$

and

$$\Phi(x, y) = G(Jj_* x, j_* y). \tag{3.4}$$

Then, Φ is a 2-form on M. If E is the contravariant form of η with respect to g, then it is a vector field on M satisfying

$$JN = -j_* E. \tag{3.5}$$

An endomorphism φ of $\mathfrak{X}(M)$ is defined by the relation

$$\Phi(x, y) = g(\varphi x, y). \tag{3.6}$$

Thus, Φ being a 2-form, φ is skew symmetric with respect to g. Moreover, by (3.3),

$$Jj_* x = j_* \varphi x + \eta(x) N. \tag{3.7}$$

It follows that

$$\varphi^2 = -I + \eta \otimes E, \tag{3.8}$$

where I is the identity transformation field of M_m, $m \in M$. In addition, (3.3) and (3.7) yield

$$\eta(\varphi x) = 0,$$

which is equivalent to

$$\varphi E = 0$$

by the skew symmetry of φ. Consequently, M is an almost contact manifold [2].

4. Invariant submanifolds of codimension 2 of a cosymplectic space.

In the sequel, M is an *invariant submanifold* of $\widetilde{M}$, that is

$$\tilde{\varphi}\, \iota_* x = \iota_* \varphi x, \qquad \iota = i \circ j,$$

namely, at each point of M, the tangent space is invariant under the action of $\tilde{\varphi}$. Then, by means of (2.1), (2.4), (2.6), (2.14) and (3.7),

$$\eta(x) N = \tilde{\eta}(\iota_* x) A$$

and

$$j^* \alpha = 0.$$

Putting $x = E$, we obtain $N = A$. For, by (3.5), since $JE' = A$,

$$E' = \pm j_* E.$$

Hence, $N = \tilde{\eta}(\iota_* E) A = \eta'(j_* E) A = A$, by choosing $\eta'(j_* E) = 1$, since N and A are each of length 1. Thus, *if M is an invariant submanifold of an almost contact manifold with immersion ι, the vector field A coincides with the normal field N and $j^* \alpha = 0$.*

PROPOSITION 4. *If $\widetilde{M}$ is a cosymplectic manifold, then M is also cosymplectic.*

PROOF. Since $\eta = \iota^* \tilde{\eta}$, $(\nabla_x \eta)(y) + \eta(\nabla_x y) = (\tilde{\nabla}_{\iota_* x} \tilde{\eta})(\iota_* y) + \tilde{\eta}(\tilde{\nabla}_{\iota_* x} \iota_* y) = \tilde{\eta}(\tilde{\nabla}_{\iota_* x} \iota_* y)$. For, in a cosymplectic manifold the covariant derivative of the contact form is zero. From (2.12) and (3.1), we obtain

$$\widetilde{\nabla}_{\iota_* x}\, \iota_* y = \iota_* \nabla_x y - k(x, y)\, \widetilde{\varphi} \widetilde{N} + h'(x, y)\, \widetilde{N},$$

where $h'(x, y) = h(j_* x, j_* y)$. Applying (2.2) and (2.3), we get $\nabla_x \eta = 0$ (see also § 5).

Defining the (1,1) tensor field H' by $h'(x, y) = g(H' x, y)$, we get

$$Hj_* x = j_* H' x - \omega(x) N$$

for some 1-form ω on M.

PROPOSITION 5. *Let M be an invariant submanifold of the cosymplectic space $\widetilde{M}$ with the immersion ι. Then,*

$$K = -\varphi H' = H' \varphi.$$

PROOF. We differentiate the function $\alpha(j_* y)$ in the direction x, then apply Proposition 2, formulae (2.14) and (3.7), and observe that $j^* \alpha = 0$:

$$\begin{aligned} x(\alpha(j_* y)) &= (D_{j_* x}\alpha)(j_* y) + \alpha(D_{j_* x} j_* y) \\ &= -h(j_* x, fj_* y) + k(x, y) \\ &= -h(j_* x, Jj_* y - \eta'(j_* y) A) + k(x, y) \\ &= -h(j_* x, j_* \varphi y) + k(x, y) \\ &= -G(Hj_* x, j_* \varphi y) + k(x, y) \\ &= -G(j_* H' x, j_* \varphi y) + k(x, y) \\ &= -g(H' x, \varphi y) + g(Kx, y), \end{aligned}$$

so

$$g(Kx, y) = -g(\varphi H' x, y)$$

and

$$g(x, Ky) = g(x, H' \varphi y).$$

COROLLARY. *Under the conditions in the proposition,*

$$K^2 = H'^2$$

and

$$trace\ H' = trace\ K = 0,$$

so M is a minimal submanifold.

PROOF. By the proposition, $K^2 = - H' \varphi^2 H' = H'^2 - (\eta \circ H') \otimes H' E$. But, by Proposition 2, $h(X, E') = 0$, so since $G(HX, E') = G(X, HE) = G(X, Hj_* E)$, we get $G(j_* x, Hj_* E) = G(j_* x, j_* H' E) = g(x, H' E) = 0$. That M is a minimal submanifold is a consequence of the fact that the second fundamental tensors are symmetric and Φ is skew symmetric.

THEOREM 1. *Let M be an invariant submanifold of a cosymplectic manifold $\widetilde{M}$. If M is immersed in $\widetilde{M}$ as an orientable hypersurface of a hypersurface with the property (T), and if the field of unit normals N on P is a Killing vector field, then M is a totally geodesic submanifold of $\widetilde{M}$.*

PROOF. By Proposition 2, $h(X, fY) + h(Y, fX) = 0$ which is equivalent to the statement that H commutes with f. Applying this to the vector field $j_* x$, we get

$$H' \varphi = \varphi H', \quad \omega \circ \varphi = 0.$$

For,

$$\begin{aligned} Hfj_* x &= H\{Jj_* x - \eta'(j_* x) A\} \\ &= H\{j_* \varphi x + \eta(x) N\} - \eta(x) HA \\ &= Hj_* \varphi x \\ &= j_* H' \varphi x - \omega(\varphi x) N, \end{aligned}$$

and

$$\begin{aligned} fHj_* x &= f\{j_* H' x - \omega(x) N\} \\ &= fj_* H' x \\ &= Jj_* H' x - \eta'(j_* H' x) A \\ &= j_* \varphi H' x + \eta(H' x) N - \eta(H' x) A \\ &= j_* \varphi H' x. \end{aligned}$$

Applying Proposition 5, $K = 0$ and $H' = 0$, the latter being due to the Corollary to Proposition 5.

COROLLARY. *Under the conditions in the theorem, the hypersurface P is a Kaehler manifold.*

PROOF. Since H and f commute $D_A A = fHA = HfA = 0$. Applying Proposition 3, the induced globally framed structure on P is normal. Hence, J is integrable (see [6]). By (2.15), P is Kaehlerian if η' and α are closed. That η' is closed is immediate since $\widetilde{\eta}$ is closed. That α is closed is a consequence of the fact that A is a parallel field. To see this, we express any C^∞ vector field on P as $j_* X + \mu N$ for some $x \in \mathfrak{X}(M)$ and C^∞ function μ on P, and show that $(D_{j_* x}\alpha)(j_* y), (D_{j_* x}\alpha)(A), (D_{j_* x}\alpha)(E'), (D_N \alpha)(j_* y)$, $(D_N \alpha)(E')$, and $(D_N \alpha)(A)$ vanish. That this is the case follows from Proposition 2, the vanishing of H' and the fact that $j^* \alpha$ is zero.

If the induced almost complex structure tensor J is integrable there exists an affine connection D on P such that $DJ = 0$. If this is the Riemannian connection induced by $\widetilde{g}$, then the geometrical condition on N may be replaced by the condition that J be integrable. For, then by [4], Proposition 20, K and φ commute.

5. Invariant submanifolds of codimension 2 of a Sasakian space.

Theorem 1 has an analogue for normal contact metric spaces, that is for *Sasakian manifolds*. To this end, we state the appropriate analogue of Proposition 2 (see [5]).

PROPOSITION 6. *Let $\widetilde{M}$ be a Sasakian manifold. Then, the relations*

$$(D_X f)\, Y = -\, G(X, Y)\, E' + \eta'(Y)\, X + \alpha(Y)\, HX - h(X, Y)\, A,$$

$$D_X E' = fX, \quad D_X A = fHX,$$

$$(D_X \eta')(Y) = F(X, Y), \quad (D_X \alpha)(Y) = -\, h(X, fY),$$

$$h(X, E') = \eta'(HX) = \alpha(X), \quad h(X, A) = \alpha(HX)$$

hold on P.

Observe that E' is a killing vector field.

REMARK. We have shown (Proposition 4) that an invariant submanifold of a cosymplectic manifold with the immersion ι is also a cosymplectic manifold. A more general statement can be made, namely, *an invariant submanifold of a quasi-Sasakian manifold with the immersion ι is a quasi-Sasakian manifold.* To see this, observe that $\Phi = \iota^* \widetilde{\Phi}$, since $\Phi = j^* \Omega$, $i^* \widetilde{\Phi} = \Omega - 2\eta' \wedge \alpha$ and j^* is a ring homomorphism. Moreover, the condition $[\widetilde{\varphi}, \widetilde{\varphi}] + d\widetilde{\eta} \otimes \widetilde{E} = 0$ implies $[\varphi, \varphi] + d\eta \otimes E = 0$. However, Theorem 1 and

Theorem 2 (below) do not extend to quasi-Sasakian spaces in general. The key statement required is that if A is a Killing vector field, then H' and φ commute. Observe also that

$$(\widetilde{V}_{\iota_* x} \widetilde{\varphi}) \widetilde{N} = \iota_* (\varphi H' x + Kx).$$

This is an identity if the ambient space is either cosymplectic or Sasakian (see Proposition 8 for the latter) since $\widetilde{V}_{\tilde{X}} \widetilde{\varphi}$ vanishes in the former case, and although this is not so for normal contact manifolds $(\widetilde{V}_{i_* X} \widetilde{\varphi}) \widetilde{N} = \widetilde{\eta} (\widetilde{N}) i_* X - \widetilde{g} (i_* X, \widetilde{N}) \widetilde{E} = 0$ by virtue of (2.3).

For quasi-Sasakian manifolds of different rank, $K \neq - \varphi H'$ unless the immersion is further restricted (see [1], Proposition 5.1).

PROPOSITION 7. *If $\widetilde{M}$ is a Sasakian manifold, then M is also a Sasakian manifold.*

PROOF. The structure tensors of $\widetilde{M}$ are related by

$$(\widetilde{V}_{\tilde{X}} \widetilde{\varphi}) \widetilde{Y} = \widetilde{\eta} (\widetilde{Y}) \widetilde{X} - \widetilde{g} (\widetilde{X}, \widetilde{Y}) \widetilde{E}.$$

Hence, $(\widetilde{V}_{\iota_* x} \widetilde{\varphi}) (\iota_* y) = \iota_* \{\eta (y) x - g (x, y) E\}$. Since M is invariant $\widetilde{V}_{\iota_* x} (\widetilde{\varphi} \iota_* y) = \widetilde{V}_{\iota_* x} (\iota_* \varphi y) = \iota_* \{(V_x \varphi) y + \varphi V_x y\} - k (x, \varphi y) \widetilde{\varphi} \widetilde{N} + h' (x, \varphi y) \widetilde{N}$. But $\widetilde{V}_{\iota_* x} (\widetilde{\varphi} \iota_* y) = (\widetilde{V}_{\iota_* x} \widetilde{\varphi}) (\iota_* y) + \widetilde{\varphi} \{\iota_* V_x y - k (x, y) \widetilde{\varphi} \widetilde{N} + h' (x, y) \widetilde{N}\} = \iota_* \{\eta (y) x - g (x, y) E + \varphi V_x y\} + k (x, y) \widetilde{N} + h' (x, y) \widetilde{\varphi} \widetilde{N}$. Thus,

$$(V_x \varphi) y = \eta (y) x - g (x, y) E$$

which says that M is a Sasakian manifold.

Observe that the above proof also yields the formulae

$$k (x, \varphi y) = - h' (x, y)$$

and

$$k (x, y) = h' (x, \varphi y).$$

Hence,

PROPOSITION 8. *Let M be an invariant submanifold of the Sasakian manifold $\widetilde{M}$ with the immersion ι. Then,*

$$K = H' \varphi, \quad H' = \varphi K.$$

COROLLARY. *Under the conditions in the proposition,*

$$K^2 = H'^2$$

and

$$\text{trace } H' = \text{trace } K = 0,$$

so M is a minimal submanifold.

The proof of Theorem 2 below parallels that of Theorem 1, Propositions 2 and 5 being replaced by Propositions 6 and 8, respectively.
The following fact is also required.

LEMMA. *If the ambient space is a normal contact manifold, then $H'E$ vanishes.*

PROOF. By Proposition 6, $h(j_* x, E') = 0$ since $j^* \alpha = 0$. The remainder of the proof may be found in the proof of the Corollary to Proposition 5.

THEOREM 2. *Let M be an invariant hypersurface of a Sasakian manifold $\widetilde{M}$. If M is immersed in $\widetilde{M}$ as an orientable hypersurface of a hypersurface with the property (T), and if the field of unit normals N on P is a Killing vector field, then M is a totally geodesic submanifold of $\widetilde{M}$.*

COROLLARY. *The hypersurface P is a non-Kaehlerian hermitian manifold.*
J is integrable by Theorem 10 of [5] and Theorem 1 of [6].

That P is not Kaehlerian is a consequence of the fact that η' is not closed. For, by Proposition 6, if η' were closed, then F would vanish and this is not possible.

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