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ON RELATIONS OF CURVATURE TENSORS OVER SEN'S SYSTEM OF AFFINE CONNECTIONS

RANJAN KUMAR GARAI

In a paper Sen (Sen, 1959) obtained some relations of curvature tensors over Sen's system of affine connections. In this paper we obtain some other relations of this nature.

Sen's system of affine connections (Sen, 1950 a) may briefly be described as follows: Let Γ_{ij}^{t} be an arbitrary affine connection and g_{ij} the fundamental tensor in a Riemannian space. Denoting Γ_{ij}^{t} by a, let $a^* = \Gamma_{ij}^{t} + g^{lt} g_{il,j}$ where comma denotes covariant derivative with respect to Γ_{ij}^{t} and $a' = \Gamma_{ji}^{t}$. a^* and a' are called respectively the associate and conjugate of a. The affine connection a is self associate if $a = a^*$ and is self conjugate if $a = a'$. It is seen that these affine connections have involutory property $a^{**} = a'' = a$.

If we put $a_1 = a,\ a_2 = a^*,\ a_3 = a^{*\prime},\ a_4 = a^{*\prime *}, \ldots,$

$$\alpha = g^{lt} g_{il,j}, \quad \alpha_c = g^{lt} g_{jl,i}, \quad \gamma = g^{lt} g_{ij,l},$$

$$\beta = g^{lt} g_{im} (\Gamma_{lj}^{m} - \Gamma_{jl}^{m}), \quad \beta_c = g^{lt} g_{mj} (\Gamma_{li}^{m} - \Gamma_{il}^{m}),$$

we obtain the following cyclic sequence of 12 terms (if they are all distinct)

$$(1.1) \quad \begin{cases} a_1 = a,\ a_2 = a + \alpha,\ a_3 = a' + \alpha_c,\ a_4 = a + \alpha + \beta - \gamma, \\ a_5 = a' + \alpha_c + \beta_c - \gamma,\ a_6 = a + \alpha + \alpha_c + \beta + \beta_c - \gamma, \\ a_7 = a' + \alpha + \alpha_c + \beta + \beta_c - \gamma,\ a_8 = a' + \alpha_c + \beta + \beta_c - \gamma, \\ a_9 = a + \alpha + \beta + \beta_c - \gamma,\ a_{10} = a' + \alpha_c + \beta_c, \\ a_{11} = a + \alpha + \beta,\ a_{12} = a'. \end{cases}$$

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The sequence (1.1) is then used to construct the following coefficients of affine connections:

$$(1.2) \qquad \left(\frac{1}{2}(a_p + a_q)\right)^* = \frac{1}{2}(a_p^* + a_q^*), \quad \left(\frac{1}{2}(a_p + a_q)\right)' = \frac{1}{2}(a_p' + a_q'),$$

where a_p and a_q belong to the sequence (1.1). Sen's system of affine connections generated by a is finally formed to consist of all affine connections which are generated by repeated applications of *, / on the set (1.2). In this system, the Christoffel symbols (which define the Levi-Civita parallelism) are given by

$$(1.3) \qquad \left\{ {t \atop ij} \right\} = \frac{1}{2}(a_p + a_{p+6}), \qquad p = 1, 2, \ldots .$$

The Levi-Civita parallelism is the only parallelism in Sen's system which is both self-associate and self conjugate.

In another paper Sen (Sen, 1950 b) obtained some fundamental relations connecting curvature tensors formed by the coefficients of affine connections of the system. We state the following results from his work as we shall use them frequently in our discussion.

Let Γ_{ij}^t and L_{ij}^t correspond to two arbitrary connections and let

$$T_{ij}^t = \Gamma_{ij}^t - L_{ij}^t, \quad \Delta_{ij}^t = \frac{1}{2}(\Gamma_{ij}^t + L_{ij}^t).$$

If Γ_{ijk}^t, L_{ijk}^t and Δ_{ijk}^t, be the curvature tensors formed with Γ_{ij}^t, L_{ij}^t and Δ_{ij}^t respectively, then

$$(1.4) \qquad \Delta_{ijk}^t - \frac{1}{2}(\Gamma_{ijk}^t + L_{ijk}^t) = \frac{1}{4}(T_{sk}^t T_{ij}^s - T_{sj}^t T_{ik}^s).$$

Consequently if $a = \Gamma_{ij}^t$, $b = L_{ij}^t$, $c = \Delta_{ij}^t$, $d = \Omega_{ij}^t$ correspond to any four affine connections such that

$$|a - b| = |c - d|,$$

then

$$(1.5) \qquad C(a) + C(b) - C(c) - C(d) = 2\left[C\left(\frac{1}{2}(a+b)\right) - C\left(\frac{1}{2}(c+d)\right)\right],$$

where

$$(1.6) \qquad C(a) = \Gamma_{ijk}^t = \frac{\partial \Gamma_{ik}^t}{\partial x^j} - \frac{\partial \Gamma_{ij}^t}{\partial x^k} + \Gamma_{hj}^t \Gamma_{ik}^h - \Gamma_{hk}^t \Gamma_{ij}^h,$$

and similarly for the other curvature tensors in (1.5). Since, by (1.3)

$$\frac{1}{2}(a_p + a_{p+6}) = \frac{1}{2}(a_q + a_{q+6}),$$

it follows immediately from (1.5) that

$$C(a_p) + C(a_q) - C(a_{p+6}) - C(a_{q+6}) = \tag{1.7}$$

$$= 2\left[C\left(\frac{1}{2}(a_p + a_q)\right) - C\left(\frac{1}{2}(a_{p+6} + a_{q+6})\right)\right].$$

We shall also use the following relation obtained by Sen (Sen, 1959)

$$2C(u) = 2C\left(\frac{1}{2}(a_p + a_q)\right) - C(a_p) - C(a_q) + \tag{1.8}$$

$$+ C\left(\frac{1}{2}(a_{p+6} + a_q)\right) + C\left(\frac{1}{2}(a_{q+6} + a_p)\right).$$

where

$$u = \frac{1}{2}(a_p + a_{p+6}) = \frac{1}{2}(a_q + a_{q+6}) = \left\{\begin{matrix} t \\ ij \end{matrix}\right\}.$$

We know that the Riemannian curvature tensor R_{hijk} satisfies the following relations with regard to indices.

$$R_{hijk} + R_{ihjk} = 0, \tag{1.9}$$

$$R_{hijk} + R_{hjki} + R_{hkij} = 0, \tag{1.10}$$

$$R^t_{ijk\,;\,l} + R^t_{ikl\,;\,j} + R^t_{ilj\,;\,k} = 0, \tag{1.11}$$

$$R_{hijk\,;\,l} + R_{hikl\,;\,j} + R_{hilj\,;\,k} = 0, \tag{1.12}$$

$$R^t_{jkl\,;\,m} + R^t_{ljm\,;\,k} + R^t_{mlk\,;\,j} + R^t_{kmj\,;\,l} = 0, \tag{1.13}$$

and

$$R_{hjkl\,;\,m} + R_{hljm\,;\,k} + R_{hmlk\,;\,j} + R_{hkmj\,;\,l} = 0, \tag{1.14}$$

where semi colon denotes covariant derivatives with respect to Levi-Civita parallelism. Prof. R. N. Sen (Sen, 1950 b) generalized the relation (1.9) and Dr. H. Sen (Sen, 1959), the relations (1.10) to (1.12) over Sen's system of affine connections. In this paper we have done the same thing for the relations (1.13) and (1.14). Some other relations are also obtained which can be looked upon as another way of generalising (1.10).

2. We shall first prove a useful formula relating covariant derivatives of curvature tensors. In order to do so, it seems convenient to use the following notations. With reference to the sequence (1.1), put

$$(2.1)\qquad a_p = {}^{a_p}\Gamma^t_{ij}\,, \quad \frac{1}{2}({}^{a_1}\Gamma^s_{jm} - {}^{a_{12}}\Gamma^s_{jm}) = V^s_{jm}\,, \quad \frac{1}{2}({}^{a_7}\Gamma^s_{jm} - {}^{a_{12}}\Gamma^s_{jm}) = U^s_{jm}\,.$$

In what follows a solidus followed by an index indicates covariant differentiation with respect to the affine connections with which the respective curvature tensors are formed, whereas a semicolon, as before, denotes covariant differentiation with respect to the Christoffel symbol. The following relations can be verified by straightforward calculations.

$$(2.2)\qquad {}^{\frac{1}{2}(a_6+a_7)}\Gamma^i_{jkl\,/\,m} = {}^{\frac{1}{2}(a_6+a_7)}\Gamma^i_{jkl\,;\,m} + {}^{\frac{1}{2}(a_6+a_7)}\Gamma^s_{jkl}\,U^i_{sm} - \{{}^{\frac{1}{2}(a_6+a_7)}\Gamma^i_{skl}\,U^s_{jm} + {}^{\frac{1}{2}(a_6+a_7)}\Gamma^i_{jsl}\,U^s_{km} + {}^{\frac{1}{2}(a_6+a_7)}\Gamma^i_{jks}\,U^s_{lm}\}.$$

$$(2.3)\qquad {}^{\frac{1}{2}(a_1+a_{12})}\Gamma^i_{jkl\,/\,m} = {}^{\frac{1}{2}(a_1+a_{12})}\Gamma^i_{jkl\,;\,m} - {}^{\frac{1}{2}(a_1+a_{12})}\Gamma^s_{jkl}\,U^i_{sm} + \{{}^{\frac{1}{2}(a_1+a_{12})}\Gamma^i_{skl}\,U^s_{jm} + {}^{\frac{1}{2}(a_1+a_{12})}\Gamma^i_{jsl}\,U^s_{km} + {}^{\frac{1}{2}(a_1+a_{12})}\Gamma^i_{jks}\,U^s_{lm}\}.$$

$$(2.4)\qquad {}^{\frac{1}{2}(a_{12}+a_7)}\Gamma^i_{jkl\,/\,m} = {}^{\frac{1}{2}(a_{12}+a_7)}\Gamma^i_{jkl\,;\,m} - {}^{\frac{1}{2}(a_{12}+a_7)}\Gamma^s_{jkl}\,V^i_{sm} + \{{}^{\frac{1}{2}(a_{12}+a_7)}\Gamma^i_{skl}\,V^s_{jm} + {}^{\frac{1}{2}(a_{12}+a_7)}\Gamma^i_{jsl}\,V^s_{km} + {}^{\frac{1}{2}(a_{12}+a_7)}\Gamma^i_{jks}\,V^s_{lm}\}.$$

$$(2.5)\qquad {}^{\frac{1}{2}(a_1+a_6)}\Gamma^i_{jkl/m} = {}^{\frac{1}{2}(a_1+a_6)}\Gamma^i_{jkl\,;\,m} + {}^{\frac{1}{2}(a_1+a_6)}\Gamma^s_{jkl}\,V^i_{sm} - \{{}^{\frac{1}{2}(a_1+a_6)}\Gamma^i_{skl}\,V^s_{jm} + {}^{\frac{1}{2}(a_1+a_6)}\Gamma^i_{jsl}\,V^s_{km} + {}^{\frac{1}{2}(a_1+a_6)}\Gamma^i_{jks}\,V^s_{lm}\}.$$

$$(2.6)\qquad {}^{a_1}\Gamma^i_{jkl/m} = {}^{a_1}\Gamma^i_{jkl\,;\,m} + {}^{a_1}\Gamma^s_{jkl}\,V^i_{sm} - \{{}^{a_1}\Gamma^i_{skl}\,V^s_{jm} + {}^{a_1}\Gamma^i_{jsl}\,V^s_{km} + {}^{a_1}\Gamma^i_{jks}\,V^s_{lm}\} - {}^{a_1}\Gamma^s_{jkl}\,U^i_{sm} + \{{}^{a_1}\Gamma^i_{skl}\,U^s_{jm} + {}^{a_1}\Gamma^i_{jsl}\,U^s_{km} + {}^{a_1}\Gamma^i_{jks}\,U^s_{lm}\}.$$

$$(2.7)\qquad {}^{a_6}\Gamma^i_{jkl/m} = {}^{a_6}\Gamma^i_{jkl\,;\,m} + {}^{a_6}\Gamma^s_{jkl}\,V^i_{sm} - \{{}^{a_6}\Gamma^i_{skl}\,V^s_{jm} + {}^{a_6}\Gamma^i_{jsl}\,V^s_{km} + {}^{a_6}\Gamma^i_{jks}\,V^s_{lm}\} + {}^{a_6}\Gamma^s_{jkl}\,U^i_{sm} - \{{}^{a_6}\Gamma^i_{skl}\,U^s_{jm} + {}^{a_6}\Gamma^i_{jsl}\,U^s_{km} + {}^{a_6}\Gamma^i_{jks}\,U^s_{lm}\}.$$

$$(2.8)\quad {}^{a_7}\Gamma^i_{jkl/m} = {}^{a_7}\Gamma^i_{jkl\,;\,m} - {}^{a_7}\Gamma^s_{jkl}\, V^i_{sm} + \{{}^{a_7}\Gamma^i_{skl}\, V^s_{jm} + {}^{a_7}\Gamma^i_{jsl}\, V^s_{km} + {}^{a_7}\Gamma^i_{jks}\, V^s_{lm}\}$$
$$+ {}^{a_7}\Gamma^s_{jkl}\, U^i_{sm} - \{{}^{a_7}\Gamma^i_{skl}\, U^s_{jm} + {}^{a_7}\Gamma^i_{jsl}\, U^s_{km} + {}^{a_7}\Gamma^i_{jks}\, U^s_{lm}\}.$$

$$(2.9)\quad {}^{a_{12}}\Gamma^i_{jkl/m} = {}^{a_{12}}\Gamma^i_{jkl\,;\,m} - {}^{a_{12}}\Gamma^s_{jkl}\, V^i_{sm} + \{{}^{a_{12}}\Gamma^i_{skl}\, V^s_{jm} + {}^{a_{12}}\Gamma^i_{jsl}\, V^s_{km} + {}^{a_{12}}\Gamma^i_{jks}\, V^s_{lm}\}$$
$$- {}^{a_{12}}\Gamma^s_{jkl}\, U^i_{sm} + \{{}^{a_{12}}\Gamma^i_{skl}\, U^s_{jm} + {}^{a_{12}}\Gamma^i_{jsl}\, U^s_{km} + {}^{a_{12}}\Gamma^i_{jks}\, U^s_{lm}\}.$$

By (1.8) we have

$$(2.10)\quad 2C(u) = 2C\left(\frac{1}{2}(a_6 + a_7)\right) - C(a_6) - C(a_7) +$$
$$+ C\left(\frac{1}{2}(a_{12} + a_7)\right) + C\left(\frac{1}{2}(a_1 + a_6)\right)$$
$$= 2C\left(\frac{1}{2}(a_1 + a_{12})\right) - C(a_1) - C(a_{12}) + C\left(\frac{1}{2}(a_{12} + a_7)\right) + C\left(\frac{1}{2}(a_1 + a_6)\right).$$

Theлefore, using all the relations from (2.2) to (2.10), we get

$$(2.11)\quad (2\,{}^{\frac{1}{2}(a_6+a_7)}\Gamma^i_{jkl/m} - {}^{a_6}\Gamma^i_{jkl/m} - {}^{a_7}\Gamma^i_{jkl/m} + {}^{\frac{1}{2}(a_{12}+a_7)}\Gamma^i_{jkl/m} + {}^{\frac{1}{2}(a_1+a_6)}\Gamma^i_{jkl/m})$$
$$+ (2\,{}^{\frac{1}{2}(a_1+a_{12})}\Gamma^i_{jkl/m} - {}^{a_1}\Gamma^i_{jkl/m} - {}^{a_{12}}\Gamma^i_{jkl/m} + {}^{\frac{1}{2}(a_{12}+a_7)}\Gamma^i_{jkl/m} + {}^{\frac{1}{2}(a_1+a_6)}\Gamma^i_{jkl/m})$$
$$= 4R^i_{jkl\,;\,m} + U^i_{sm}\, K^s_{jkl} + V^i_{sm}\, P^s_{jkl} - \{U^s_{jm}\, K^i_{skl} + V^s_{jm}\, P^i_{skl}$$
$$+ U^s_{km}\, K^i_{jsl} + V^s_{km}\, P^i_{jsl} + U^s_{lm}\, K^i_{jks} + V^s_{lm}\, P^i_{jks}\},$$

where

$$K^i_{jkl} = 2\,({}^{\frac{1}{2}(a_6+a_7)}\Gamma^i_{jkl} - {}^{\frac{1}{2}(a_1+a_{12})}\Gamma^i_{jkl}) - {}^{a_6}\Gamma^i_{jkl} - {}^{a_7}\Gamma^i_{jkl} + {}^{a_1}\Gamma^i_{jkl} + {}^{a_{12}}\Gamma^i_{jkl}$$

and

$$P^i_{jkl} = 2\,({}^{\frac{1}{2}(a_1+a_6)}\Gamma^i_{jkl} - {}^{\frac{1}{2}(a_7+a_{12})}\Gamma^i_{jkl}) - {}^{a_6}\Gamma^i_{jkl} + {}^{a_7}\Gamma^i_{jkl} - {}^{a_1}\Gamma^i_{jkl} + {}^{a_{12}}\Gamma^i_{jkl}.$$

But, by (1.7), we have $K^i_{jkl} = 0$ and $P^i_{jkl} = 0$.

Hence, from (2.11), we obtain finally the relation

$$(2.12)\quad 2\,[{}^{\frac{1}{2}(a_6+a_7)}\Gamma^i_{jkl/m} + {}^{\frac{1}{2}(a_1+a_{12})}\Gamma^i_{jkl/m} + {}^{\frac{1}{2}(a_7+a_{12})}\Gamma^i_{jkl/m} + {}^{\frac{1}{2}(a_1+a_6)}\Gamma^i_{jkl/m}]$$
$$- [{}^{a_1}\Gamma^i_{jkl/m} + {}^{a_6}\Gamma^i_{jkl/m} + {}^{a_7}\Gamma^i_{jkl/m} + {}^{a_{12}}\Gamma^i_{jkl/m}] = 4R^i_{jkl\,;\,m}.$$

There are three such formulae over Sen's sequence (1.1), Viz :

$$(2.13)\quad 2\,[{}^{\frac{1}{2}(a_p+a_q)}\Gamma^i_{jkl/m} + {}^{\frac{1}{2}(a_{p+6}+a_{q+6})}\Gamma^i_{jkl/m} + {}^{\frac{1}{2}(a_p+a_{q+6})}\Gamma^i_{jkl/m} + {}^{\frac{1}{2}(a_q+a_{p+6})}\Gamma^i_{jkl/m}]$$

$$- [{}^{a_p}\Gamma^i_{jkl/m} + {}^{a_q}\Gamma^i_{jkl/m} + {}^{a_{p+6}}\Gamma^i_{jkl/m} + {}^{a_{q+6}}\Gamma^i_{jkl/m}] = 4R^i_{jkl\,;\,m}\,,$$

where (p, q) is any one of (1,12), (2,3) and (4,5).

We now proceed to generalise the identity (1.13) over Sen's system of affine connections. We put

$$(2.14)\qquad \{{}^{a_p}\Gamma^i_{jklm}\} = {}^{a_p}\Gamma^i_{jklm} + {}^{a_p}\Gamma^i_{ljm/k} + {}^{a_p}\Gamma^i_{mlk/j} + {}^{a_p}\Gamma^i_{kmj/l}\,.$$

Applying (2.12) and using the notation (2.14), we get

$$(2.15)\qquad 2\,[\{{}^{\frac{1}{2}(a_6+a_7)}\Gamma^i_{jklm}\} + \{{}^{\frac{1}{2}(a_1+a_{12})}\Gamma^i_{jklm}\} + \{{}^{\frac{1}{2}(a_7+a_{12})}\Gamma^i_{jklm}\} + \{{}^{\frac{1}{2}(a_1+a_6)}\Gamma^i_{jklm}\}]$$

$$- [\{{}^{a_1}\Gamma^i_{jklm}\} + \{{}^{a_6}\Gamma^i_{jklm}\} + \{{}^{a_7}\Gamma^i_{jklm}\} + \{{}^{a_{12}}\Gamma^i_{jklm}\}] = 4\,\{{}^{\frac{1}{2}(a_p+a_{p+6})}\Gamma^i_{jklm}\}.$$

Again, since $\frac{1}{2}(a_1 + a_{12})$, $\frac{1}{2}(a_6 + a_7)$ and $\frac{1}{2}(a_p + a_{p+6})$ are self conjugate (Eisenhart, 1927), we have

$$\{{}^{\frac{1}{2}(a_1+a_{12})}\Gamma^i_{jklm}\} = \{{}^{\frac{1}{2}(a_6+a_7)}\Gamma^i_{jklm}\} = \{{}^{\frac{1}{2}(a_p+a_{p+6})}\Gamma^i_{jklm}\} = 0.$$

Hence (2.15) reduces to

$$\{{}^{a_1}\Gamma^i_{jklm}\} + \{{}^{a_6}\Gamma^i_{jklm}\} + \{{}^{a_7}\Gamma^i_{jklm}\} + \{{}^{a_{12}}\Gamma^i_{jklm}\} - 2\,[\{{}^{\frac{1}{2}(a_1+a_6)}\Gamma^i_{jklm}\}$$

$$+ \{{}^{\frac{1}{2}(a_7+a_{12})}\Gamma^i_{jklm}\}] = 0.$$

Thus we have

$$(2.16)\qquad \{{}^{a_p}\Gamma^i_{jklm}\} + \{{}^{a_q}\Gamma^i_{jklm}\} + \{{}^{a_{p+6}}\Gamma^i_{jklm}\} + \{{}^{a_{q+6}}\Gamma^i_{jklm}\}$$

$$- 2\,[\{{}^{\frac{1}{2}(a_p+a_{q+6})}\Gamma^i_{jklm}\} + \{{}^{\frac{1}{2}(a_q+a_{p+6})}\Gamma^i_{jklm}\}] = 0.$$

where (p, q) is any one of (1,12), (2,3) and (4,5).

The three equations (2.16) constitute the required generalisation of the identity (1.13). Here we note that using the relation (2.13), putting $\{{}^{a_p}\Gamma^t_{ijkl}\}$

for ${}^{a_p}\Gamma^t_{ijk/l} + {}^{a_p}\Gamma^t_{ikl/j} + {}^{a_p}\Gamma^t_{ilj/k}$ and adopting the above process, one can obtain another proof of the identities (5.9) obtained by Sen (Sen, 1959) which generalises the Bianchi's identity (1.11) over Sen's system of affine connections.

Now we proceed to generalise the identity (1.14). Denoting the covariant derivative with respect to a_p by the notation $/_{a_p}$ and putting

$$\{{}^{a_p}\Gamma_{hjklm}\} = {}^{a_p}\Gamma_{\underset{a_p}{hjkl/m}} + {}^{a_p}\Gamma_{\underset{a_p}{hljm/k}} + {}^{a_p}\Gamma_{\underset{a_p}{hmlk/j}} + {}^{a_p}\Gamma_{\underset{a_p}{hkmj/l}},$$

we have

$$(2.17) \qquad \{{}^{a_p}\Gamma_{hjklm}\} = g_{\underset{a_p}{ht/m}}\, {}^{a_p}\Gamma^t_{jkl} + g_{\underset{a_p}{ht/k}}\, {}^{a_p}\Gamma^t_{ljm} + g_{\underset{a_p}{ht/j}}\, {}^{a_p}\Gamma^t_{mlk} + g_{\underset{a_p}{ht/l}}\, {}^{a_p}\Gamma^t_{kmj} + g_{ht}\{{}^{a_p}\Gamma^t_{jklm}\}.$$

Putting ${}^{a_1}\Gamma^t_{ij} - {}^{a_{12}}\Gamma^t_{ij} = \overline{V}^t_{ij}$, the following results are seen to hold. The covariant derivative of g_{ij}

$$(2.18) \quad \begin{cases} \text{With respect to } a_1, a_8 \text{ is } g_{ij,k}, \\ \ldots \quad \ldots \quad \ldots a_3, a_{10} \text{ is } g_{ij,k} - g_{ik,j} - g_{jk,i} - g_{is}\overline{V}^s_{kj} - g_{js}\overline{V}^s_{ki}, \\ \ldots \quad \ldots \quad \ldots a_5, a_{12} \text{ is } g_{ij,k} + g_{is}\overline{V}^s_{jk} + g_{js}\overline{V}^s_{ik}, \\ \ldots \quad \ldots \quad \ldots a_p \text{ and } a_{p+6} \text{ are negative of one another.} \end{cases}$$

Now applying (2.17) and (2.18) and remembering the first of the equations (2.16), we get,

$$(2.19) \quad \{{}^{a_1}\Gamma_{hjklm}\} + \{{}^{a_6}\Gamma_{hjklm}\} + \{{}^{a_7}\Gamma_{hjklm}\} + \{{}^{a_{12}}\Gamma_{hjklm}\} - \\ - 2\left[\{{}^{\frac{1}{2}(a_1+a_6)}\Gamma_{hjklm}\} + \{{}^{\frac{1}{2}(a_7+a_{12})}\Gamma_{hjklm}\}\right]$$

$$= g_{ht,m}({}^{a_1}\Gamma^t_{jkl} + {}^{a_{12}}\Gamma^t_{jkl} - {}^{a_7}\Gamma^t_{jkl} - {}^{a_6}\Gamma^t_{jkl}) + g_{ht,k}({}^{a_1}\Gamma^t_{ljm} + {}^{a_{12}}\Gamma^t_{ljm} - {}^{a_7}\Gamma^t_{ljm} - {}^{a_6}\Gamma^t_{ljm})$$

$$+ g_{ht,j}({}^{a_1}\Gamma^t_{mlk} + {}^{a_{12}}\Gamma^t_{mlk} - {}^{a_7}\Gamma^t_{mlk} - {}^{a_6}\Gamma^t_{mlk}) + g_{ht,l}({}^{a_1}\Gamma^t_{kmj} + {}^{a_{12}}\Gamma^t_{kmj} - {}^{a_7}\Gamma^t_{kmj} - {}^{a_6}\Gamma^t_{kmj})$$

$$+ (g_{hs}\overline{V}^s_{tm} + g_{ts}\overline{V}^s_{hm})({}^{a_{12}}\Gamma^t_{jkl} - {}^{a_6}\Gamma^t_{jkl} + {}^{\frac{1}{2}(a_1+a_6)}\Gamma^t_{jkl} - {}^{\frac{1}{2}(a_7+a_{12})}\Gamma^t_{jkl})$$

$$+ (g_{hs}\overline{V}^s_{tk} + g_{ts}\overline{V}^s_{hk})({}^{a_{12}}\Gamma^t_{ljm} - {}^{a_6}\Gamma^t_{ljm} + {}^{\frac{1}{2}(a_1+a_6)}\Gamma^t_{ljm} - {}^{\frac{1}{2}(a_7+a_{12})}\Gamma^t_{ljm})$$

$$+ (g_{hs}\overline{V}^s_{tj} + g_{ts}\overline{V}^s_{hj})({}^{a_{12}}\Gamma^t_{mlk} - {}^{a_6}\Gamma^l_{mlk} + {}^{\frac{1}{2}(a_1+a_6)}\Gamma^t_{mlk} - {}^{\frac{1}{2}(a_7+a_{12})}\Gamma^t_{mlk})$$

$$+ (g_{hs}\overline{V}^s_{tl} + g_{ts}\overline{V}^s_{hl})({}^{a_{12}}\Gamma^t_{kmj} - {}^{a_6}\Gamma^t_{kmj} + {}^{\frac{1}{2}(a_1+a_6)}\Gamma^t_{kmj} - {}^{\frac{1}{2}(a_7+a_{12})}\Gamma^t_{kmj}).$$

By (1.7) we have

$$(2.20)\quad \begin{cases} {}^{a_1}\Gamma^t_{ijk} + {}^{a_{12}}\Gamma^t_{ijk} - {}^{a_7}\Gamma^t_{ijk} - {}^{a_6}\Gamma^t_{ijk} = 2\,[{}^{\frac{1}{2}(a_1+a_{12})}\Gamma^t_{ijk} - {}^{\frac{1}{2}(a_6+a_7)}\Gamma^t_{ijk}] \\ \text{and} \\ {}^{a_{12}}\Gamma^t_{ijk} - {}^{a_6}\Gamma^t_{ijk} + {}^{\frac{1}{2}(a_1+a_6)}\Gamma^t_{ijk} - {}^{\frac{1}{2}(a_7+a_{12})}\Gamma^t_{ijk} = [{}^{\frac{1}{2}(a_1+a_{12})}\Gamma^t_{ijk} - {}^{\frac{1}{2}(a_6+a_7)}\Gamma^t_{ijk}]. \end{cases}$$

Therefore the right hand side of (2.19) becomes

$$\begin{aligned}(2g_{ht,\,m} + g_{hs}\,\overline{V}^s_{tm} + g_{ts}\,\overline{V}^s_{hm})\,({}^{\frac{1}{2}(a_1+a_{12})}\Gamma^t_{jkl} - {}^{\frac{1}{2}(a_6+a_7)}\Gamma^t_{jkl})\\ + (2g_{ht,\,k} + g_{hs}\,\overline{V}^s_{tk} + g_{ts}\,\overline{V}^s_{hk})\,({}^{\frac{1}{2}(a_1+a_{12})}\Gamma^t_{ljm} - {}^{\frac{1}{2}(a_6+a_7)}\Gamma^t_{ljm})\\ + (2g_{ht,\,j} + g_{hs}\,\overline{V}^s_{tj} + g_{ts}\,\overline{V}^s_{hj})\,({}^{\frac{1}{2}(a_1+a_{12})}\Gamma^t_{mlk} - {}^{\frac{1}{2}(a_6+a_7)}\Gamma_{mlk})\\ + (2g_{ht.\,l} + g_{hs}\,\overline{V}^s_{tl} + g_{ts}\,\overline{V}^s_{hl})\,({}^{\frac{1}{2}(a_1+a_{12})}\Gamma^t_{kmj} - {}^{\frac{1}{2}(a_6+a_7)}\Gamma^t_{kmj}).\end{aligned}$$

Again, since $\frac{1}{2}(a_1 + a_{12})$ and $\frac{1}{2}(a_6 + a_7)$ are self conjugate, we have from (2.17) and (2.18)

$$\begin{aligned}2\,\{{}^{\frac{1}{2}(a_1+a_{12})}\Gamma_{hjklm}\} = {} & [2g_{ht,\,m} + g_{hs}\,\overline{V}^s_{tm} + g_{ts}\,\overline{V}^s_{hm}]\,{}^{\frac{1}{2}(a_1+a_{12})}\Gamma^t_{jkl}\\ & + [2g_{ht,\,k} + g_{hs}\,\overline{V}^s_{tk} + g_{ts}\,\overline{V}^s_{hk}]\,{}^{\frac{1}{2}(a_1+a_{12})}\Gamma^t_{ljm}\\ & + [2g_{ht,\,j} + g_{hs}\,\overline{V}^s_{tj} + g_{ts}\,\overline{V}^s_{hj}]\,{}^{\frac{1}{2}(a_1+a_{12})}\Gamma^t_{mlk}\\ & + [2g_{ht,\,l} + g_{hs}\,\overline{V}^s_{tl} + g_{ts}\,\overline{V}^s_{hl}]\,{}^{\frac{1}{2}(a_1+a_{12})}\Gamma^t_{klm}\,.\end{aligned}$$

Similarly for

$$2\,\{{}^{\frac{1}{2}(a_6+a_7)}\Gamma_{hjklm}\}.$$

Hence (2.19) finally reduces to

$$\begin{aligned}&\{{}^{a_1}\Gamma_{hjklm}\} + \{{}^{a_6}\Gamma_{hjklm}\} + \{{}^{a_7}\Gamma_{hjklm}\} + \{{}^{a_{12}}\Gamma_{hjklm}\}\\ &\quad - 2\,[\{{}^{\frac{1}{2}(a_1+a_6)}\Gamma_{hjklm}\} + \{{}^{\frac{1}{2}(a_7+a_{12})}\Gamma_{hjklm}\} + \{{}^{\frac{1}{2}(a_1+a_{12})}\Gamma_{hjklm}\} + \{{}^{\frac{1}{2}(a_6+a_7)}\Gamma_{hjklm}\}] = 0.\end{aligned}$$

Thus we have

$$(2.21) \qquad \{{}^{a_p}\Gamma_{hjklm}\} + \{{}^{a_q}\Gamma_{hjklm}\} + \{{}^{a_{p+6}}\Gamma_{hjklm}\} + \{{}^{a_{q+6}}\Gamma_{hjklm}\}$$

$$- 2\,[\{{}^{\frac{1}{2}(a_p+a_{q+6})}\Gamma_{hjklm}\} + \{{}^{\frac{1}{2}(a_{p+6}+a_q)}\Gamma_{hjklm}\} + \{{}^{\frac{1}{2}(a_p+a_q)}\Gamma_{hjklm}\}$$

$$+ \{{}^{\frac{1}{2}(a_{p+6}+a_{q+6})}\Gamma_{hjklm}\}] = 0,$$

where (p, q) is any one of (1,12), (2,3) and (4,5). The three equations (2.21) constitute the required generalisation of the identity (1.14) over Sen's system of affine connections.

3. In this section we proceed to prove some relations connecting curvature tensors which may be considered as generalisation of (1.10). For this purpose we adopt the following notations. Put

$$(3.1) \quad \text{and} \qquad \begin{aligned} {}^{a_p}_{1}A_{hijk} &= {}^{a_p}\Gamma_{hijk} + {}^{a_p}\Gamma_{hjki} + {}^{a_p}\Gamma_{hkij} \\ {}^{a_p}_{2}A_{hijk} &= {}^{a_p}\Gamma_{hijk} + {}^{a_p}\Gamma_{jikh} + {}^{a_p}\Gamma_{kihj}\,. \end{aligned}$$

Dr. H. Sen (Sen, 1959) generalised the identity (1.10) and obtained two sets of relations which are as follows:

$$(3.2) \qquad {}^{a_p}_{1}A_{hijk} + {}^{a_p}_{1}A_{hijk} - {}^{a_{p+6}}_{1}A_{hijk} - {}^{a_{q+6}}_{1}A_{hijk} = 0,$$

where (p, q) is any one of (1,12), (2,3) and (4,5).

$$(3.3) \qquad {}^{a_p}_{2}A_{hijk} + {}^{a_q}_{2}A_{hijk} - {}^{a_{p+6}}_{2}A_{hijk} - {}^{a_{q+6}}_{2}A_{hijk} = 0,$$

where (p, q) is any one of (2,11), (1,4) and (3,6).

We propose to obtain some other relations of this type.

It is known that ${}^{a_p}_{1}A_{hijk} = 0$ if a_p is self conjugate. Therefore, applying the formula (1.8) directly, we have

$$(3.4) \quad 2\,{}^{\frac{1}{2}(a_p+a_q)}_{1}A_{hijk} - {}^{a_p}_{1}A_{hijk} - {}^{a_q}_{1}A_{hijk} + {}^{\frac{1}{2}(a_{p+6}+a_q)}_{1}A_{hijk} + {}^{\frac{1}{2}(a_p+a_{q+6})}_{1}A_{hijk} = 0.$$

Accordingly, from (3.4) we arrive at the following sets of formalae.

$$
(3.5a)\qquad \begin{cases} {}^{a_1}_{1}A_{hijk} + {}^{a_{12}}_{1}A_{hijk} - {}^{\frac{1}{2}(a_{12}+a_7)}_{1}A_{hijk} - {}^{\frac{1}{2}(a_1+a_6)}_{1}A_{hijk} = 0, \\ {}^{a_6}_{1}A_{hijk} + {}^{a_7}_{1}A_{hijk} - {}^{\frac{1}{2}(a_7+a_{12})}_{1}A_{hijk} - {}^{\frac{1}{2}(a_1+a_6)}_{1}A_{hijk} = 0, \end{cases}
$$

$$
(3.5b)\qquad \begin{cases} {}^{a_2}_{1}A_{hijk} + {}^{a_3}_{1}A_{hijk} - {}^{\frac{1}{2}(a_3+a_8)}_{1}A_{hijk} - {}^{\frac{1}{2}(a_2+a_9)}_{1}A_{hijk} = 0, \\ {}^{a_8}_{1}A_{hijk} + {}^{a_9}_{1}A_{hijk} - {}^{\frac{1}{2}(a_3+a_8)}_{1}A_{hijk} - {}^{\frac{1}{2}(a_2+a_9)}_{1}A_{hijk} = 0, \end{cases}
$$

$$
(3.5c)\qquad \begin{cases} {}^{a_4}_{1}A_{hijk} + {}^{a_5}_{1}A_{hijk} - {}^{\frac{1}{2}(a_5+a_{10})}_{1}A_{hijk} - {}^{\frac{1}{2}(a_4+a_{11})}_{1}A_{hijk} = 0, \\ {}^{a_{10}}_{1}A_{hijk} + {}^{a_{11}}_{1}A_{hijk} - {}^{\frac{1}{2}(a_5+a_{10})}_{1}A_{hijk} - {}^{\frac{1}{2}(a_4+a_{11})}_{1}A_{hijk} = 0. \end{cases}
$$

The three formulae (3.2) can be obtained by subtracting one equation from the other from (3.5a), (3.5b) and (3.5c).

The second member of (2.20) gives another relation

$$
{}^{a_{12}}_{1}A_{hijk} - {}^{a_6}_{1}A_{hijk} + {}^{\frac{1}{2}(a_1+a_6)}_{1}A_{hijk} - {}^{\frac{1}{2}(a_7+a_{12})}_{1}A_{hijk} = 0.
$$

Similarly using the formula

$$
{}^{a_1}\Gamma_{hijk} - {}^{a_7}\Gamma_{hijk} + {}^{\frac{1}{2}(a_7+a_{12})}\Gamma_{hijk} - {}^{\frac{1}{2}(a_1+a_6)}\Gamma_{hijk} = {}^{\frac{1}{2}(a_1+a_{12})}\Gamma_{hijk} - {}^{\frac{1}{2}(a_6+a_7)}\Gamma_{hijk},
$$

we get

$$
{}^{a_1}_{1}A_{hijk} - {}^{a_7}_{1}A_{hijk} + {}^{\frac{1}{2}(a_7+a_{12})}_{1}A_{hijk} - {}^{\frac{1}{2}(a_1+a_6)}_{1}A_{hijk} = 0.
$$

Thus we have

$$
(3.6)\qquad \begin{cases} {}^{a_{12}}_{1}A_{hijk} - {}^{a_6}_{1}A_{hijk} + {}^{\frac{1}{2}(a_1+a_6)}_{1}A_{hijk} - {}^{\frac{1}{2}(a_7+a_{12})}_{1}A_{hijk} = 0, \\ {}^{a_1}_{1}A_{hijk} - {}^{a_7}_{1}A_{hijk} + {}^{\frac{1}{2}(a_7+a_{12})}_{1}A_{hijk} - {}^{\frac{1}{2}(a_1+a_6)}_{1}A_{hijk} = 0, \\ {}^{a_2}_{1}A_{hijk} - {}^{a_8}_{1}A_{hijk} + {}^{\frac{1}{2}(a_8+a_3)}_{1}A_{hijk} - {}^{\frac{1}{2}(a_2+a_9)}_{1}A_{hijk} = 0, \\ {}^{a_3}_{1}A_{hijk} - {}^{a_9}_{1}A_{hijk} + {}^{\frac{1}{2}(a_2+a_9)}_{1}A_{hijk} - {}^{\frac{1}{2}(a_8+a_3)}_{1}A_{hijk} = 0, \\ {}^{a_4}_{1}A_{hijk} - {}^{a_{10}}_{1}A_{hijk} + {}^{\frac{1}{2}(a_{10}+a_5)}_{1}A_{hijk} - {}^{\frac{1}{2}(a_4+a_{11})}_{1}A_{hijk} = 0, \\ {}^{a_5}_{1}A_{hijk} - {}^{a_{11}}_{1}A_{hijk} + {}^{\frac{1}{2}(a_{11}+a_4)}_{1}A_{hijk} - {}^{\frac{1}{2}(a_{10}+a_5)}_{1}A_{hijk} = 0. \end{cases}
$$

Now writing $g_{ht}\, C(a^*) = \Gamma^*_{hijk}$ and using $\Gamma_{hijk} + \Gamma^*_{ihjk} = 0$ (Sen, 1950b), we get

$$ {}^{a_p}_{1}A_{hijk} = - {}^{a_p^*}_{2}A_{ihjk} $$

and consequently all the equations of (3.5) and (3.6) reduce to

$$ \begin{cases} {}^{a_5}_{2}A_{hijk} + {}^{a_8}_{2}A_{hijk} - {}^{\frac{1}{2}(a_{11}+a_8)}_{2}A_{hijk} - {}^{\frac{1}{2}(a_5+a_2)}_{2}A_{hijk} = 0, \\ {}^{a_2}_{2}A_{hijk} + {}^{a_{11}}_{2}A_{hijk} - {}^{\frac{1}{2}(a_8+a_{11})}_{2}A_{hijk} - {}^{\frac{1}{2}(a_5+a_2)}_{2}A_{hijk} = 0, \\ {}^{a_1}_{2}A_{hijk} + {}^{a_4}_{2}A_{hijk} - {}^{\frac{1}{2}(a_7+a_4)}_{2}A_{hijk} - {}^{\frac{1}{2}(a_1+a_{10})}_{2}A_{hijk} = 0, \\ {}^{a_7}_{2}A_{hijk} + {}^{a_{10}}_{2}A_{hijk} - {}^{\frac{1}{2}(a_7+a_4)}_{2}A_{hijk} - {}^{\frac{1}{2}(a_1+a_{10})}_{2}A_{hijk} = 0, \\ {}^{a_3}_{2}A_{hijk} + {}^{a_6}_{2}A_{hijk} - {}^{\frac{1}{2}(a_3+a_{12})}_{2}A_{hijk} - {}^{\frac{1}{2}(a_6+a_9)}_{2}A_{hijk} = 0, \\ {}^{a_9}_{2}A_{hijk} + {}^{a_{12}}_{2}A_{hijk} - {}^{\frac{1}{2}(a_3+a_{12})}_{2}A_{hijk} - {}^{\frac{1}{2}(a_6+a_9)}_{2}A_{hijk} = 0, \end{cases} \tag{3.7} $$

and

$$ \begin{cases} {}^{a_{11}}_{2}A_{hijk} - {}^{a_5}_{2}A_{hijk} + {}^{\frac{1}{2}(a_2+a_5)}_{2}A_{hijk} - {}^{\frac{1}{2}(a_8+a_{11})}_{2}A_{hijk} = 0, \\ {}^{a_2}_{2}A_{hijk} - {}^{a_8}_{2}A_{hijk} + {}^{\frac{1}{2}(a_8+a_{11})}_{2}A_{hijk} - {}^{\frac{1}{2}(a_2+a_5)}_{2}A_{hijk} = 0, \\ {}^{a_1}_{2}A_{hijk} - {}^{a_7}_{2}A_{hijk} + {}^{\frac{1}{2}(a_7+a_4)}_{2}A_{hijk} - {}^{\frac{1}{2}(a_1+a_{10})}_{2}A_{hijk} = 0, \\ {}^{a_4}_{2}A_{hijk} - {}^{a_{10}}_{2}A_{hijk} + {}^{\frac{1}{2}(a_1+a_{10})}_{2}A_{hijk} - {}^{\frac{1}{2}(a_7+a_4)}_{2}A_{hijk} = 0, \\ {}^{a_3}_{2}A_{hijk} - {}^{a_9}_{2}A_{hijk} + {}^{\frac{1}{2}(a_9+a_6)}_{2}A_{hijk} - {}^{\frac{1}{2}(a_3+a_{12})}_{2}A_{hijk} = 0, \\ {}^{a_6}_{2}A_{hijk} - {}^{a_{12}}_{2}A_{hijk} + {}^{\frac{1}{2}(a_{12}+a_3)}_{2}A_{hijk} - {}^{\frac{1}{2}(a_6+a_9)}_{2}A_{hijk} = 0. \end{cases} \tag{3.8} $$

respectively. The equations (3.5), (3.6), (3.7) and (3.8) may be considered as the generalisation of (1.10). The three formulae (3.3) can be obtained by adding two suitable equations from (3.7) or (3.8).

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