

ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3^e série, tome 24,
n° 4 (1970), p. 703-715

http://www.numdam.org/item?id=ASNSP_1970_3_24_4_703_0

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MANIFOLDS OF THE HOMOTOPY TYPE OF A BOUQUET OF SPHERES

By D. D. J. HACON

1. Introduction.

This note is concerned with manifolds homotopy equivalent to $\vee S_i$ (a bouquet of spheres of varying dimensions). In this connection a useful concept is that of *thickening* which is a homotopy generalization of the idea of regular neighbourhood. The reader is referred to [7] for definitions in the general case. Here we shall be concerned with the specific problem of describing the set of thickenings of $\vee S_i$ and it will be convenient to adopt a definition of thickening that differs slightly from that to be found in [6] (see § 2).

In [3] Haefliger classified thickenings of (simply-connected) bouquets of spheres subject to certain dimensional restrictions. Our purpose here is to improve on Haefliger's result in the piecewise-linear case and deal with the non-simply-connected case, reducing it to the problem of classifying concordance classes of embeddings of a number of solid tori in a certain manifold, as follows.

Denote by P^n the solid n -pretzel i. e. an n -ball with a finite number of 1-handles attached orientably. Then the classification of thickenings is reduced to the classification of concordance classes of embeddings of the disjoint union of solid tori in ∂P , which is a simpler question. For instance, if the bouquet in question is simply-connected then ∂P is a sphere and the problem is now to classify concordance classes of embeddings of solid tori in a sphere. See [3]. If, on the other hand, the bouquet consists of a circle and a sphere we need to look at knots of a solid torus in $S^1 \times S^q$ (q being $\dim P - 2$).

Suppose $f: \vee S_i \rightarrow W$ is a homotopy equivalence (and hence a simple homotopy equivalence since the Whitehead group of a free group is trivial). If f is homotopic to a piecewise linear embedding $g: \vee S_i \rightarrow W$ proceed as follows. Take a regular neighbourhood N of $g \vee S_i$ in $\text{Int } W$, the interior of

W (if $g \vee S_i$ meets ∂W isotop it into $\text{Int } W$). N is homeomorphic to W , for by the s -cobordism theorem [1], $W - \text{Int } N$ is homeomorphic to $\partial N \times [0, 1]$. We thus obtain a handlebody decomposition of W suffixed by the cell structure of $\vee S_i$.

In general, however, there exist homotopy equivalences $f: \vee S_i \rightarrow W$ which are *not* homotopic to an embedding and consequently the above procedure cannot be followed. But a theorem of Stallings [5] allows us to factor $f: \vee S_i \rightarrow W$ up to homotopy through a simple homotopy equivalence $f': \vee S_i \rightarrow N$ where N is a $p_N + 1$ -dimensional polyhedron in W . We seek a simple description of N in terms of $\vee S_i$ which will (as in the case when f is an embedding) provide a handlebody decomposition of W suffixed by the cell structure of $\vee S_i$. In fact it will be shown (§ 3) that, if $f: \vee S_i \rightarrow W$ is a homotopy equivalence, W may be expressed as P plus handles of index two or more and that handles of sufficiently large index are attached disjointly i. e. after a certain point in the construction of W the order in which handles are subsequently attached is immaterial.

2. The main theorem.

Throughout we restrict ourselves to the piecewise linear (PL) category [7].

Write $\cup S_i$ for the disjoint union $S_1 \cup, \dots, \cup S_N$ of spheres S_1, \dots, S_N of dimensions p_1, \dots, p_N subject to the condition $1 \leq p_1 \leq, \dots, \leq p_N$. Let $* = (*, \dots, *)$ be a point of $S_1 \times, \dots, \times S_N$. Then $\vee S_i = S_1 \vee, \dots, \vee S_N$ is the subpolyhedron

$$(S_1 \times \{*\} \times, \dots, \times \{*\}) \cup, \dots, \cup (\{*\} \times, \dots, \times \{*\} \times S_N) \text{ of } S_1 \times, \dots, \times S_N.$$

Let $\pi: \cup S_i \rightarrow \vee S_i$ be the obvious identification map. If we write $\vee S_i$ in the form $S^1 \vee, \dots, \vee S^1 \vee S_1 \vee, \dots, \vee S_N$ it is understood that $p_i \geq 2$.

Now let M be a compact, connected, oriented manifold with nonempty boundary ∂M and such that

$$(1) \quad \dim M \geq \text{Max}(6, \dim \cup S_i + 3)$$

$$(2) \quad \partial M \subset M \text{ induces an isomorphism of fundamental groups.}$$

We will be considering pairs (M, f) , M as above and $f: \cup S_i \rightarrow M$ homotopic to $g \circ \pi$ where $g: \vee S_i \rightarrow M$ is a homotopy equivalence. We call such a pair (M, f) a *thickening*.

REMARKS (1). Since M is assumed connected any map $f: \cup S_i \rightarrow M$ factors up to homotopy through $\pi: \cup S_i \rightarrow \vee S_i$.

(2) Suppose $f: \cup S_i \rightarrow M$ factors up to homotopy through a homotopy equivalence $g: \vee S_i \rightarrow M$. Let $g': \vee S_i \rightarrow M$ be any other homotopy factorization. Then g' is also a homotopy equivalence.

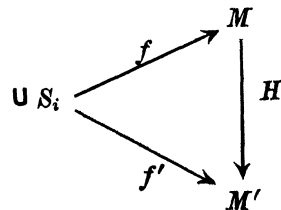
(3) If S_1, \dots, S_N are all circles then all thickenings (M, f) are equivalent in the sense below.

(M, f) and (M', f') are said to be *equivalent* if $M \cong M'$ are homeomorphic and the homeomorphism H can be chosen to preserve all the data i. e..

(a) H is orientation-preserving

(b) the diagram

homotopy commutes.



REMARK. In the simply-connected case ($p_1 \geq 2$) these definitions coincide with those of Haefliger [3].

Let P be the manifold defined in the introduction. That is a ball plus 1-handles. Let $\dim P = q$ where $q \geq p_N + 3$ and $p_1 \geq 2$. Any inessential orientation-preserving embedding $h: \bigcup_1^N \partial \Delta^{p_i} \times \Delta^{q-p_i} \rightarrow \partial P$ determines an oriented manifold of the homotopy type of a bouquet of spheres. The manifold is P plus handles $\Delta^{p_i} \times \Delta^{q-p_i}$ attached by means of the given embedding. Since the embedding h was assumed inessential we obtain a well-defined homotopy class of maps $f: \cup S_i \rightarrow P + \text{handles}$. It is easily seen that this f is in fact a thickening.

REMARK. The restriction that h be inessential is only a restriction if $p_1 = 2$, since $\pi_i(P) = 0$ for $i > 1$.

Suppose h, k are concordant embeddings. It is an immediate consequence of the concordance extension theorem [4] that the two thickenings defined by h and k are equivalent.

Thus there is defined a function Φ from the set of concordance classes of embeddings of solid tori in ∂P to the set of equivalence classes of thickenings of $\vee S_i$.

Here is the main result of this note.

THEOREM. If, for the bouquet $S^1 \vee \dots \vee S^1 \vee S_1 \vee \dots \vee S_N$, $\dim P = q \geq \text{Max}(6, p_N + 3)$ then

(1) Φ is surjective if $2p_N - q + 1 < p_1$

(2) Φ is injective if $2p_N - q + 2 < p_1$.

The proof is deferred to section 3.

3. The factorization lemma.

In this section it will be shown that any thickening $f: \cup S_i \rightarrow W$ factors up to homotopy through a homotopy equivalence $g: \vee S_i \rightarrow N$ where N is a subpolyhedron of $\text{Int } W$ with special properties. But first some notation. Write X for $\vee S_i$ and filter X by $* = X_0 \subset, \dots, \subset X_N = X$ where $X_i = S_i \vee, \dots, \vee S_i$ and $*$ is the base point of X . Let $f_k = f|S_k$. Finally, denote by Σf the singular set of f i. e. the closure of the set $\{x \in X | f^{-1}fx \neq x\}$.

LEMMA 1. (factorization lemma) Let $f: X \rightarrow \text{Int } W$ be a thickening (or more accurately let $f \circ \pi$ be one). Suppose that f is nondegenerate and that for $k = 1, \dots, N$, $\dim \Sigma f \cap S_k \leq p_k + p_N - q$. Then there exist polyhedra $Y_0 \subset Y_1 \subset, \dots, \subset Y_N$ in $\text{Int } W$ such that

- (1) $f|X_k \rightarrow Y_k \cup fS_k$ is a homotopy equivalence
- (2) Y_{k+1} collapses to $Y_k \cup fS_k$
- (3) $\dim Y_k \cap fS_{k+i} \leq p_k + p_N - q$ ($i > 0$)
- (4) $\Sigma f \cap S_k \subset f_k^{-1}Y_k$ and the latter collapses to $*$.

REMARK 1. If the first few S_i of X have small enough dimension then f embeds them (by the general position hypotheses on f) and the first few Y_i are defined by $Y_i = fX_{i-1}$ satisfying conditions (1), ..., (4).

REMARK 2. Lemma 1 yields a minimal handlebody decomposition for W as follows. Define inductively handlebodies H_k in $\text{Int } W$ ($k = 1, \dots, N$) by first triangulating $f: X \rightarrow W$ so that Y_0, \dots, Y_N appear as subcomplexes (to be denoted by the same symbols). Take the barycentric second-derived subdivisions of X, W . f remains simplicial being non-degenerate. Define $H_k = N^2(Y_k \cup fS_k; W)$, the simplicial neighbourhood of $Y_k \cup fS_k$ in the second-derived subdivision of W . Then H_{k+1} is H_k plus a handle. For $(Y_{k+1} \cup fS_{k+1}) - \text{Int } N^2(Y_{k+1}; W) = fS_{k+1} - \text{Int } N^2(Y_{k+1}; W)$ and the latter is a p_{k+1} -disk in $\text{Int } W$ that meets $N^2(Y_{k+1}; W)$ in its boundary. This follows from condition (4).

And $H_{k+1} - \text{Int } N^2(Y_{k+1}; W)$ is a ball meeting $N^2(Y_{k+1}; W)$ in $N^2(fS_{k+1}; W) \cap \partial N^2(Y_{k+1}; W)$ which is a solid torus. By regular neighbo-

uhoods theory, H_{k+1} is homeomorphic to

$$N^2(Y_{k+1}; W) \cup_{\psi} \Delta^{2k+1} \times \Delta^{q-2k+1}$$

where $\psi: \partial\Delta \times \Delta \rightarrow \partial N^2(Y_{k+1}; W)$ is an embedding. But by condition (2) Y_{k+1} collapses to $Y_k \cup fS_k$ and so $N^2(Y_{k+1}; W)$ and $H_k = N^2(Y_k \cup fS_k; W)$ are homeomorphic. Therefore H_{k+1} is H_k plus a handle and we obtain a handlebody decomposition of H_N (and hence of W) suffixed by the cell structure of X . Furthermore, (1) implies that the thickening $f: X \rightarrow W$ is filtered by a series of thickenings $f|X_k \rightarrow N^2(Y_k \cup fS_k; W)$.

As observed in the introduction, it is possible (under certain dimensional restrictions) to find a handlebody decomposition of W in which the handles are attached independently of one another after a certain stage. To show this we need a modification of the factorization lemma.

LEMMA 2. Let f satisfy the hypothesis of lemma 1. Suppose, in addition, that $p_{k+1} \geq 2p_N - q + 2$ for some $k (1 \leq k < N)$, and that Y_0, \dots, Y_k have been found satisfying conditions (1) through (4) of Lemma 1. Then there exists a polyhedron Y in $\text{Int } W$ such that

- (a) Y collapses to $Y_k \cup fS_k$
- (b) $f: X \rightarrow Y \cup fX$ is a homotopy equivalence
- (c) $\Sigma f \subset f^{-1}Y$
- (d) $f_{k+i}^{-1}Y$ collapses to $*$ for all $i > 0$.

REMARK. As before we have a handlebody decomposition of W . Triangulate $f: X \rightarrow W$ so that Y_0, \dots, Y_k, Y are subcomplexes of W . Define

$$H_j = N^2(Y_j \cup fS_j; W) \quad (0 \leq j \leq k)$$

and

$$H = N^2(Y \cup fX; W).$$

Then the handles $N^2(fS_j; W) - \text{Int } N^2(Y; W)$ are attached independently to $N^2(Y; W)$ i.e. $N^2(fS_j; W) \cap \partial N^2(Y; W)$ are disjoint solid tori ($k+1 \leq j \leq N$). For, by (c) of Lemma 2, f embeds

$$\begin{aligned} f^{-1} \left\{ \bigcup_{j=k+1}^N fS_j - \text{Int } N^2(Y; W) \right\} &\text{ and } S_j - f^{-1} \text{Int } N^2(Y; W) \\ &= S_j - \text{Int } N^2(f_j^{-1}Y; W) \\ &= S_j \text{ minus the interior of a ball, by (d)} \\ &= a \text{ ball, } (j = k+1, \dots, N); \end{aligned}$$

and the ball $fS_j - \text{Int } N^2(Y; W)$ meets $N^2(Y; W)$ in its boundary only, by (e). Thus $N^2(Y; W) \cup fX$ is $N^2(Y; W)$ plus balls $fS_j - \text{Int } N^2(Y; W)$ attached disjointly to $\partial N^2(Y; W)$. This completes the proof that H is H_k plus disjointly attached handles.

To prove Lemma 1 and 2 we will need some general position and engulfing lemmas.

DEFINITION. If Y_0, Y, Z are polyhedra in the manifold M and $Y_0 \subset Y$, then $Y - Y_0$ is said to be in *general position with respect to Z* if $\dim(Y - Y_0) \cap Z \leq \dim Y - Y_0 + \dim Z - \dim M$.

DEFINITION. If Y is a polyhedron and M a manifold, a map $f: Y \rightarrow M$ is in *general position* if

- (1) f is non-degenerate
- (2) $\dim \Sigma f < 2 \dim Y - \dim M$.

COROLLARY TO THEOREM 15 [7]. If Y_0, Y, A_1, \dots, A_n are polyhedra in a manifold M with $Y_0 \subset Y$ and $Y - Y_0 \subset \text{int } M$, then there exists a homeomorphism $h: M \rightarrow M$ such that

- (1) $h|_{Y_0 \cup \partial M} = \text{Identity}$
- (2) $h(Y - Y_0)$ is in general position with respect to A_1, \dots, A_n .

PROOF. By induction on $\dim A_1 \cup \dots \cup A_n$.

COROLLARY TO THEOREM 18 [7]. Let $f: Y \rightarrow \text{Int } M$ be a map and Y_0 a subpolyhedron of Y . Suppose $f|_{Y_0}$ is in general position. Then f is homotopic to g , a map in general position, by an arbitrarily small homotopy that keeps Y_0 fixed.

LEMMA 3. If $f: X \rightarrow M$, X a sphere-bouquet, M a manifold, then f is homotopic to $g: X \rightarrow \text{Int } M$ where g is in general position and $\dim \Sigma g \cap S_k \leq \dim S_k + \dim X - \dim M$ ($k = 1, \dots, N$).

PROOF. First homotop fX into $\text{Int } M$ and then use induction on N , the number of spheres in the bouquet. If $N = 1$ apply the second corollary above. If not, the inductive step is proved by homotoping $f|_{S_N}$ into general position keeping f^* fixed and then applying the first corollary to minimize the dimension of $fS_N \cap fX_{N-1}$ by putting $f(S_N - *)$ into general position with respect to fX_{N-1} keeping f^* fixed.

To state the engulfing lemmas we need

DEFINITION. A subpolyhedron C of a manifold M is called a k -spine of M if the pair M, C is k -connected.

DEFINITION. A polyhedron is called t -collapsible if it can be collopsed to a polyhedron of dimension not greater than t . The following lemma is a special case of Theorem 21 [7].

LEMMA 4 (Zeeman). Let C be an $m-3$ -collapsible k -spine of the manifold M ($\dim M$ being m), Y a polyhedron in M and

$$\dim Y \cap \partial M < \dim Y \leq k \leq m - 3.$$

Then Y may be engulfed from C relative to ∂M i. e. there exists C^+ in M such that $C \cup Y \subset C^+$, $(C \cup Y) \cap \partial M = C^+ \cap \partial M$, C^+ collapses to C , and $\dim C^+ - C \leq \dim Y + 1$.

ADDENDUM TO LEMMA 4. Suppose that A_1, \dots, A_n are polyhedra in M . By the corollary to Theorem 15 we may insist that $C^+ - (C \cap Y)$ be in general position with respect to A_1, \dots, A_n .

LEMMA 5. Let C be an $m-3$ -collapsible k -spine of M and D a $q-3$ -collapsible $k+1$ -spine of Q and let $f: M, C \rightarrow Q, D$ be non-degenerate and proper (i.e. $f^{-1} \partial Q = \partial f^{-1} Q$). Suppose that $\dim (f^{-1} D - C) = x \leq k \leq m - 3 \leq q - 6$ and that $\partial M \cap (f^{-1} D - C)$ is empty.

Then there exist polyhedra $C^+ \subset M, D^+ \subset Q$ such that

- (A) $C^+ = f^{-1} D^+$ (i.e. $\dim f^{-1} D^+ - C^+ < 0$)
- (B) $C^+ \cap \partial M = C \cap \partial M; D^+ \cap \partial Q = D \cap \partial Q$
- (C) C^+ collapses to $C; D^+$ collapses to D
- (D) $\dim C^+ - C \leq x + 1; \dim D^+ - D \leq x + 2$.

If, further, $A_1, \dots, A_n \subset Q$ are polyhedra in general position with respect to fM , then C^+, D^+ may be chosen to satisfy (A), ..., (D) and the extra condition

- (E) $D^+ - D$ is in general position with respect to A_1, \dots, A_n .

PROOF. The proof resembles that of Lemma 63 [7]. We will define inductively polyhedra $C_i \subset M, D_i \subset Q$ such that

- (a) $fC_i \subset D_i$ and $\dim f^{-1} D_i - C_i \leq x - i$.
- (b) C_i collapses to $C; D_i$ collapses to D .

- (c) $C_i \cap \partial M = C \cap \partial M$; $D_i \cap \partial Q = D \cap \partial Q$.
 (d) $\dim C_i - C_{i-1} \leq x + 2 - i$; $\dim D_i - D_{i-1} \leq x + 3 - i$.
 (e) $D_i - D_{i-1}$ is in general position with respect to A_1, \dots, A_n .

The induction starts with $C_i = C$, $D_i = D$ ($i \leq 0$) and finishes with $i = x + 1$ because then $\dim f^{-1} D_i - C_i < 0$. Condition (E) will be satisfied because $D^+ - D = \bigcup_{i \geq 0} (D_{i+1} - D_i)$ and each $D_{i+1} - D_i$ is in general position with respect to A_1, \dots, A_n .

The inductive step ($i \geq 0$).

Assume that C_j, D_j have been chosen satisfying (a), ..., (e) for $j \leq i$.

By (a) $\dim f^{-1} D_i - C_i \leq x - i$.

By (b) C_i is an $m - 3$ -collapsible k -spine of M (since C is).

So by Lemma 4 there exists $C_{i+1} \subset M$ such that C_{i+1} collapses to C_i , $f^{-1} D_i \subset C_{i+1}$, $\partial M \cap C_{i+1} = \partial M \cap f^{-1} D_i$, $\dim C_{i+1} - C_i \leq x + 1 - i$ and $C_{i+1} - f^{-1} D_i$ is in general position with respect to $f^{-1} A_1, \dots, f^{-1} A_n$. This implies that $\dim f C_{i+1} - D_i \leq \dim f(C_{i+1} - C_i) \leq x + 1 - i$; also that $\partial M \cap C_{i+1} = \partial M \cap C_i \cup \partial M \cap f^{-1} D_i$. But

$$\begin{aligned}
 \partial M \cap f^{-1} D_i &= \\
 &= f^{-1}(\partial Q \cap D_i) \quad (f \text{ is proper}) \\
 &= f^{-1}(\partial Q \cap D) \quad (\text{by (c)}) \\
 &= \partial M \cap f^{-1} D \\
 &= \partial M \cap C \quad (\text{by initial hypothesis}).
 \end{aligned}$$

By (b) D_i is a $q - 3$ -collapsible $k + 1$ -spine of Q . So by Lemma 4, there exists $D_{i+1} \subset Q$ such that D_{i+1} collapses to D_i , $f C_{i+1} \subset D_{i+1}$, $\dim D_{i+1} - D_i \leq x + 2 - i$, $\partial Q \cap D_{i+1} = \partial Q \cap (D_i \cup f C_{i+1})$ and $D_{i+1} - (D_i \cup f C_{i+1})$ is in general position with respect to fM, A_1, \dots, A_n . This implies that $\dim f^{-1} D_{i+1} - C_{i+1} \leq \dim f f^{-1} D_{i+1} - f C_{i+1} = \dim fM \cap (D_{i+1} - f C_{i+1}) = \dim fM \cap (D_{i+1} - (f C_{i+1} \cup D_i)) \leq x + 2 - i - 3$. Also we have that $\partial Q \cap D_{i+1} = \partial Q \cap D_i \cup \partial Q \cap f C_{i+1}$. But $\partial Q \cap f C_{i+1} = f(\partial M \cap C_{i+1}) = f(\partial M \cap C) \subset \subset \partial Q \cap D$. So $\partial Q \cap D_{i+1} = \partial Q \cap D$. We have thus defined C_{i+1}, D_{i+1} satisfying (a), ..., (d). The Δ also satisfies (e); for, $D_{i+1} - (f C_{i+1} \cup D_i)$ is in general position with respect to A_1, \dots, A_n ; and $f C_{i+1} - D_i = f(C_{i+1} - f^{-1} D_i)$ is in general position with respect to A_1, \dots, A_n since A_1, \dots, A_n are (by hypothesis) in general position with respect to fM and $C_{i+1} - f^{-1} D_i$ was chosen

to be in general position with respect to $f^{-1}A_1, \dots, f^{-1}A_n$. This completes the proof of the inductive step and hence of lemma 5.

PROOF OF LEMMA 1. Let us write $Z_k = Y_k \cup fS_k$. Construct Y_k (and hence Z_k) inductively starting with $Y_0 = Z_0 = fX_0 = f*$. Suppose that we have found Y_0, \dots, Y_k satisfying conditions (1), ..., (4). By lemma 4 and the fact that $\dim \Sigma f \cap S_{k+1} \leq p_{k+1} - 3$ there exists C_{k+1} in S_{k+1} such that $\Sigma f \cap S_{k+1} \subset C_{k+1}$, C_{k+1} collapses to $*$, and $\dim C_{k+1} \leq 1 + p_N + p_{k+1} - q$. Now Z_k is a $p_{k+1} - 1$ -spine of $\text{Int } W$ and $1 + p_N + p_{k+1} - q \leq p_{k+1} - 2$. Therefore by lemma 4 there exists D_{k+1} in $\text{Int } W$ such that $fC_{k+1} \subset D_{k+1}$, D_{k+1} collapses to Z_k , $\dim D_{k+1} - Z_k \leq 2 + p_N + p_{k+1} - q$ and $D_{k+1} - (Z_k \cup fC_{k+1})$ is in general position with respect to fS_{k+1}, \dots, fS_N . This and condition (3) imply that for $i > 1$

$$\begin{aligned} & p_N + p_{k+1} - q \geq \\ & \geq \dim fS_{k+i} \cap [D_{k+1} - (Z_k \cup fC_{k+1}) \cup Z_k \cup fC_{k+1}] \\ & \geq \dim fS_{k+i} \cap D_{k+1}. \end{aligned}$$

Now $f_{k+1} : S_{k+1}, C_{k+1} \rightarrow W$, D_{k+1}, C_{k+1} is a $p_{k+1} - 2$ -spine of S_{k+1} , D_{k+1} is a $p_{k+1} - 1$ -spine of $\text{Int } W$ and $\dim f_{k+1}^{-1} D_{k+1} - C_{k+1} \leq p_N + p_{k+1} - q$. So, by lemma 5, there exists Y_{k+1} in $\text{Int } W$ such that Y_{k+1} collapses to D_{k+1} , $f_{k+1}^{-1} Y_{k+1}$ collapses to $*$ and $\dim fS_{k+i} \cap (Y_{k+1} - D_{k+1}) \leq p_N + p_{k+1} - q$ ($i > 1$). It follows that $\dim fS_{k+i} \cap Y_{k+1} \leq p_N + p_{k+1} - q$ ($i > 1$). Thus Y_{k+1} is defined and satisfies (2) (3) and (4).

The proof of the induction step will be complete once it has been shown that $f|X_{k+1} \rightarrow Z_{k+1}$ is a homotopy equivalence. First triangulate $f : X \rightarrow W$ and pass to the barycentric second derived triangulations of X, W . f remains simplicial.

We showed that $f_{k+1}^{-1} Y_{k+1}$ collapsed to $*$. Thus $N^2(f_{k+1}^{-1} Y_{k+1}; S_{k+1}) = f_{k+1}^{-1} N^2(Y_{k+1}; W)$ is a ball.

Further, $\Sigma f \cap S_{k+1} \subset f_{k+1}^{-1} Y_{k+1}$ and so f_{k+1} maps

$$S_{k+1} - \text{Int } N^2(f_{k+1}^{-1} Y_{k+1}; S_{k+1})$$

homeomorphically onto $Z_{k+1} - \text{Int } N^2(Y_{k+1}; W)$.

To prove that $f|X_{k+1} \rightarrow Z_{k+1}$ is a homotopy equivalence, we show that

(*) $f|X_{k+1} \rightarrow N^2(Y_{k+1}; W) \cup fS_{k+1}$ is a homotopy equivalence.

(**) $N^2(Y_{k+1}; W) \cup fS_{k+1}$ collapses to $Y_{k+1} \cup fS_{k+1}$.

Composing (**) with (*), we obtain a homotopy equivalence:

$$X_{k+1} \xrightarrow{f|} N^2(Y_{k+1}; W) \cup fS_{k+1} \supset Y_{k+1} \cup fS_{k+1}.$$

PROOF OF (*). f maps the pair $X_{k+1}, X_k \cup N^2(f_{k+1}^{-1} Y_{k+1}; S_{k+1})$ into the pair $fS_{k+1} \cup N^2(Y_{k+1}; W), N^2(Y_{k+1}; W)$.

$f| X_k \rightarrow N^2(Y_{k+1}; W)$ is a homotopy equivalence because $f| X_k \rightarrow Z_k$ is one and $N^2(Y_{k+1}; W)$ collapses to Z_k via Y_{k+1} .

Let us write $U(\)$ for «universal cover of». All spaces to which $U(\)$ is applied will have isomorphic fundamental groups for [by Remark (1) following Lemma 1] p_{k+1} may be assumed to be greater than one. Therefore the map $f| X_{k+1}$ induces homology excision isomorphisms between $H_*(U(X_{k+1}), U(X_k \cup N^2(f_{k+1}^{-1} Y_{k+1}; S_{k+1})))$ and

$$H_*(U(fS_{k+1} \cup N^2(Y_{k+1}; W)), U(N^2(Y_{k+1}; W))).$$

So, by the 5-Lemma and Whitehead's theorem, the map $fX_{k+1} \rightarrow N^2(Y_{k+1}; W) \cup fS_{k+1}$ induces isomorphisms of homotopy groups in all dimensions and is thus a homotopy equivalence.

PROOF OF (**). $N^2(Y_{k+1}; W) \cup fS_{k+1}$ collapses to $Y_{k+1} \cup fS_{k+1}$ because we may factor the collapse from $N^2(Y_{k+1}; W)$ to Y_{k+1} through $Y_{k+1} \cup U N^2(Y_{k+1} \cap fS_{k+1}; fS_{k+1})$. This proves (**) and completes the proof of Lemma 1.

PROOF OF LEMMA 2. Suppose polyhedra Y_0, \dots, Y_k have been found satisfying conditions (1), ..., (4) of Lemma 1. Recall that for $i > 0$ $\dim \Sigma f \cap S_{k+i} \leq p_N + p_{k+i} - q$ and $\dim f_{k+i}^{-1} Z_k \leq p_N + p_k - q$. So by Lemma 4 there exists C in $S_{k+1} \cup \dots \cup S_N$ such that $(\Sigma f \cup f^{-1} Z_k) \cap (S_{k+1} \cup \dots \cup S_N) \subset C$, C collapses to $*$ and $\dim C \cap S_{k+i} \leq p_N + p_{k+i} - q + 1$. Now Z_k is a $p_{k+1} - 1$ -spine of $\text{Int } W$ and by hypothesis $1 + 2p_N - q \leq p_{k+1} - 1$. Therefore by Lemma 4 there exists D such that $fC \subset D$, D collapses to Z_k $\dim D - Z_k \leq 2 + 2p_N - q$ and $D - (fC \cup Z_k)$ is in general position with respect to fS_{k+1}, \dots, fS_N .

Then

$$\begin{aligned} \dim f_{k+i}^{-1} D - (C \cap S_{k+i}) &= \dim f S_{k+i} \cap (d - (f C \cup Z_k)) \\ &\leq p_{k+i} + 2 + 2p_N - q - q \\ &\leq p_{k+1} - 3. \end{aligned}$$

Now $C \cap S_{k+i}$ is a (collapsible) $p_{k+1} - 2$ -spine of S_{k+i} and D is a $p_{k+1} -$

— 1-spine of $\text{Int } W$ and so there exists Y in $\text{Int } W$ such that Y collapses to Z_k , $\Sigma f \subset f^{-1}Y$, and $f_{k+i}^{-1}Y$ collapses to $*$ ($i > 0$). The proof of lemma 2 is completed by showing that (as in lemma 1) $f: X \rightarrow Y \cup fX$ is a homotopy equivalence.

It remains to prove the theorem of § 2.

PROOF OF THEOREM. (1) Surjectivity of Φ . If $f: X \rightarrow W$ is a thickening, homotop f into general position in the sense of lemma 3 and use lemma 2 to obtain a manifold W_0 in $\text{Int } W$ such that $fX \subset \text{Int } W$ and $f: X \rightarrow W_0$ is a thickening representing an element in the image of Φ . See Remark after lemma 2. The S cobordism theorem provides us with an equivalence between the thickenings $f: X \rightarrow W_0$ and $f: X \rightarrow W$ and so Φ is surjective.

(2) Injectivity of Φ . Consider the special case $X = S^1 \vee, \dots, \vee S^1 \vee S^p$; the proof for more spheres is similar.

Let $*$ be the barycenter of the simplex Δ . Let h_0, h_1 be two embeddings of the solid torus $\partial \Delta \times \Delta$ in ∂P ($\dim \partial \Delta = p - 1$ and $\dim \partial \Delta \times \Delta = \dim \partial P$). Let the handlebody corresponding to h_i be $H(h_i) = P \cup_{h_i} \Delta \times \Delta$ ($i = 0, 1$). Let $\delta_i: \Delta \times \Delta \rightarrow H(h_i)$ and $p_i: P \rightarrow H(h_i)$ be the associated embeddings (thus $p_i^{-1} \delta_i = h_i$ i. e. $\forall x \in \partial \Delta \times \Delta, \delta_i x = p_i h_i x$). Suppose that h_0, h_1 determine equivalent thickenings (the equivalence being a homeomorphism $G: H(h_1) \rightarrow H(h_0)$). Then a relative version of the proof of surjectivity shows that there exist embeddings

$$\alpha: \Delta \times \Delta \times [0, 1] \rightarrow H(h_0) \times [0, 1]$$

$$\beta: P \times [0, 1] \rightarrow H(h_0) \times [0, 1] \text{ such that}$$

$$\begin{aligned} 1) \quad & \alpha(x, 0) = (\delta_0 x, 0) \\ & \alpha(x, 1) = (G \delta_1 x, 1) \end{aligned}$$

$$\beta(x, 0) = (p_0 x, 0)$$

and

$$\beta P \times \{1\} = G p_1 P \times \{1\}$$

$$2) \quad \alpha^{-1} \text{Im } \beta = \partial \Delta \times \Delta \times [0, 1].$$

Thus we have a concordance $\alpha^{-1} \circ \beta | \partial \Delta \times \Delta \times [0, 1] \rightarrow \partial P \times [0, 1]$ between h_0 and $\lambda \circ h_1$ where $\lambda: P \rightarrow P$ is a self equivalence (i. e. an orientation — preserving homeomorphism homotopic to the identity). We need to show that $\lambda \circ h_1$ and h_1 are concordant.

First we choose λ of a special type. Let $I^1 = [-1, +1]$ and $I^k = I^1 \times \dots \times I^1 \subset \mathbb{R}^k$. Then if $q \geq 3$ we take $P^q = P^3 \times I^{q-3}$. $P^3 = B \cup H$ is the union of a 3-ball B and disjointly-attached 1-handles.

Let C be the union of the set of cores of these handles. Then the reader may verify the following.

PROPOSITION. Any self-equivalence $\lambda: P^3 \times I^k \rightarrow P^3 \times I^k$ is concordant to one of the form $\mu \times \text{Id}$, where $\mu|_{B \cup C} = \text{Id}$. As for h_1 , we may clearly assume that $\text{Im } h_1$ lies in $\text{Int } P^q \times \{-1\} \subset \partial(P^q \times I^1)$. It will suffice then to prove the following.

LEMMA. If $\lambda: P^q \rightarrow P^q$ is a self-equivalence and $\Sigma^p \subset \text{Int } P$ a sphere ($p \leq q - 3$) then λ is concordant to λ' where λ' fixes (pointwise) a neighbourhood of Σ in P .

PROOF. By the proposition above choose $\lambda = \mu \times \text{Id}$, with $\mu|_{B \cup C} = \text{Id}$. Thus $\lambda|_{B \times I^{q-3} \cup C \times I^{q-3}} = \text{Id}$. The result of [2] is easily generalized to show that Σ can be compressed (by an ambient isotopy) into $B \times I^{q-3} \cup C \times I^{q-3}$ [the intersection of Σ with $C \times I^{q-3}$ being a set of disjoint cylinders (= homeomorphs of $S^{p-1} \times [0, 1]$). Thus λ fixes Σ . It remains to show that after an isotopy λ fixes not only Σ but some neighbourhood of Σ in P .

Let \tilde{P} be the universal cover of P with covering projection $\pi: \tilde{P} \rightarrow P$. Since Σ is inessential in P choose a connected component $\tilde{\Sigma}$ of $\pi^{-1}\Sigma$; thus $\Sigma, \tilde{\Sigma}$ are homeomorphic via π . Furthermore, in a neighbourhood of $\tilde{\Sigma}$, π is (1-1). Let $\tilde{\lambda}: \tilde{P} \rightarrow \tilde{P}$ be the lift of λ that fixes $\tilde{\Sigma}$ pointwise i. e. $\pi \circ \tilde{\lambda} = \lambda \circ \pi$ and $\tilde{\lambda}|_{\tilde{\Sigma}} = \text{Id}$.

Since $\lambda|_{B \times I^{q-3}} = \text{Id}$ there is a q -ball R in \tilde{P} with $\tilde{\lambda}R = R$ and $\tilde{\Sigma} \subset \text{Int } R$.

It follows from Lemma 59 of [7] that $\tilde{\lambda}$ is isotopic (fixing $\tilde{\Sigma}$ to $A: \tilde{P} \rightarrow \tilde{P}$ that fixes R pointwise. Projecting down by π we see that there is an ambient isotopy of P that takes λ to λ' where λ' is the inclusion in a neighbourhood of Σ . This completes the proof of the lemma and hence of the theorem.

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